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On the behaviour of elastic materials with scattered extending cracks

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BULK constitutive relations (in local approximation) are derived for an elastic material with numerous random cracks which may extend at stress increase. Scattered non-interacting cracks are assumed and a simplified crack extension criterion restricted to forces normal to plane cracks is applied. To stationary cracks which may open or close depending on the sense of internal forces there corresponds a pseudo-linear elastic domain and the constitutive relations given in [1] and [2]. The shape of this domain and of the bounding surface depending on the stress path is analysed and its representations in the 3-dimensional space and in the stress space are given. The damage function is defined, yielding the crack intensity distribution after spatial orientation. Incremental stress/strain relations are derived in the interior and at the border of the elastic domain, that is for stationary and for extending cracks.

Wyprowadza się makroskopowe związki konstytutywne w przybliżeniu lokalnym dla materiału sprężystego z licznymi losowymi rysami, które mogą narastać przy wzroście naprężeń. Zakłada się rysy rozproszone bez interakcji oraz uproszczone kryterium propagacji rys ograniczone do sił normalnych do rysy płaskiej. Rysom stacjonarnym mogącym się otwierać lub zamykać w zależności od wzrostu sił wewnętrznych odpowiada obszar sprężystości pseudoliniowej w przestrzeni naprężeń i związki konstytutywne podane w [1 i 2]. Analizuje się postać tego obszaru i powierzchni ograniczającej w zależności od drogi (historii) naprężenia i podaje się sposoby jego przedstawienia w przestrzeni trójwymiarowej i w przestrzeni naprężeń. Definiuje się funkcję uszkodzenia podającą rozkład zagęszczenia rys według orientacji przestrzennej. Wyprowadza się związki przyrostowe naprężeń i odkształceń wewnątrz i na granicy obszaru sprężystego, tj. dla rys stacjonarnych i rys propagujących się.

Выводятся макроскопические определяющие соотношения, в локальном приближении, для упругого материала с многими случайными трещинами, которые могут нарастать при росте напряжений. Предполагаются рассеянные трещины без взаимодействия и упрощенный критерий распространения трещин, ограниченный нормальными силами к плоской трещине. Стационарным трещинам, могущим открываться или закрываться в зависимости от направления внутренних сил, отвечает область псевдолинейной упругости в пространстве напряжений и определяющие соотношения приведены в [1 и 2]. Анализируется вид этой области и ограничивающей поверхности в зависимости от пути (истории) напряжения и приводятся способы его представления в трехмерном пространстве и в пространстве напряжений. Определяется функция повреждения, приводящая распределение плотности трещин по пространственной ориентировке. Выводятся соотношения в приростах напряжений и деформаций внутри и на границе упругой области, т.е. для стационарных и распространяющихся трещин.

1. Preliminaries

THE BEHAVIOUR of elastic cracked materials features certain similarity to elastic-plastic materials with strain hardening. There exists an interior domain in the stress space (analogous to the elastic domain in the theory of plasticity) where the cracks are stationary, that is do not extend, and a bounding hypersurface (analogous to the plastic yield surface) where these can extend [1, 3]. Observe that crack extension may be looked upon as the counterpart of plastic slip. Many available theories recognize more or less this behaviour and assume as a rule, irrespectively of introduced generalizations, that the interior domain is linear (in the simplest case linear elastic, may be anisotropic because of cracks), and that the incremental relations at the bounding surface for active stress increments are provided by a linear transformation, like in plasticity. In order to specialize these relations and to establish simple rules of "expansion" of the hypersurface (analogously to strain hardening in plasticity) one readily makes use of the concept of internal state parameters.

The present author has shown [1] that such assumptions cannot be correct since neither the interior domain nor the incremental relations (under crack extension) can be linear. This follows from unilateral internal constraints yielded by the cracks which may open or close depending on local internal forces and leads to nonlinearity of a special kind (called pseudo-linearity). The general form of the relevant constitutive relations for elastic materials with stationary frictionless cracks is given in [1] while in [3] the incremental relations for extending cracks are derived and a detailed discussion of the theory as compared with classical plasticity is carried out.

Now the question arises how these general purely phenomenological relations (which hold for any crack geometry, amount and distribution) can be specialized for different crack patterns. A comparatively simple solution can be found for non-interacting cracks and in [2] the respective relations have been derived for stationary cracks. The present paper spreads the argument over extending cracks (bounding hypersurface) and completes the theory for non-interacting cracks. Even in this simple case a scalar or tensor damage parameter has proved insufficient for crack pattern description in the context of deriving pseudo-linear elastic tensor functions (generalization of the elastic tensor) and had to be replaced by crack intensity distribution functions after direction [2]. Thus the generalization depends on using the said nonlinear relations accompanied by a more comprehensive crack pattern description. The essential point is that an increase of damage bears not only on the behaviour at the bounding hypersurface but it also affects pseudo-linear properties in the interior domain (stationary cracks) which must be accounted for by the theory. Since we want to derive bulk properties from micro-phenomena, the quantities describing the latter should appear explicitly in the constitutive macro-relations (say local conditions of crack extension), unlike phenomenological theories using hypothetical macro-quantities (e.g. internal state parameters).

The theory provides an analysis of the successive following interdepencies: the shape of the bounding hypersurface as depending on the loading path (cf. the fundamental proposition in Sect. 4); the crack pattern (described by the crack intensity distribution function) as depending on the shape of the bounding surface; the elastic functions as depending on the crack intensity distribution functions; the stress increment—strain increment relations the transformation depending on the elastic functions, etc. According to the above, the layuot of paper is the following. In Sect. 2 the phenomenological background is briefly recalled (without proofs and detailed discussion which can be found in [1, 3])(1). In Sect. 3 the bounding conditions for the interior domain are discussed and a representation in the three-dimensional space is given. In Sect. 4 this problem is analysed

(1) The reader is recommended to get acquainted with the papers [1, 2] beforehand since we make extensive use of them.

in the stress space which allows us to show explicitly how the bounding hypersurface depends on stress history. In Sect. 5 we pass to incremental relations and define the crack intensity distribution function. In Sect. 6 the dependence of the elastic tensor function on the said function is given, again under recourse to [2] where these relations have been derived. Section 7 provides constitutive incremental relations. In Sect. 8 practical applications to brittle materials are briefly discussed.

Before we proceed let us discuss the simplifying assumption about non-interacting cracks which (i) are small scattered flat (approximately plane) cracks and (ii) extend only under normal-to-crack tension (the generalization of the latter assumption is discussed in Sect. 5). Observe that only such far-reaching restrictions make the theory simple enough to bring to light explicitly essential interdependencies and, at the same time, numerically manageable. The assumption (i) holds approximately for scattered cracks at distances equal about $(3 \div 4)$ — fold average crack diameter (observe that in multiphase materials neighbour grains prevent immediate joining of cracks). This enables one to use solutions for a single crack in the infinite medium, disregarding the interaction terms. The assumption (ii) will be seen to simplify the theory greatly (avoiding criteria of crack extension under complicated stresses). It is motivated by shear resistance exceeding as a rule many times the tensile one (say in concrete $3 \div 4$ times). According to this the cracks are assumed to be perpendicular to the principal tensile stress trajectories. The said assumption allows to disregard the in-plane shapes of cracks produced by directional shear so that the "averaged" crack may be supposed circular. This holds evidently for initial (say shrinkage) cracks in a macro-isotropic medium while the load cracks tend to grow into circular shapes since the latter yield the largest critical extension stress [4].

2. Phenomenological background

The phenomenological (macro) relations in local approximation are derived in [1 and 3], and here we only summarize the most essential results. Like in plasticity (with strain hardening) we have, in the stress space representation, an elastic domain bounded by a limit (yield) hyper-surface which may "expand" in the course of loading (Fig. 1). The

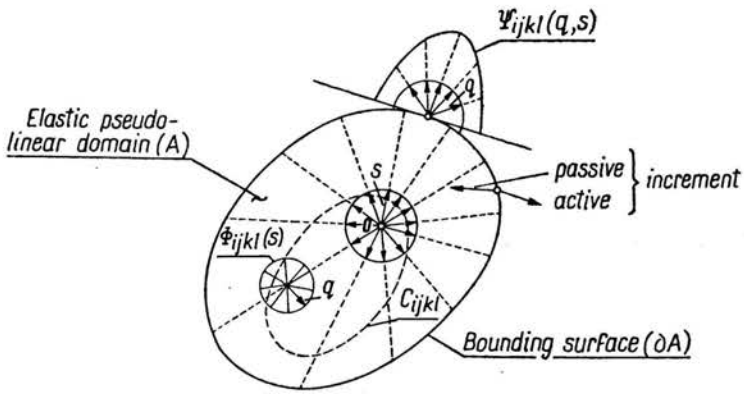

FIG. 1.

elastic domain corresponds to a stationary (not extending) crack system whereas in the bounding surface, for the outward ("active") stress increment directions, there is an extension of cracks accompanied by energy dissipation.

The stationary random cracks may open or close, following local forces, hence the material provides unilateral internal constraints and its behaviour is nonlinear of a special type called pseudolinear [1]. The medium has been shown to obey the following constitutive law [1]:

$$\boldsymbol{\sigma} = C(\mathbf{e})\boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} = S(\mathbf{s})\boldsymbol{\sigma},$$

(2.1)
$$\mathbf{e} = \frac{\boldsymbol{\epsilon}}{|\boldsymbol{\epsilon}|}, \quad \mathbf{s} = \frac{\boldsymbol{\sigma}}{\boldsymbol{\sigma}}, \quad |\boldsymbol{\epsilon}| = \sqrt{\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}}, \quad |\boldsymbol{\sigma}| = \sqrt{\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}},$$

where $\boldsymbol{\epsilon}$, $\boldsymbol{\sigma}$, C, S are consequtively strain and stress tensors, elastic stiffness and elastic compliance fourth order tensors; $\mathbf{e}$, $\mathbf{s}$ are the respective reduced (divided by the norm) tensors represented by unit vectors in the stress (strain) space. The C and S tensors are seen to depend on the direction in the relevant 6-space; in a fixed direction (that is under proportional loading, all stress components increasing proportionally) they preserve constant values. Thus the elastic domain features a distinct zero point and a "stellate" structure; upon assigning to each direction the corresponding value C_{ijkl} or S_{ijkl} we can plot the respective elastic tensor diagrams (Fig. 1). In the particular case of constant elastic tensors the diagrams would be spherical.

The incremental relation inside the elastic domain or for passive (inward) directions on the limit surface reads [3]

(2.2)
$$d\boldsymbol{\epsilon} = \phi(\mathbf{s})d\boldsymbol{\sigma} \quad \text{or} \quad \dot{\boldsymbol{\epsilon}} = \phi(\mathbf{s})\dot{\boldsymbol{\sigma}}$$

and, similarly for Eq. (2.1), the dot denoting rate terms, where $\phi(\mathbf{s})$ is a fourth order tensor-valued function of $\mathbf{s}$, the only argument. This means that for a given $\boldsymbol{\sigma}$ the momentary elastic tensor is constant (that is independent of the increment direction, consequently the transformation is linear); however, it depends on the point $\boldsymbol{\sigma}$, more precisely on the relevant direction $\mathbf{s}$. In particular, for different $\mathbf{s}$ the material may show different anisotropy following the system of the open and the closed cracks.

The simple relationship (2.2) holds no more for the active increment directions on the bounding surface, associated with crack increase and the following nonlinear transformation law is valid [3]:

$$\dot{\boldsymbol{\epsilon}} = [\phi(\mathbf{s})+\psi(\mathbf{q},\mathbf{s})]\dot{\boldsymbol{\sigma}},$$

(2.3)
$$\mathbf{q} = \frac{\dot{\boldsymbol{\sigma}}}{|\dot{\boldsymbol{\sigma}}|}, \quad |\dot{\boldsymbol{\sigma}}| = \sqrt{\dot{\boldsymbol{\sigma}}\cdot\dot{\boldsymbol{\sigma}}},$$

where Ψ is again a fourth order tensor depending this time on the increment direction $\mathbf{q}$ (cf. Fig. 1). In the particular case when the expression in the brackets does not depend on $\mathbf{q}$ the relation would become analogous with the respective one in strain-hardening plasticity.

The active stress increments give rise to changes in the bounding surface of the elastic domain and in the C, S-diagrams because then the crack system is changed. Under the assumption of irreversibility of the crack damage (we do not admit, say, a crack healing) the active incerements may only cause an "increase" of damage, i.e. of the (single and total) crack surface. Irrespectively of what mathematical definition of "damage" will be introduced, we should be able, according to the above observation, to set an order relation in the abstract space of "damages". As long as the increase of damage gives rise to "expansion" of the elastic domain in the stress space, the crack extension is called stable; then in (2.3) $\dot{\boldsymbol{\sigma}} \cdot \dot{\boldsymbol{\epsilon}} \geqslant 0$ and the transformation is positive definite. Both the bounding surface in the $\boldsymbol{\sigma}$-space and the corresponding one in the $\boldsymbol{\epsilon}$-space extend, by which we understand that the new elastic domain includes the former one, in the set-theoretical meaning. Upon attaining a certain critical value of damage we arrive at a bounding surface a part of which (at least one point) reaches the limit of stability. The active $\boldsymbol{\sigma}$-increments are no more possible in these points while the active $\boldsymbol{\epsilon}$-increments make the $\boldsymbol{\sigma}$-surface "shrink" (at least partially). This corresponds to the declining part in the one-dimensional $\boldsymbol{\epsilon}$-controlled $\boldsymbol{\sigma}(\boldsymbol{\epsilon})$ curve. The "increase" in $\boldsymbol{\epsilon}$ is associated with a "decrease" of $\boldsymbol{\sigma}$ (the transformation (2.3) is no more positive definite) and the volume element progressively gets over carrying forces.

The functions $S(\mathbf{s})$ (together with $\phi(\mathbf{s})$ obtained by derivation) ψ $(\mathbf{q}; \mathbf{s})$ and the bounding surface of the elastic domain (and similarly for Eq. $(2.1)_1$) are implicitly interconnected by the primary quantity of crack damage which depends, in turn, on the loading path. There arises the question whether they could be determined theoretically on certain "structural" premises. This turns out to be possible in a fairly simple way only under the assumptions discussed in Sect. 1, that is for the scattered cracks and stable crack extension. We discuss first the bounding surface of the elastic domain and its dependence on the loading path since this will be seen not to require the notion of damage (Sects. 3 and 4). Then we introduce the latter concept and examine how the crack system will be changed under an active stress increase (Sect. 5). Finally we analyse how this bears upon the pseudo-linear elastic properties of the cracked material (Sect. 6) and what are the stress increment—strain increment relations under increase of damage (Sect. 7). All the mentioned factors concur to bring about what we call bulk behaviour of the cracked material.

3. Elastic domain

Consider an elementary plane crack with spatial orientation provided by the unit vector $\mathbf{n}$ normal to crack surface. The arrows of these vectors point a unit hemisphere since we assimilate opposite crack surfaces. Consequently, if $\mathbf{a}$ is an arbitrary fixed vector, we take into account only the $\mathbf{n}$ satisfying $\mathbf{n} \cdot \mathbf{a} \geqslant 0$. Consider a random system of such cracks and separate the subsystem of all cracks with the orientation $\mathbf{n}$ (more precisely, in the infinitesimal cone about $\mathbf{n}$). This system will be called the partial system or in brief the $\mathbf{n}$-system (for a more detailed discussion of this concept cf. [2]). If the cracks are non-interacting, so are the partial systems.

Let $\boldsymbol{\sigma}^{(\mathbf{n})} = \boldsymbol{\sigma}\mathbf{n}$ denote the macro-stress vector in the cross-section $\mathbf{n}$, with $\boldsymbol{\sigma}$ the macro-stress tensor. Let, in turn, $\boldsymbol{\sigma}_{\mathbf{n}}^{(\mathbf{n})}$ and $\boldsymbol{\sigma}_{t}^{(\mathbf{n})}$ denote consecutively the normal and the tangential stress vector component of $\boldsymbol{\sigma}^{(\mathbf{n})}$, with the respective intensities $\sigma_{\mathbf{n}}^{(\mathbf{n})}$, $\sigma_{\mathbf{t}}^{(\mathbf{n})}$; obviously

$$
\begin{aligned}
\sigma_{\mathbf{n}}^{(\mathbf{n})} &= \mathbf{n}\cdot\boldsymbol{\sigma}\mathbf{n} = \boldsymbol{\sigma}\cdot(\mathbf{n}\otimes\mathbf{n}) = \sigma_{ij}n_in_j,\\
\boldsymbol{\sigma}_{\mathbf{n}}^{(\mathbf{n})} &= \sigma_{\mathbf{n}}^{(\mathbf{n})}\mathbf{n} = (\mathbf{n}\otimes\mathbf{n})\boldsymbol{\sigma}^{(\mathbf{n})},\\
\sigma_{\mathbf{t}}^{(\mathbf{n})} &= \boldsymbol{\sigma}\mathbf{n}-(\mathbf{n}\cdot\boldsymbol{\sigma}\mathbf{n})\mathbf{n} = (I-\mathbf{n}\otimes\mathbf{n})\boldsymbol{\sigma}^{(\mathbf{n})},\\
(\sigma_{\mathbf{t}}^{(\mathbf{n})})^2 &= (\sigma^{(\mathbf{n})})^2-(\sigma_{\mathbf{n}}^{(\mathbf{n})})^2,
\end{aligned}
\tag{3.1}
$$

where $(\sigma^{(\mathbf{n})})^2 = \boldsymbol{\sigma}^{(\mathbf{n})}\cdot\boldsymbol{\sigma}^{(\mathbf{n})}$. In the 3-space $\boldsymbol{\sigma}$ is looked upon as a transformation over the vector $\mathbf{n}$ whereas $\mathbf{n}\otimes\mathbf{n}$, $I-\mathbf{n}\otimes\mathbf{n}$ (with I the identity operator) are projective operators over the stress vector $\boldsymbol{\sigma}^{(\mathbf{n})}$. In the 9-space the diadic symmetric tensors $\mathbf{n}\otimes\mathbf{n}$ provide unit vectors in the symmetric stress 6-subspace. No dot products are interpreted as matrix-on-matrix (in patricular vector) operations while dot (scalar) products implicate overlapping over all indices.

Suppose the crack $\mathbf{n}$-system is stationary in the elastic domain $\mathscr{A}_{\mathbf{n}}$, that is none of the $\mathbf{n}$-cracks extends at any increment $d\boldsymbol{\sigma}$ provided $\boldsymbol{\sigma}\in\mathscr{A}_{\mathbf{n}}$; in other words, $\mathscr{A}_{\mathbf{n}}$ is the partial stationarity domain. The (overall) elastic domain $\mathscr{A}$ is defined as the set of all $\boldsymbol{\sigma}$ where the crack system (as a whole) is stationary, consequently $\mathscr{A} = \bigcap\limits_{\mathbf{n}}\mathscr{A}_{\mathbf{n}}$. Let $\bar{\mathscr{A}}$ denote the closure of $\mathscr{A}$ and $\partial\mathscr{A} = \bar{\mathscr{A}}\setminus\mathscr{A}$ the bounding hyper-surface (the envelope of all the partial ones). For $\boldsymbol{\sigma}\in\partial\mathscr{A}$ the cracks may or may not extend depending on the direction of $d\boldsymbol{\sigma}$.

According to the assumptions discussed in Sect. 1 the $\mathbf{n}$-cracks do not extend if

$$
\sigma_{\mathbf{n}}^{(\mathbf{n})} < \begin{cases} \hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})} & \text{for} \quad \hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})} > 0,\\ 0 & \text{for} \quad \hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})} \leqslant 0,\end{cases}
\tag{3.2}
$$

where $\hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})} = \max\limits_{\tau}\sigma_{\mathbf{n}}^{(\mathbf{n})}(\tau)$ denotes the maximum normal-to-crack stress in the loading path (history), τ denoting the time parameter (up to the present moment) and $\sigma_{\mathbf{n}}^{(\mathbf{n})}$ taken algebraically (positive for tension). Should the material possess certain inherent (initial) crack resistance, we would replace (3.2) by

$$
\sigma_{\mathbf{n}}^{(\mathbf{n})} < \begin{cases} \hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})} & \text{for} \quad \hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})} > \varrho(\mathbf{n}),\\ \varrho(n) & \text{for} \quad \hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})} \leqslant \varrho(\mathbf{n}),\end{cases}
\tag{3.3}
$$

where $\varrho(\mathbf{n})$ denotes the initial crack resistance for the $\mathbf{n}$-system; for strength isotropy $\varrho(\mathbf{n}) = \varrho = \text{const}$. Consequently, the elastic domain for (3.2) and (3.3), respectively, is given by

$$
\begin{aligned}
\mathscr{A} &= \bigcap_{\mathbf{n}}\{\boldsymbol{\sigma}:\ \sigma_{\mathbf{n}}^{(\mathbf{n})} < \max[0, \hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})}]\},\\
\mathscr{A} &= \bigcap_{\mathbf{n}}\{\boldsymbol{\sigma}:\ \sigma_{\mathbf{n}}^{(\mathbf{n})} < \max[\varrho(\mathbf{n}):\ \hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})}]\},
\end{aligned}
\tag{3.4}
$$

where $\sigma_{\mathbf{n}}^{(\mathbf{n})}$ is provided by (2.1). Observe that the dependence on the stress path is restricted to the values of $\boldsymbol{\sigma}$ yielding max $\sigma_{\mathbf{n}}^{(\mathbf{n})}$ and no functional strength criterion (interconnecting, say, the quantities $\sigma_{\mathbf{n}}^{(\mathbf{n})}$'s and $\sigma_{\mathbf{t}}^{(\mathbf{n})}$) is involved. This holds obviously for scattered self-similar (say circular) cracks. Instead, taking account of $\sigma_{\mathbf{t}}^{(\mathbf{n})}$ and the transversal shapes of cracks would require keeping track of the stress path for each $\mathbf{n}$-system with no simple superposition of effects.

The most direct representation of the elastic domain $\mathscr{A}$ can be given in a three-dimensional space if we assign to each direction $\mathbf{n}$ a guiding radius with the length $\sigma_{\mathbf{n}}^{(\mathbf{n})}$. To each $\boldsymbol{\sigma}(\tau)$ in the stress path there corresponds a domain of the "admissible" $\sigma_{\mathbf{n}}^{(\mathbf{n})}$,

$$\sigma_{\mathbf{n}}^{(\mathbf{n})}(\mathbf{n}) \leqslant \mathbf{n}\cdot\boldsymbol{\sigma}(\tau)\mathbf{n} = \sigma_{ij}(\tau)n_i n_j. \tag{3.5}$$

The union of these domains for all τ provides the domain of admissible $\sigma_{\mathbf{n}}^{(\mathbf{n})}$, bounded by the surface $\sigma_{\mathbf{n}}^{(\mathbf{n})} = \hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})}(\mathbf{n}) = \max\,[\sigma_{ij}(\tau)n_i n_j]$. The surfaces in (3.5) (obtained with the equality sign) form a one-parametric family with τ the parameter. For differentiable functions $\sigma_{ij}(\tau)$ the necessary extremum condition reads

$$\dot{\boldsymbol{\sigma}}(\tau)\cdot(\mathbf{n}\otimes\mathbf{n}) = \dot{\sigma}_{ij}(\tau)n_i n_j = 0 \tag{3.6}$$

and the family possesses an envelope surface, found by eliminating τ between Eq. (3.5) (with the equality sign) and Eq. (3.6).

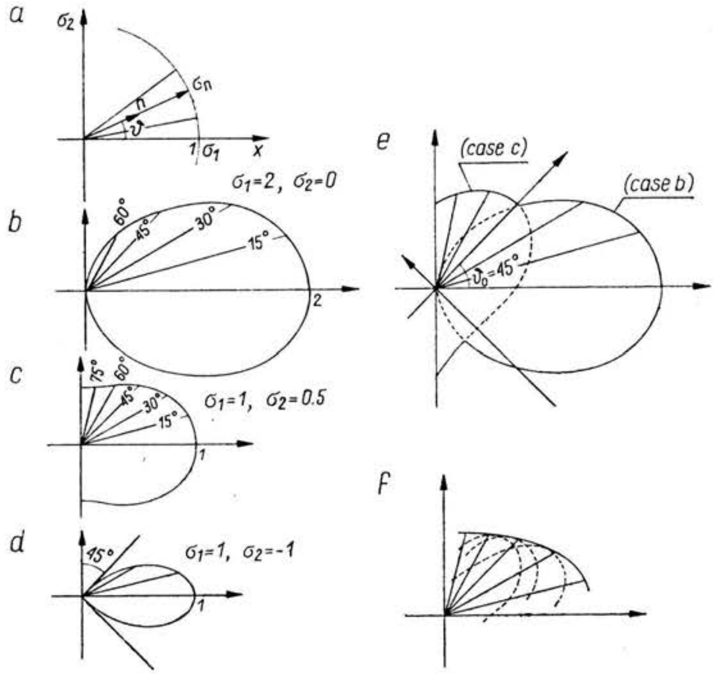

FIG. 2.

For the practical plotting at the bounding surface it is convenient to express Eq. (3.5) in principal stresses

$$\sigma_{\mathbf{n}}^{(\mathbf{n})} = \boldsymbol{\sigma}\cdot(\mathbf{n}\otimes\mathbf{n}) = \Big(\sum_I \sigma_I \mathbf{n}_I\otimes\mathbf{n}_I\Big)\cdot(\mathbf{n}\otimes\mathbf{n}) = \sum_I \sigma_I(\tau)n_I^2(\tau). \tag{3.7}$$

$I = 1, 2, 3$, where σ_I are the principal stresses and $\mathbf{n}_I$ the principal directions of $\boldsymbol{\sigma}$; $n_I = \mathbf{n}_I\cdot\mathbf{n}$ are components of $\mathbf{n}$ in the orthogonal frame of $\mathbf{n}_I$. Figure 2 shows the diagrams of $\sigma_{\mathbf{n}}^{(\mathbf{n})}$ vs. $\mathbf{n}$ in the two-dimensional case where, in polar coordinates, $\mathbf{n} = [\cos\vartheta, \sin\vartheta]^T$. According to Eq. (3.7) with $I = 1, 2$ we get

$$\sigma_{\mathbf{n}}^{(\mathbf{n})}(\vartheta) = \sigma_1\cos^2(\vartheta-\vartheta_1)+\sigma_2\sin^2(\vartheta-\vartheta_1),$$

where ϑ_1 corresponds to n_1. In Figs. 2a÷d curves for different pairs (σ_1, σ_2) are plotted, one of the principal directions coinciding with the x-axis (one should keep in mind that $\sigma_{\mathbf{n}}^{(\mathbf{n})} > 0$ otherwise the bounding value is 0). Figure 2e shows a composition of the domain for two competent loads in the stress path and Fig. 2f illustrates the formation of the envelope curve. For $\varrho(\mathbf{n}) \neq 0$ the respective diagram should be plotted (yielding, say, a circle for strength isotropy) and it would interfere like in Fig. 2e.

4. Representation in the stress space

Let us examine now a more customary (at least, per analogy to plasticity) representation of the elastic domain $\mathscr{A}$ (cf. (3.4)), in the stress space. Consider, first, a stress path where each $\dot{\boldsymbol{\sigma}}(\tau)$ is a pure compression, that is for any $\mathbf{n}$, $\sigma_{\mathbf{n}}^{(\mathbf{n})} = \boldsymbol{\sigma}\cdot(\mathbf{n}\otimes\mathbf{n}) \leqslant 0$ (it is convenient to include $\sigma_{\mathbf{n}}^{(\mathbf{n})} = 0$). This is equivalent, according to Eq. (3.7), to a $\boldsymbol{\sigma}$ with non-positive principal stresses, $\sigma_I \leqslant 0$ and we call such a path *purely compressive*. Then the elastic domain (cf. (3.4))

$$\mathscr{A} = \bigcap_{\mathbf{n}} \{\boldsymbol{\sigma}\colon \boldsymbol{\sigma}\cdot(\mathbf{n}\otimes\mathbf{n}) \leqslant 0\} \tag{4.1}$$

is a convex cone in the stress space, with 0 the apex point called the *compressive cone*. Indeed, $\boldsymbol{\sigma} \in \bar{\mathscr{A}}$ implies $c\boldsymbol{\sigma} \in \bar{\mathscr{A}}$ for any real $c > 0$ and $\bar{\mathscr{A}}$ is convex since it is an intersection of the half-spaces $\boldsymbol{\sigma}\cdot(\mathbf{n}\otimes\mathbf{n}) \leqslant 0$. The cone apex angle is right. In fact, for any $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \bar{\mathscr{A}}$,

$$\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 = \Big(\sum_I \sigma_{1I}\mathbf{n}_{1I}\otimes\mathbf{n}_{1I}\Big)\cdot\Big(\sum_K \sigma_{2K}\mathbf{n}_{2K}\otimes\mathbf{n}_{2K}\Big)$$
$$= \sum_{I,K} \sigma_{1I}\sigma_{2K}(\mathbf{n}_{1I}\cdot\mathbf{n}_{2K})^2 \geqslant 0, \qquad I, K = 1, 2, 3$$

in view of $\sigma_{1I}, \sigma_{2K} \leqslant 0$, consequently the angle is not obtuse. On the other hand purely compressive $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$ may be orthogonal (take for instance uniaxial compressions in perpendicular directions), so the angle cannot be acute. Observe that each of the vectors $\mathbf{n}\otimes\mathbf{n}$ spans a one-dimensional subspace of uniaxial loading in the direction $\mathbf{n}$. The scalar product $\boldsymbol{\sigma}\cdot(\mathbf{n}\otimes\mathbf{n})$ in (4.1) yields the perpendicular projection of $\boldsymbol{\sigma}$ into this subspace resulting in the vector $\sigma_{\mathbf{n}}^{(\mathbf{n})}(\mathbf{n}\otimes\mathbf{n})$ of uniaxial stress (this vector may be looked upon as a 9-vector or a 6-vector in the symmetric 6-subspace, $\mathbf{n}\otimes\mathbf{n}$ being symmetric). The operation is isomorphic to forming the 3-vector $\sigma_{\mathbf{n}}^{(\mathbf{n})}\mathbf{n}$ in the Euclidean space.

The form of the elastic domain yielded by $(3.4)_1$ for the stress path (trajectory) $\boldsymbol{\sigma}(\tau)$ is defined in the following proposition: $\bar{\mathscr{A}}$ is the convex hull generated by the trajectory $\boldsymbol{\sigma}(\tau)$ and the compressive cone.

Here we assume that the virgin point $\boldsymbol{\sigma} = 0$ (which yields the compressive cone as the admissible region) belongs to $\boldsymbol{\sigma}(\tau)$; therefore points in infinity determined by the generatrices of the compressive cone surface are admissible and may be looked upon as points of the trajectory. Let $\hat{\boldsymbol{\sigma}}^{(\mathbf{n})}(\tau)$ denote the point (points) of $\boldsymbol{\sigma}(\tau)$ yielding $\hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})}$, i.e. $\hat{\boldsymbol{\sigma}}^{(\mathbf{n})}(\tau)\cdot(\mathbf{n}\otimes\mathbf{n}) = \hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})}$; then, according to $(3.4)_1$,

$$\bar{\mathscr{A}} = \bigcap_{\mathbf{n}} \bar{\mathscr{A}}_n = \bigcap_{\mathbf{n}} \{\boldsymbol{\sigma}\colon \boldsymbol{\sigma}\cdot(\mathbf{n}\otimes\mathbf{n}) \leqslant \hat{\boldsymbol{\sigma}}^{(\mathbf{n})}(\tau)\cdot(\mathbf{n}\otimes\mathbf{n})\}. \tag{4.2}$$

Since $\bar{\mathscr{A}}$ is the intersection of the closed half-spaces it is convex, and since each point of $\boldsymbol{\sigma}(\tau)$ belongs to $\bar{\mathscr{A}}$ it includes the path $\boldsymbol{\sigma}(\tau)$. Each of the points in $\partial\mathscr{A}$ belongs to a hyperplane passing through a point (points) of the trajectory and perpendicular to a vector $\mathbf{n}\otimes\mathbf{n}$; consequently, $\bar{\mathscr{A}}$ is the least convex set including $\boldsymbol{\sigma}(\tau)$ (any point outside it would belong to a plane not passing through $\boldsymbol{\sigma}(\tau)$ and would yield for a $\mathbf{n}\otimes\mathbf{n}$, $\boldsymbol{\sigma}\cdot(\mathbf{n}\otimes\mathbf{n}) > \max \sigma_{\mathbf{n}}^{(\mathbf{n})}$).

Obviously the argument can be extended to the case (3.3) and $(3.4)_2$. The compressive cone would be replaced by an "initial" domain bounded by the envelope surface of the planes perpendicular to the $\mathbf{n}\otimes\mathbf{n}$, at the respective distances $\varrho(\mathbf{n})$ from the origin point. The domain $\bar{\mathscr{A}}$ would be the convex hull generated by the path $\boldsymbol{\sigma}(\tau)$ and the said initial domain. In any case the bounding hyper-surface $\partial\mathscr{A}$ is formed by a hyper-plane osculating on the stress path, only the "extremum" parts of the latter being competent.

Consider, in particular, in the context of $(3.4)_1$ a path where for each τ and each $\mathbf{n}$

$$\boldsymbol{\sigma}(\tau)\cdot(\mathbf{n}\otimes\mathbf{n}) \leqslant \boldsymbol{\sigma}(\tau_0)(\mathbf{n}\otimes\mathbf{n}).$$

Then

$$\mathscr{A} = \bigcap_{\mathbf{n}} \{\boldsymbol{\sigma}\colon\ \boldsymbol{\sigma}\cdot(\mathbf{n}\otimes\mathbf{n}) \leqslant \boldsymbol{\sigma}_0\cdot(\mathbf{n}\otimes\mathbf{n})\} = \bigcap_{\mathbf{n}} \{\boldsymbol{\sigma}\colon(\boldsymbol{\sigma}-\boldsymbol{\sigma}_0)\cdot(\mathbf{n}\otimes\mathbf{n}) \leqslant 0\},$$

where $\boldsymbol{\sigma}_0 = \boldsymbol{\sigma}(\tau_0)$ is the stress that has been attained in the (previous or present) moment τ_0. The right hand expression shows that $\bar{\mathscr{A}}$ arises by a parallel shifting of the compressive cone up to the apex position $\boldsymbol{\sigma}_0$. Since $(\boldsymbol{\sigma}-\boldsymbol{\sigma}_0)$, the vector belonging to a convex cone generates an order relation $\boldsymbol{\sigma} \leqslant \boldsymbol{\sigma}_0$ in the stress space (a relation which is reflective and transitive; however, in general not total), we obtain the following proposition: the domain $\bar{\mathscr{A}}$ is a cone (namely a shifted compressive cone) with the apex point $\boldsymbol{\sigma}_0$ if for each τ, $\boldsymbol{\sigma}(\tau) \leqslant \boldsymbol{\sigma}_0$.

Analytically we obtain points of $\partial\mathscr{A}$, again, by solving the system of equations (3.5) (with the equality sign) and Eq. (3.6) the former of which is now expressed more explicitly

$$(\mathbf{n}\otimes\mathbf{n})\cdot\boldsymbol{\sigma} = (\mathbf{n}\otimes\mathbf{n})\cdot\boldsymbol{\sigma}(\tau) \tag{4.3}$$

or in components

$$\sigma_{ij}n_in_j = \sigma_{ij}(\tau)n_in_j,$$

where $\boldsymbol{\sigma}(\tau)$ denotes a point in the stress path while $\boldsymbol{\sigma}$ is (without explicitly writing up the argument) the current point of the respective hyper-plane. The interpretation of the equations is now different as compared with Sect. 3. A tensor $\boldsymbol{\sigma}$ was represented there by a surface in the 3-space and now it is a 9- or 6-vector. Equation (3.6) now yields the $\mathbf{n}\otimes\mathbf{n}$-vectors orthogonal to the element $d\boldsymbol{\sigma} = \dot{\boldsymbol{\sigma}}(\tau)\,d\tau$ of the stress trajectory. With Eq. (4.3) it provides hyper-planes passing through $d\boldsymbol{\sigma}$.

From the solutions of Eq. (3.6) with respect to τ (and for a fixed $\mathbf{n}$) we select the one (ones) making $\sigma_{\mathbf{n}}^{(\mathbf{n})}$ maximum, $\sigma_{\mathbf{n}}^{(\mathbf{n})} = \hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})}$. Suppose τ' and τ'' are two such solutions (Fig. 3); then in the "regular" case, according to Eqs. (3.6) and (4.3)′, the system of equations

$$\dot{\sigma}_{ij}(\tau')n_in_j = 0, \quad \dot{\sigma}_{ij}(\tau'')n_in_j = 0, \quad \sigma_{ij}(\tau')n_in_j = \sigma_{ij}(\tau'')n_in_j \tag{4.4}$$

supplemented by the condition $n_in_i = 1$ makes possible to express the quantities involved as functions of a parameter, say τ', i.e. $n_i = n_i(\tau')$, $\tau'' = \tau''(\tau')$, and yields a right- linear sector of $\partial\mathscr{A}$ passing through the points $\boldsymbol{\sigma}(\tau')$, $\boldsymbol{\sigma}(\tau'')$, while Eq. (4.3) yields the osculating hyper-plane. The latter planes form a one-parametric family (with the parameter τ'). The envelope surface of this family is determined by the system of equations

$$(4.5) \qquad n_i(\tau')n_j(\tau')\sigma_{ij} = \hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})}(\tau'), \qquad \frac{\partial}{\partial\tau'}[n_i(\tau')n_j(\tau')]\sigma_{ij} = \frac{\partial}{\partial\tau'}\hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})}(\tau')$$

the first of which is the mentioned Eq. (4.3) with the abbreviated notation of the right hand part and the second one is the usual envelope condition. Both equations yield the

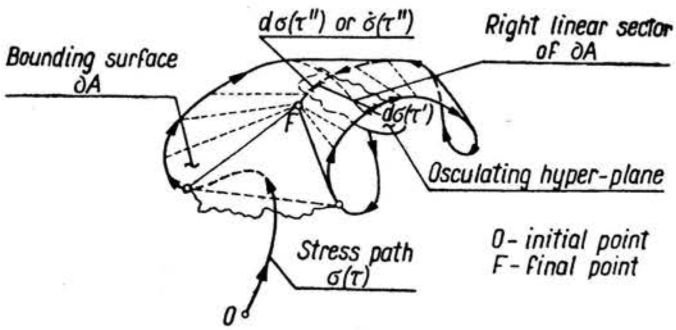

FIG. 3.

intersection of the neighbour hyper-planes, that is the right-linear generating sector as a function of the parameter τ'. Upon eliminating τ' between them we would obtain the equation of the respective part of $\partial\mathscr{A}$ (cf. Fig. 3). Observe that Eq. (3.6) may not have solutions. Using a substitution analogous to (3.7), that is expressing $\mathbf{n}$ in the frame of local principal axes of the tensor $\dot{\boldsymbol{\sigma}}$, we may obtain all proper values positive implying no solution. This is the case of all principal stresses increasing, i.e. for $d\boldsymbol{\sigma}$ in the tensile cone (the counterpart of the compressive cone). If this holds for each point of $\boldsymbol{\sigma}(\tau)$ (a simple path with increasing $\boldsymbol{\sigma}$ with respect to the previously introduced order relation), the elastic domain will be the shifted cone with the final (present) τ determining the apex point.

The condition (3.6) does not hold, obviously, for nondifferentiable loci of $\boldsymbol{\sigma}(\tau)$ and must be replaced by the general one $\sigma_{\mathbf{n}}^{(\mathbf{n})} = \hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})}$. If a point $\sigma(\tau)$ yields $\hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})}$ for many $\mathbf{n}$, it is a vertex point or an edge point (for a differentiable locus) of $\partial\mathscr{A}$. If, vice versa, for a given $\mathbf{n}$ there are many τ where $\boldsymbol{\sigma}(\tau)$ provides $\hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})}$, then the hyper-surface $\partial\mathscr{A}$ includes respectively: a right-linear sector for two solutions, a two-dimensional "triangle" (cf. Fig. 3) for three linearly independent solutions etc. up to a hyper-plane in the stress space.

5. Increase of damage

Let the internal state of a cracked material be described by the crack intensity function $p_{\mathbf{n}} = p_{\mathbf{n}}(\mathbf{n})$. The intensity may be defined as the total area of the elementary cracks (in the unit volume) with the orientation $\mathbf{n}$ (that is with the unit normal-to-crack vector $\mathbf{n}$

within the elementary cone about $\mathbf{n}$) per the elementary body angle $d\omega$. Then the total area of all cracks amounts to $P = \int_{\Omega} p_{\mathbf{n}}(\mathbf{n})\, d\omega$ where we integrate over a hemisphere (i.e. the solid angle of Ω is 2π) since the opposite crack edges are assimilated to a surface element and $\mathbf{n}$ points into a fixed (but arbitrary) hemisphere. Consequently, we may also write $p_{\mathbf{n}}(\mathbf{n}) = P\pi(\mathbf{n})$ where $\pi(\mathbf{n}) = p_{\mathbf{n}}/P$ is a normalized intensity distribution after orientation. Since $\int_{\Omega} \pi(\mathbf{n})\, d\omega = 1$ it may be looked upon as a probability distribution of cracks after orientation.

The crack intensity $p_{\mathbf{n}}$ is seen to correspond with the $\hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})}$-criterion (disregarding the in-plane crack shapes). We assume that there exists a functional dependence $p_{\mathbf{n}} = f(\hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})})$, more explicitly,

$$p_{\mathbf{n}}(\mathbf{n}) = f[\hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})}(\mathbf{n})]. \tag{5.1}$$

That is to say, the damage $p_{\mathbf{n}}$ depends on the maximum $\sigma_{\mathbf{n}}^{(\mathbf{n})}$ in the stress history and not on the way it has been attained. The crack increase is supposed not to be a spontaneous process (like, say, in the Griffith theory) so that we remain within the stability domain. This is justified by plastic screening at crack border extension and crossing grains with variable strength (first of all the reinforcement). Consequently, the function f in Eq. (5.1) is an increasing function and the function $p_{\mathbf{n}}(\mathbf{n})$ may be plotted, by means of Eq. (5.1), on graphs like in Fig. 2. Should the material exhibit crack resistance anisotropy, then Eq. (5.1) will assume the more general form

$$p_{\mathbf{n}}(\mathbf{n}) = f[\hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})}(\mathbf{n}), \mathbf{n}]. \tag{5.2}$$

In case of the extended criterion (3.3) we have $p_{\mathbf{n}} = 0$ for $\hat{\sigma}_{\mathbf{n}}^{(\mathbf{n})} \leqslant \varrho(\mathbf{n})$. Observe that the function $p_{\mathbf{n}}(\mathbf{n})$ plays the role of an internal state parameter, a scalar or tensor parameter being generalized to a function.

In case of non-interacting plane cracks the function $p_{\mathbf{n}}(\mathbf{n})$ will be seen to suffice for deriving constitutive relations for stationary and for extending cracks. The function f in Eqs. (5.1) and (5.2) is to be found from the theory of a single plane crack in the infinite medium providing the dependence of crack size on external (perpendicular to crack) tension. This problem (of crack extension) has an extensive literature and many well-known approaches are available; however, it does not belong to the present theory. Our objective is to show how the bulk relations depend on micro-phenomena described by the function f (presupposed to be known) which, therefore, should enter explicity constitutive relations.

6. Elastic functions

Having expressed by (5.1) or (5.2) the state of damage, we have to find the elastic functions $C(\mathbf{e})$, $S(\mathbf{s})$ in (2.1) (cf. the respective diagrams in Fig. 1). This problem has been solved in [2] where the reader is referred to and we only summarize certain basic results. The tensor function $S(\mathbf{s})$, say, (analogus results hold for $C(\mathbf{c})$) is provided by (cf. [2], Eq. (2.3))

$$S(\mathbf{s}) = \widehat{S} + S_a(\mathbf{s}), \tag{6.1}$$

where $\widehat{S}$ is a constant term (independent of **s**) and S_a the additional term depending on the state of stress. $\widehat{S}$ is the elastic compliance tensor for a medium with open cracks (imagine narrow slits yet wide enough not to be made closed) while $S_a(\mathbf{s})$ is brought about by response forces at closed cracks.

For non-interacting cracks the terms in question are expressed by the damage function in the following way ([2], Eqs. (2.3), (2.5) and (3.5)):

$$(6.2)\qquad \begin{aligned}\widehat{S} &= S_0 + \int_{\Omega} p_{\mathbf{n}}(\mathbf{n}) S_b^{(\mathbf{n})}(\mathbf{n}) d\omega,\\ S_a(\mathbf{s}) &= \int_{\Omega} p_{\mathbf{n}}(\mathbf{n}) \boldsymbol{\eta}^{(\mathbf{n})}(\mathbf{n}) \otimes \mathbf{n} \otimes \mathbf{n} d\omega.\end{aligned}$$

S_0 is the elastic compliance tensor for plain (non-cracked) material and the domain of integration, Ω is the whole hemisphere. $S_b(\mathbf{n})$ denotes the damage compliance tensor called "basic" for the partial **n**-oriented crack system; it accounts for the open **n**-cracks per unit damage. $\boldsymbol{\eta}^{(\mathbf{n})}(\mathbf{n})$ is the "additional" strain due to response forces at closed cracks in the partial **n**-system. The domain of integration Ω_s is provided, for a given $\boldsymbol{\sigma}$, by the condition $\boldsymbol{\sigma} \cdot (\mathbf{n} \otimes \mathbf{n}) < 0$ or equivalently $\mathbf{s} \cdot (\mathbf{n} \otimes \mathbf{n}) < 0$. The functions $S_b^{(\mathbf{n})}(\mathbf{n})$, $\boldsymbol{\eta}^{(\mathbf{n})}(\mathbf{n})$ are material constants (independent of the state of damage). In case strength istotropy they do not depend on **n** (to orthogonal transformation); then $S_b^{(\mathbf{n})}$ involves 5 constants and $\boldsymbol{\eta}^{(\mathbf{n})}$ 2 constants ([2], (2.4) and (3.4)). They are defined for unit crack concentration (resolved crack systems, [2]) i.e. for $\pi(\mathbf{n})$ (Sect. 5), consequently, the integral terms in (6.2) are proportional to P. That is to say, we consider a medium with **n**-oriented cracs having a unit total area in the unit volume and employ approximately the solution for a typical single crack in the unbounded medium. For example, for $\boldsymbol{\eta}^{(\mathbf{n})}$ to be calculated theoretically we use the solution for a medium loaded with $\boldsymbol{\sigma} = \mathbf{n} \otimes \mathbf{n}$, i.e.$\sigma_{\mathbf{n}}^{(\mathbf{n})} = 1$. Then the mutual displacement of the opposite crack edges, integrated over the area of the crack and multiplied by the number of cracks per unit area yields the **n**-directed principal strain of $\boldsymbol{\eta}^{(\mathbf{n})}$ This follows from the principle of virtual work as applied to the effective displacement and virtual forces at crack edges, to be equated with the work over average strain.

7. Constitutive relations

In order to find the incremental constitutive relations (cf. (2.2) and (2.3)) and the functions ϕ, Ψ, let us form the differential of Eq. $(2.1)_2$:

$$(7.1)\qquad d\varepsilon_{ij} = d[S_{ijkl}(s_{pq})\sigma_{kl}] + \delta S_{ijkl}(s_{pq})\sigma_{kl}.$$

The first term corresponds to the increment $d\boldsymbol{\sigma}$ under a fixed function $S(\mathbf{s})$; the second one follows from the variation of this function. The first term is the only one for points inside the elastic domain $\mathscr{A}$ or for points in $\partial\mathscr{A}$ for passive directions of $d\boldsymbol{\sigma}$. By usual differentiation and suitable changing of the denotation of indices we bring it to the form

(7.2) $$d[S_{ijkl}(\mathbf{s}_{pq})\sigma_{kl}] = S_{ijkl}d\sigma_{kl} + \frac{\partial S_{ijkl}}{\partial s_{pq}} \frac{\partial s_{pq}}{\partial \sigma_{rs}} d\sigma_{rs}\sigma_{kl}$$

$$= \left(S_{ijkl} + \frac{\partial S_{ijmn}}{\partial s_{pq}} \frac{\partial s_{pq}}{\partial \sigma_{kl}} \sigma_{mn}\right) d\sigma_{kl} = \phi_{ijkl} d\sigma_{kl}.$$

The expression in the parentheses in seen to be the function ϕ in (2.2) of the argument $\mathbf{s}$ which readily may be checked by differentiation under substitution of the function $\mathbf{s}(\boldsymbol{\sigma})$ from (2.1). According to (6.1) and (6.2) only the part $S_a(\mathbf{s})$ depends on $\mathbf{s}$ and undergoes differentiation, where for strength isotropy the integrated function $\boldsymbol{\eta}^{(\mathbf{n})}(\mathbf{n})$ has been shown to take the form ([2], Eq. (3.4))

(7.3) $$\boldsymbol{\eta}^{(\mathbf{n})}(\mathbf{n}) = a\mathbf{I} + b\mathbf{n}\otimes\mathbf{n},$$

a, b being material constants and $\mathbf{I}$ the unit second order tensor. Consequently,

(7.4) $$S_{aijkl}(s_{pq}) = \int_{\Omega_s} p_{\mathbf{n}}(n_r)(a\delta_{ij}n_k n_l + b n_i n_j n_k n_l) d\omega,$$

where Ω_s is the elliptical cone $s_{pq}n_p n_q < 0$ (it comes out explicitly on the basis of principal directions of $\mathbf{s}$; then $\sum_{k=1}^{3} s_k n_k^2 < 0$ with s_K the principal stresses and n_K the principal directions). In simpler cases explicit formulae can be obtained upon integrating in spherical coordinates; however, in general elliptic-type integrals are involved (for detailed discussion cf. [2]).

The second term in (7.1), upon substitution of (6.1) and (6.2), assumes the form

(7.5) $$\delta S_{ijkl}(s_{pq})\sigma_{kl} = \left[\int \delta p_{\mathbf{n}}(n_r) S^{(\mathbf{n})}_{\mathbf{b}ijkl} d\omega + \int \delta p_{\mathbf{n}}(n_r) \eta^{(\mathbf{n})}_{ij} n_k n_l d\omega\right] \sigma_{kl},$$

where in case of strength isotropy $S^{(\mathbf{n})}_{\mathbf{b}}$ and $\boldsymbol{\eta}^{(\mathbf{n})}$ do not depend on $\mathbf{n}$ to an orthogonal transformation (in particular $\boldsymbol{\eta}^{(\mathbf{n})}$ is provided by Eq. (7.3) and $S_{\mathbf{b}ijkl} = n_{ip}n_{jq}n_{kr}n_{ls}S^{0}_{\mathbf{b}pqrs}$ where $S^0_{\mathbf{b}}$ is referred to a fixed coordinate system conveniently connected with the axes of symmetry). By (5.1)

$$\delta p_{\mathbf{n}}(\mathbf{n}) = \frac{dp_{\mathbf{n}}(\hat{\sigma}^{(\mathbf{n})}_{\mathbf{n}})}{d\hat{\sigma}^{(\mathbf{n})}_{\mathbf{n}}} \delta\hat{\sigma}^{(\mathbf{n})}_{\mathbf{n}} = p'[\sigma^{(\mathbf{n})}_{\mathbf{n}}(\mathbf{n})](\mathbf{n}\otimes\mathbf{n}) \cdot d\boldsymbol{\sigma},$$

where $p'_{\mathbf{n}}$ denotes the derivative function of the arguments n_r and $d\boldsymbol{\sigma}$ is an active stress increment (in a point of the bounding surface $\partial\mathscr{A}$ in the stress space). Consequently, under suitable denotation of indices

(7.6) $$\delta S_{ijmn}\sigma_{mn} = \left\{\left[\int p'_{\mathbf{n}}(n_r) S^{(\mathbf{n})}_{\mathbf{b}ijmn} n_k n_l d\omega \right.\right.$$

$$\left.\left. + \int p'_{\mathbf{n}}(n_r)\eta^{(\mathbf{n})}_{ij} n_m n_n n_k n_l d\omega\right] \sigma_{mn}\right\} d\sigma_{kl} = \Psi_{ijkl} d\sigma_{kl},$$

where Ψ is the tensor function given in (2.3) corresponding to the expression in the brackets { }. The domain of integration in (7.5) and (7.6) follows from three conditions:

(i) The point is in the bounding surface, i.e. (cf. (3.2) and (3.3))

$$\boldsymbol{\sigma}\cdot(\mathbf{n}\otimes\mathbf{n})=\begin{cases}\hat{\sigma}_{\mathrm{n}}^{(\mathrm{n})} & \text{for}\quad \hat{\sigma}_{\mathrm{n}}^{(\mathrm{n})}>0 \quad (\text{or } \varrho(\mathbf{n})),\\ 0 \ (\text{or } \varrho(\mathbf{n})) & \text{for}\quad \hat{\sigma}_{\mathrm{n}}^{(\mathrm{n})}<0 \quad (\text{or } \varrho(\mathbf{n}));\end{cases}$$

(ii) $d\boldsymbol{\sigma}$ is active, that is

$$(\boldsymbol{\sigma}+d\boldsymbol{\sigma})\cdot(\mathbf{n}\otimes\mathbf{n})>\begin{cases}\hat{\sigma}_{\mathrm{n}}^{(\mathrm{n})} & \text{for}\quad \hat{\sigma}_{\mathrm{n}}^{(\mathrm{n})}>0 \quad (\text{or } \varrho(\mathbf{n})),\\ 0 \ (\text{or } \varrho(\mathbf{n})) & \text{for}\quad \hat{\sigma}_{\mathrm{n}}^{(\mathrm{n})}\leqslant 0 \quad (\text{or } \varrho(\mathbf{n})),\end{cases}$$

which, in connection with (i), leads to the condition

$$d\boldsymbol{\sigma}\cdot(\mathbf{n}\otimes\mathbf{n})>0 \quad \text{or} \quad \mathbf{q}\cdot(\mathbf{n}\otimes\mathbf{n})>0$$

with $\mathbf{q}$ defined in (2.3). Only the active $d\boldsymbol{\sigma}$ give rise to an increase of $p_{\mathrm{n}}(\mathbf{n})$ (which is an increasing function of $\hat{\sigma}_{\mathrm{n}}^{(\mathrm{n})}$) while the passive ones cause $\delta p_{\mathrm{n}}=0$.

(iii) Additional quantities would appear only at closed cracks, consequently

$$\mathbf{s}\cdot(\mathbf{n}\otimes\mathbf{n})<0$$

as explained in Sect. 6. One must keep in mind that $\boldsymbol{\sigma}$, $d\boldsymbol{\sigma}$ belong to the stress path while $\mathbf{s}$ does not (it is not connected with $\boldsymbol{\sigma}$) because it is only the variable argument of the constitutive function $S_{\mathrm{a}}(\mathbf{s})$ searched for.

Let us denote the sets of $\mathbf{n}$ defined by the above three conditions consecutively by Ω_{σ}, Ω_{q}, Ω_{s}. Then the domain of integration in the first integral in (7.5) and (7.6) is $\Omega_{\sigma}\cap\Omega_{\mathrm{q}}$ and in the second integral $\Omega_{\sigma}\cap\Omega_{\mathrm{q}}\cap\Omega_{\mathrm{s}}$. If we consider a fixed point in $\partial\mathscr{A}$ (i.e. satisfying (i)) Ψ is a function of the arguments $\mathbf{q}$, $\mathbf{s}$ (cf. (2.3) and Fig. 1).

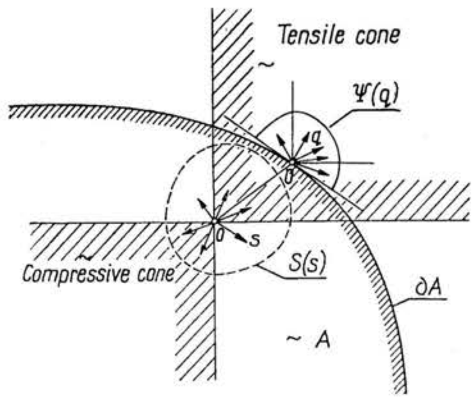

FIG. 4.

In the 3-space representation (cf. Sect. 3), in general, the region Ω_{σ} is cut out from the "terminal" $\sigma_{\mathrm{n}}^{(\mathrm{n})}$ -surface by the intersection curves with the envelope surfaces and by the generatrix curve for the final τ provided by the conical surface (3.6). The regions Ω_{q}, Ω are bounded, in turn, by conical surfaces in (ii) and (iii), the inequality signs being replaced by equalities. In the stress space representation (cf. Sect. 4 and Fig. 3), $\boldsymbol{\sigma}$ being a vertex point, Ω_{σ} includes all the $\mathbf{n}$ for which the hyper-planes passing through $\boldsymbol{\sigma}$ and orthogonal to vectors $\mathbf{n}\otimes\mathbf{n}$ do not intersect the stress path, Ω_{q}, Ω_{s} include the $\mathbf{n}\otimes\mathbf{n}$'s in the half spaces yielded by the inequalities in (ii) and (iii), respectively.

The crack extension criterion (3.2) or (3.3) leads to some simplifications in the functions $S(\mathbf{s})$, $\phi(\mathbf{s})$, $\Psi(\mathbf{s})$ (Fig. 4, cf. also Fig. 1 where instead of S we plotted C). The stress space is divided into the characteristic subregions: the compressive cone (according to (4.1)), the tensile cone the counterpart of the former, with reversed inequality in (4.1) and the remaining (complementary) part. For $\mathbf{s}$ in the tensile cone (all $\sigma_I > 0$, cf. (3.7)) the set $\Omega_{\mathbf{s}}$ is empty. It follows that $S_{\mathbf{a}}(\mathbf{s})$ disappears (cf. (6.1) and (6.2)) and only the constant term is retained, consequently the $S(\mathbf{s})$-diagram is a hyper-sphere segment. For $\mathbf{s}$ in the compressive cone $\Omega_{\mathbf{s}}$ includes all the $\mathbf{n}$ and the $S_{\mathbf{a}}$ terms are constant, so we obtain again a hyper-sphere with a different radius. Next, in (7.2) only the first term remains, therefore the tensor ϕ is the same for any point in the tensile cone and the same (however, different from the former) for points in the compressive cone. Finally, the second integral in (7.6) is independent of $\mathbf{s}$; consequently so is the expression in the brackets [ ] while the function Ψ depends linearly on $\boldsymbol{\sigma} \in \partial\mathscr{A}$.

The admissible region $\mathscr{A}$ includes the compressive cone; consequently in case of our simplified critetion (3.2) it is unbounded (for directions in the compressive cone). Consider now a point in $\partial\mathscr{A}$ and suppose that the 6-vector $\mathbf{q}$ is in the tensile cone, that is all principal stresses increase (cf. (3.7)). The $\Omega_{\mathbf{q}}$ includes all the $\mathbf{n}$ and the integrals in (7.6) do not depend on $\mathbf{q}$; consequently the Ψ-diagram is a hyper-sphere segment and the transformation in (2.3) (in the brackets [ ]) is constant (like in the theory of plasticity). However, this does not hold any more for the increments outside the tensile cone and, more to it, the (constant) transformation depends on the point in $\partial\mathscr{A}$.

8. Applicability of the theory

The field of application of any theory assuming non-interacting cracks is, of course, limited and covers, first of all, such phenomena as temperature or shrinkage cracks controlled by initial strains (and not by active loads). However, also for strength analysis the role of such cracks can be essential insomuch as this incipient stage affects the posterior mode of failure.

In brittle materials — where by "brittleness" we mean, first of all, a strength many times lower in tension than in compression, say, in concrete about 10 times — we have to distinguish roughly speaking between cracks at compression (say in the compressive zone of a girder) and those at tension (in the tensile zone). The former are brought about mainly by shear at closed cracks whereas the latter by perpendicular tension while the cracks are open what results in compressive strength many times exceeding the tensile one. The "tensile" crack system arises at relatively small loads and often the crack pattern undergoes no essential changes under subsequent loading (e.g. in standard reinforced structures). Thus the material keeps in memory the incipient loading which affects posterior properties of the structure.

At this initial stage there arrives a rapid increase in crack sizes and stress concentration factors at crack borders. There appears structural instability in the brittle material giving rise to an overall instability of the structure unless a reinforcement prevents it. In consequence of a release in stiffness of a damaged volume element, neighbour elements take over the forces and, first of all, so does the reinforcement. The quickness of the process preceding the onset of the tensile instability makes it difficult to keep track of the structural susceptibility to small changes in loads. On the other hand the loading path in this short interval is, as a rule, relatively simple and we often do with the tacit assumption that the cracks depend only on the direction of the loading path at the outset point (in the stress space). Thus the theory is applicable for an approximate analysis of the failure in tension. In structural concrete engineering the latter is called crack resistance and it predominantly bears on the distribution of the reinforcement. It should be clearly distinguished from the structural (ultimate) strength connected in most cases with crack resistance at compression and plastic yield of the reinforcement.

In numerical calculations the main difficulty depends on finding the integrals in Sects. 6 and 7; this problem has been extensively discussed in [2] (Appendix).

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