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Surface energy in liquids and the Hadwiger integral theorem

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THE CONTRIBUTION of the interface curvatures to the surface energy density is discussed. The attention is focused on the restrictions imposed on the possible form of this contribution by the properties of the functional of total energy of the liquid body. The requirement of the continuity, additivness and the invariance with regard to isometry in the case of convex bodies give the linear dependence of the surface energy density on the mean curvature H and the Gaussian curvature K. This results in the additional term in the expression for the pressure discontinuity at the interface equal to αK, where α is the material constant describing the contribution of the curvature H into the surface energy density. The contribution of the Gaussian curvature is shown to be responsible for the discontinuous total energy change associated with the change of the topological connection of the body.

Rozpatruje się ograniczenia nakładane przez własności funkcjonału energii ciała na postać hipotetycznej zależności gęstości energii powierzchniowej od krzywizn powierzchni rozdziału faz. Żądanie ciągłości, addytywności, niezmienniczości względem izometrii prowadzi dla ciał wypukłych do liniowej zależności gęstości energii powierzchniowej od krzywizny średniej H i krzywizny Gaussa K. Taka zależność prowadzi do pojawienia się dodatkowego członu równego αK w wyrażeniu na nieciągłość ciśnienia na powierzchni rozdziału faz, gdzie współczynnik α opisuje wkład krzywizny średniej H do wyrażenia na gęstość energii powierzchniowej. Wykazano, że wkład krzywizny Gaussa do gęstości energii powierzchniowej powoduje skończone przyrosty energii całkowitej przy zmianie spójności ciała.

Рассматриваются ограничения накладываемые свойствами функционала энергии тела на вид гипотетической зависимости плотности поверхностной энергии от кривизн поверхностей раздела фаз. Требования неразрывности, аддитивности, инвариантности по отношению к изометрии приводят, для выпуклых тел, к линейной зависимости плотности поверхностной энергии от средней кривизны H и кривизны Гаусса K. Такая зависимость приводит к появлению дополнительного члена, равного αK, в выражении для разрыва давления на поверхности раздела фаз, где коэффициент α описывает вклад средней кривизны H в выражении для плотности поверхностной энергии. Показано, что вклад кривизны Гаусса в плотность поверхностной энергии вызывает конечные приращения полной энергии при изменении связности тела.

1. Introduction

IN ROCK analysis and in stereographic metallography the so-called first theorem on the functionals on convex bodies (Hadwiger theorem) is employed [1]. This theorem has been recently applied to the problems of mechanics by A. TRZĘSOWSKI [2, 3].

In the present paper the authors present some possibilities opened by using the Hadwiger theorem for the analysis of the nonclassical mathematical model of surface tension proposed in 1971 by A. BLINOWSKI [4].

2. Surface energy of liquid bodies

In this paper we shall consider a simple model of incompressible fluid without memory effects, whose internal energy is entirely determined by the temperature. We shall assume also that the temperature is held constant, i.e. we confine our attention to the strictly reversible isothermal processes.

In the theory of capillarity a finite energy density is attributed not only to volume but also to the surface measure, thus we should define the body in such a way that the limiting case of the thin film (infinitely thin, in the sense of the phenomenological approximation used here) should be also considered as a body.

Following HADWIGER [5] we shall define the convex body as a compact convex subset of Euclidean three-dimensional point space. If such a subset contains internal points, we shall call it a proper convex body, otherwise we shall call it an improper convex body.

These definitions are strictly geometrical. In the physical sense we shall understand bodies as the materially homogeneous one phase (say A-phase) domains (or improper domains), all of them made of the same substance and surrounded by the same homogeneous internal medium (B-phase).

In line with the classic capillarity approach, that is assuming the existence of the constant surface energy density σ, we can endow every convex body under consideration with the energy functional $E(A) = \varepsilon V(A)+\sigma S(A)$ where $V(A)$ and $S(A)$ are the volume and the surface of the body and ε denotes the volumetric energy density.

According to the Hadwiger theorem [5], such a functional meets the following conditions (cf. [2]):

a) Invariance with regard to isometry, i.e. when A_1 and A_2 are isometric bodies of the same phase then

$$E(A_1) = E(A_2).$$

b) Additiveness in the following sense:

$$E(A_1 \cup A_2) = E(A_1)+E(A_2)-E(A_1 \cap A_2).$$

c) Vanishing for empty sets

$$E(\phi) = 0.$$

d) Continuity

$$A_n \to A \Rightarrow E(A_n) \to E(A).$$

By $A_n \to A$ we mean convergence in the sense of the Blaschke metrics in the set of all convex bodies:

$$d(A_1, A_2) = \inf \varrho \{A_2 \subset A_{1\varrho}, A_1 \subset A_{2\varrho}, \varrho \geqslant 0\},$$

where

$$A_\varrho = \bigcup_{a \in A} K(a, \varrho)$$

and $K(a, \varrho)$ is a sphere of radius ϱ and with the center in a (see Fig. 1).

But, again according to the Hadwiger theorem, the functional $E(A) = \varepsilon V(A)+\sigma S(A)$ is not the only possible functional which meets the conditions a)–d).

The theorem claims that the general form of the functional which meets these conditions can be represented as follows:

$$E(A) = \varepsilon V(A)+\sigma S(A)+\alpha \overline{H}(A)+\beta \overline{K}(A), \tag{2.1}$$

where ε, σ, α, β are some constants and $\overline{H}(A)$, $\overline{K}(A)$ are, respectively, the integral mean curvature and the total Gaussian curvature of the body surface.

The relation (2.1) will serve us as the starting point in our attempt aimed for the generalization of the mathematical model of capillarity. Thus we shall assume that the energy

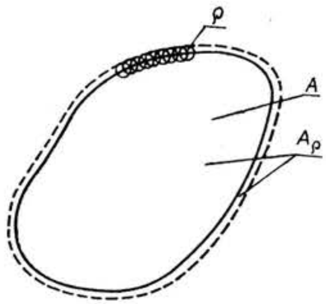

FIG. 1. Convex body A_ϱ.

functional of every convex body within the framework of our model can be expressed as in the relation (2.1) where ε, σ have the same sense as in the classical model and α, β are two additional material constants. (We remind here that the temperature is assumed to be constant and uniform).

In the case of the smooth surfaces ∂A the magnitudes $\overline{H}(A)$ and $\overline{K}(A)$ can be expressed as

$$\overline{H}(A) = \int_{\partial A} H dS, \tag{2.2}$$

where H denotes the mean (local) curvature and

$$\overline{K}(A) = \int_{\partial A} K dS, \tag{2.3}$$

where K is the Gaussian curvature.

Let us mention that the expressions (2.2) and (2.3) are also valid and strictly determined in the case when the surface is only piecewise smooth, i.e. it contains some edges and corners (cf. [1]). Here, however, for the sake of simplicity we confine our attention to the bodies with the smoth bounding surfaces. For such bodies the relation (2.1) can be rewritten as follows:

$$E(A) = \int_{A} \varepsilon dV + \int_{\partial A} (\sigma+\alpha H+\beta K) dS. \tag{2.4}$$

On one hand it is obvious that the expression (2.4) can be, at least formally, applied to describe the behaviour of an arbitrary (not obviously convex) body bounded with a smooth surface. On the other hand it can hardly be supposed that the energy functional suffers drastic changes on the transitions from the convex bodies to the other shapes. Thus we will assume that:

e) The energy functional of an arbitrary shaped body bounded with a smooth surface can be expressed as in Eq. (2.4).

In the next section we shall examine some mathematical consequences of this assumption.

3. Mechanics of bodies with curvature dependent surface energy

The model of the curvature dependent surface energy considered by A. BLINOWSKI [4] did not assume any certain form of the expression $w = w(H, K)$ describing the surface energy density as a function of the invariants of the second metric form of the surface.

It does not follow from the considerations presented in the previous section that there exists a surface energy density function, but it does follow that if it exists, then being integrated over the whole surface it should give rise to the surface integral in Eq. (2.4).

If we assume, however, that the density function exists, then again we cannot claim that it is equal to the expression under the surface integral sign in Eq. (2.4); we may suspect that the density function contains some terms which identically vanish being integrated over the arbitrary closed surface. On the other hand, however, the authors cannot point out any regular enough function of H and K having the mentioned property. This fact as well as the close correspondence with the classic model leads us to assume that:

f) The energy functional E can be defined for an arbitrary domain D of the body surface and it can be expressed as follows:

$$E(D) = \int_D (\sigma + \alpha H + \beta K) dS. \tag{3.1}$$

This assumption implies the existence of the surface energy density function $w = \sigma + \alpha H + \beta K$.

In [4] and, indirectly, in [6] it was shown that the membrane model of the surface tension is not adequate for the description of the mechanical behaviour of such an interface for which the energy density depends on the curvatures; in that case a more complex Cosserat two-dimensional model (shell model) should be used (cf. [7]).

For the sake of completeness and to avoid unnecessary complications due to excessive generality, we prefer to re-derive all the necessary relations rather than to make references to the paper by A. BLINOWSKI [4] (it should also be noticed that the mentioned paper contains some minor mistakes).

We assume that for every regular material domain S of the surface and for every velocity field, the following integral energy balance law should be obeyed:

$$\dot{\overline{\int_S w dS}} = \int_S p^i v_i dS + \int_{\partial S} T^{i\alpha} \nu_\alpha v_i dl + \int_{\partial S} M^{i\alpha} \nu_\alpha n^k \dot{n}^l \varepsilon_{ikl} dl, \tag{3.2}$$

n^i denote the components of the unit normal vector pointing outside the A-phase domain, p^i — components of the vector of the surface density of external forces acting on the surface (i.e. the resultant of the forces exerted by the A and B-phases), $T^{i\alpha}$ — components (in the mixed basis) of the surface stress tensor ($T^{i\alpha}\nu_\alpha = T^i$, where T^i — components of the contact force acting across the unit length of ∂S), $M^{i\alpha}$ — components of the moment tensor ($M^{i\alpha}\nu_\alpha = M^i$, where M^i — vector of the moment acting across the unit length of the contour) ([1]). For the other symbols — see Appendix. In the Appendix we also quote all necessary relations which will be used in the further transformations of Eq. (3.2).

Using Eqs. (A.23)–(A.27) and (A.15) we can rewrite Eq. (3.2) in the form of the surface integral

$$(3.3) \quad \int_S \Big\{ [\alpha(Ha^{\alpha\beta}-b^{\alpha\beta})t^k_\beta - \beta K a^{\alpha\beta}t^k_\beta + \sigma a^{\alpha\beta}t^k_\beta - S^{\beta\alpha}_{\ \ ,\beta}n^k - T^{k\alpha} + S^{\beta\alpha}b_{\beta\gamma}a^{\gamma\delta}t^k_\delta]v_{k,\alpha} + \left[\frac{\alpha}{2}a^{\alpha\beta}n^k + \beta(2Ha^{\alpha\beta}-b^{\alpha\beta})n^k - S^{\alpha\beta}n^k\right]v_{k,\alpha\beta} - [T^{k\alpha}_{\ \ ,\alpha}+p^k]v_k \Big\} dS = 0,$$

where

$$S^{\alpha\beta} \stackrel{\text{df}}{=} M^{i\alpha}g_{li}t^l_\gamma \varepsilon^{\gamma\beta}.$$

The relation (3.3) should hold for every material surface domain. Therefore, for every velocity field the expression under the sign of integral must identically vanish; this is possible if and only if all the the terms in Eq. (3.3) which are multiplied by the velocity gradient of the same order identically vanish (values of the gradients of each order at any given point can be chosen independently).

Thus the following relations should hold:

$$(3.4) \quad T^{k\alpha}_{\ \ ,\alpha} + p^k = 0,$$

$$(3.5) \quad T^{k\alpha} = \alpha(Ha^{\alpha\beta}-b^{\alpha\beta})t^k_\beta - \beta K a^{\alpha\beta}t^k_\beta + \sigma a^{\alpha\beta}t^k_\beta - S^{\beta\alpha}_{\ \ ,\beta}n^k + S^{\beta\alpha}b_{\beta\gamma}a^{\gamma\delta}t^k_\delta,$$

$$(3.6) \quad S^{(\alpha\beta)} = \frac{\alpha}{2}a^{\alpha\beta} + \beta(2Ha^{\alpha\beta}-b^{\alpha\beta}),$$

where the indices in parentheses denote the symmetric part of the tensor.

The foregoing considerations give no cue on the skew-symmetric part of $\mathbf{S}$ (we denote it by $S^{\langle\alpha\beta\rangle}$).

Let us suppose for now that $S^{\langle\alpha\beta\rangle}$ does not necessarily vanish. We are able then to write the following identities:

$$(3.7) \quad S^{\langle\alpha\beta\rangle} = S\varepsilon^{\alpha\beta},$$

where

$$S \stackrel{\text{df}}{=} \frac{1}{2}S^{\langle\alpha\beta\rangle}\varepsilon_{\alpha\beta}.$$

Substituting Eqs. (3.6) and (3.7) into Eq. (3.5) and taking into account Eq. (A.18), we obtain

$$(3.8) \quad T^{k\alpha} = \alpha\left(Ha^{\alpha\beta} - \frac{1}{2}b^{\alpha\beta}\right)t^k_\beta + \sigma a^{\alpha\beta}t^k_\beta - (S\varepsilon^{\alpha\beta}b_{\beta\gamma}a^{\gamma\delta}t^k_\delta - S_{,\beta}\varepsilon^{\alpha\beta}n^k).$$

([1]) On the correctness of the concepts of $T^{i\alpha}$ and $M^{i\alpha}$ tensors see [7].

Divergences of the terms in parentheses vanish (cf. Eq. (A.28) and (A.29)); therefore, substituting Eq. (3.8) into Eq. (3.4) and using Eq. (A.12) we obtain

$$p^k_{\cdot} + \alpha\left(Ha^{\alpha\beta} - \frac{1}{2}b^{\alpha\beta}\right) n^k b_{\alpha\beta} + \sigma a^{\alpha\beta} b_{\alpha\beta} n^k = 0, \tag{3.9}$$

thus

$$p^k t^l_{\delta} g_{kl} = 0 \text{ (}^2\text{)} \tag{3.10}$$

i.e. the surface cannot transmit tangential force, it means that there is no discontinuity of the tangent stress across the surface. Using Eqs. (A.16) and (A.17) we obtain the following result:

$$p^k n_k = -\alpha K - 2\sigma H, \tag{3.11}$$

which, for $\alpha = 0$, reduces to the Laplace formula.

$$\Delta p = -2H\sigma. \tag{3.12}$$

4. Discussion

In this section we present some further arguments in support of the assumptions introduced in the previous sections, as well as some consequences of these assumptions.

Let us notice that in choosing the sense of the normal we determined in fact the sign of the mean curvature in such a way, that for the convex domain containing the A-phase (A-convex body) the sign of the mean curvature will always be nonpositive and vice versa for the B-convex body — nonnegative. Generally speaking the mean curvature will be negative if the centre of the bigger curvature (we mean bigger in the sense of absolute value), lies on the A-side of the interface, and positive in the opposite case.

Defining the A-convex bodies and B-convex bodies we find at once that all the considerations of the previous section which were conducted for A-convex bodies can be repeated without change for the B-convex bodies.

We are now able to examine the assumptions made at the sections (2) and (3) from a slightly different viewpoint.

If we employ the Hadwiger theorem separately to A-convex and B-convex bodies and assume the existence of the surface energy (neglecting possible terms which do not contribute to the surface integrals), then we are left with the following situation:

$$w = \begin{cases} \sigma_A + \alpha_A H + \beta_A K & \text{for } A\text{-convex bodies,} \\ \sigma_B + \alpha_B H + \beta_B K & \text{for } B\text{-convex bodies.} \end{cases} \tag{4.1}$$

For obvious reasons we should put $\sigma_A = \sigma_B$, otherwise we would obtain two different values of w for the flat surface. The expression (4.1) do not suggest whether $\alpha_A = \alpha_B$ and $\beta_A = \beta_B$ or not; they also give no information about the expression for w in those parts of the surface of arbitrary (neither A-convex nor B-convex) bodies on which $K < 0$.

(2) This result is due to the hidden assumption of the continuity of the velocity vector across the surface, i.e. the possibility of the introduction of the unique material coordinates at the surface.

In the sections (2) and (3), by introducting the assumptions (e) and (f) we have chosen in fact the simplest possible answer for these questions, i.e. we have assumed that $\alpha_A = \alpha_B, \beta_A = \beta_B$.

Despite the whole arbitrariness of such a procedure, the simplicity and the aesthetic value of the mathematical description which has been obtained incline the authors towards the opinion that it is worthwhile to discuss some properties of the model.

As regards the local properties of the interface, we should point out that the nonclassical behaviour of the pressure discontinuity can be detected only on the double-curved (i.e. with $K \neq 0$), single interface. For obvious reasons (see Eq. (3.11) if $K = 0$, then the pressure discontinuity has the classical form. As regards the pressure discontinuity on the double surface (modelling the physical behaviour of the thin film), its behaviour does not differ from the classical one even in the case when $K \neq 0$: on the two opposite sides $\mathbf{n}$ has the opposite sense and the pressure discontinuities due to $-\alpha K$-term cancel each other (contrary to the contribution of $2\sigma H$, which is doubled in this case).

Thus the possible deviation from the Laplace model cannot be detected, e.g. in experiments with straight capillary waves at the interface or in any experiment with thin films.

Let us now turn to some other quantities introduced in Sect. 3. The vector of the moment $M^i = M^{i\alpha}\nu_\alpha$ represents a generalized force performing work on the normal vector rotation. We can see that the normal component of $\mathbf{M}$ equal to $M^i n_i$ does not contribute to the work being performed, indeed,

$$M^{p\alpha} n_p \nu_\alpha n^i n^k \dot{n}^l \varepsilon_{ikl} \equiv 0. \tag{4.2}$$

At first glance one would suppose that the quantity $M^{i\alpha} n_i$ should be determined using some additional assumptions. Let us, however, represent $M^{i\alpha}$ as follows:

$$M^{i\alpha} = (M^{k\alpha} n_k) n^i + M^{k\alpha} g_{kl} t^l_\beta a^{\beta\gamma} t^i_\gamma \tag{4.3}$$

and demand the following integral moment balance equation to be satisfied:

$$\int_{\partial S} M^{i\alpha}\nu_\alpha dl + \int_{\partial S} T^{m\alpha}\nu_\alpha g_{ml} R_k \varepsilon^{kli} dl + \int_{S} R_k p_l \varepsilon^{kli} dS = 0, \tag{4.4}$$

where R^k are the components of the position vector in the arbitrary fixed coordinate system. Performing rather long, however quite elementary transformations, and using the relations (3.4)–(3.8), we obtain

$$\begin{gathered}(M^{k\alpha} n_k)_{,\alpha} = 0,\\ (M^{k\alpha} n_k) b_{\alpha\gamma} = 0,\end{gathered} \tag{4.5}$$

hence we are able to claim that

$$M^{n\alpha} n_k = 0. \tag{4.6}$$

The other quantity $S \overset{\text{df}}{=} \dfrac{1}{2} S^{\langle\alpha\beta\rangle} \varepsilon_{\alpha\beta}$ however, still remains undefined.

Projecting Eq. (3.5) on the normal and tangent directions we obtain the following expressions:

$$T^{k\alpha} n_k = S^{\beta\alpha}_{,\beta}, \tag{4.7}$$

$$T^{\langle\mu\alpha\rangle} = HS\varepsilon^{\mu\alpha}, \tag{4.8}$$

where the brackets stand for the skew-symmetric part and $T^{\mu\alpha} \overset{\text{df}}{=} T^{i\alpha}t^k_\beta g_{ik}a^{u\beta}$. According to Eqs. (4.2) and (4.6) the quantity $M^{i\alpha}$ is completely determined by its projection $M^{\delta\alpha}$, for which, in virtue of the definition of $S^{\alpha\beta}$ and the relations (3.6) and (3.7) we obtain

$$M^{\delta\alpha} = Sa^{\delta\alpha} + \left(2\beta H + \frac{\alpha}{2}\right)\varepsilon^{\delta\alpha} - \beta b^{\alpha\beta}\varepsilon_{\mu\beta}a^{\mu\delta}. \tag{4.9}$$

Preserving $S \neq 0$ would mean that for the spherical interface we could expect from Eq. (4.8) the presence of non-vanishing $T^{\langle\mu\alpha\rangle}$ and the twisting moment (from. Eq. (4.9)) but that would mean the violation of the invariance with regard to the complete orthogonal group of space transformations. Thus, so far as we stand on the ground of the mentioned invariance, we are rather inclined to put $S = 0$.

Let us also mention that (independently of the possible value of S) we obtain the following relation:

$$p^i n_i = -T^{\alpha\beta}b_{\alpha\beta}, \tag{4.10}$$

(compare Eqs. (3.8), (3.9) and the definition of $T^{\alpha\beta}$ in Eq. (4.8)) The same formula was obtained by L. E. SCRIVEN [8] applied to the different physical situation, namely under the assumption of the anisotropy of $T^{\alpha\beta}$ due to the surface viscosity and surface tensions inhomogeneities. Thus in the authors' opinion Eq. (4.10) can be considered as the generalized Laplace formula which in the case of the classic theory reduces to

$$p^i n_i = -\sigma a^{\alpha\beta}b_{\alpha\beta}. \tag{4.11}$$

In our discussion of local properties we quote for reference the representation of $\mathbf{T}$ and $\mathbf{M}$ in the orthonormal basis of the principal curvature coordinate system $\left(\text{for which } \mathbf{b} = \begin{bmatrix} \varkappa_1 & 0 \\ 0 & \varkappa_2 \end{bmatrix}\right)$:

$$\begin{aligned} \mathbf{T} &= (\alpha H + \sigma)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{\alpha}{2}\begin{bmatrix} \varkappa_1 & 0 \\ 0 & \varkappa_2 \end{bmatrix} = \sigma\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\alpha}{2}\begin{bmatrix} \varkappa_2 & 0 \\ 0 & \varkappa_1 \end{bmatrix}, \\ \mathbf{M} &= \frac{\alpha}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \beta\begin{bmatrix} 0 & \varkappa_2 \\ -\varkappa_1 & 0 \end{bmatrix}. \end{aligned} \tag{4.12}$$

At the last point of this section we shall discuss the physical meaning of the β-coefficient in Eq. (3.1). As it has been already shown this quantity does not affect the local equilibrium, thus at first glance one would claim that it can be assumed to be equal to zero without loss of generality.

On the other hand, however, some authors [10] have pointed out that the usual dimensionless criteria which can be formulated using such quantities like surface tension, density, viscosity etc. give no cue for the description of the flow discontinuities (i.e. the changes of the topological connection of the bodies such like the rupture or the film perforation (Figs. 2, 3).

We shall show here that our β-coefficient can be possibly regarded as such a quantity (missing in the classical theory) which can supply some information about this subject.

According to the Gauss–Bonnet theorem, (cf [9]) for any smooth surface domain S bounded with the smooth contour ∂S the following relation holds:

$$\int_S K dS + \int_{\partial S} K_g dl = 2\pi, \tag{4.13}$$

where K_g denotes the geodesic curvature of the bounding contour.

Every closed, connected and orientable surface S can be characterized by a topological invariant $\chi(S)$ called the Euler-Poincare index.

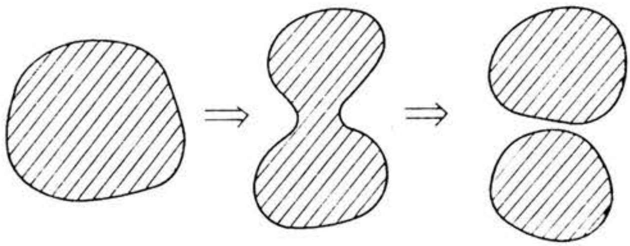

FIG. 2. A-rupture.

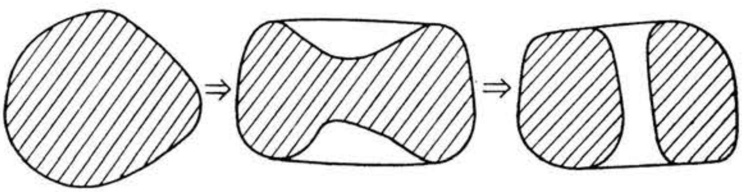

FIG. 3. A-puncture.

For the surfaces of bodies topologically equivalent to the sphere $\chi(S) = 2$ for the torus $\chi(S) = 0$ and for the most general case — surface bounding a body equivalent to the "sphere with n handles" $\chi(S) = 2(1-n)$([11, 12]). As a straight conclusion from the Gauss-Bonnet theorem the following result for the connected, closed and orientable surface can be obtained [9, 12]:

$$\int_S K dS = 2\pi\chi(S). \tag{4.14}$$

Let us consider now the two main schemes of incompressible flow discontinuities. The process resulting in the disconnection of a compact simple-connected (i.e. topologically equivalent to the sphere) domain of phase A forming two (simple-connected) domains will be called the A-rupture (Fig. 2).

Another process resulting in the perforation of the simple connected A-domain (i.e. transformation of a domain equivalent to the sphere into a domain topologically equivalent to the torus by "making a hole", e.g. by indentation with two spherical punches) we will call A-puncture (Fig. 3).

With certain caution we can generalize these notions on infinite domains — roughly speaking we should demand such processes to be semi-local, i.e. confined to the finite space domain outside of which the form of interface does not change (cf. Eq. (4.13)). With such generalizations we realize at once that the process inverse to the A-rupture can be considered as the B-puncture, the process inverse to the A-puncture in turn can be considered as the B-rupture.

We do not consider here such a flaw discontinuity as the isolated void opening inside the one phase domain — this case does not occur for the incompressible media. Of course the situation when the A-body includes a domain of B-phase can appear, but it can be easily seen that this can happen by "swallowing" a piece of B by "A-meduse", i.e. by

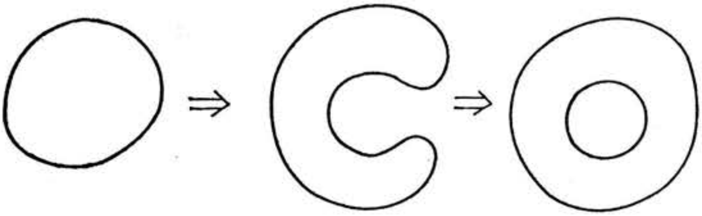

FIG. 4. Transformation of a sphere-like body into a torus-shaped, body by B-puncture.

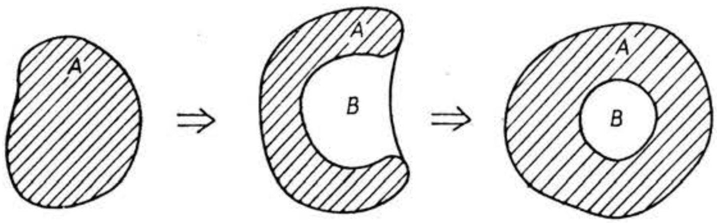

FIG. 5. „Swallowing" of B-body by A-body with B-rupture.

B-rupture (Fig. 5). If we considered the phase transformations we would not be able to exclude the mentioned case of discontinuity modelling a nucleation of the second phase, but such a situation lies beyond the bounds of the present model. This is so since if the phase transformation process occurs, then the interface can no longer be considered as a material surface. The only exception here is the case when the second phase can be considered as the vacuum, then all the considerations of the previous section remain valid and we have two additional schemes of flaw discontinuities: void opening and void closure.

Comparing Eqs. (4.13) and (4.14) with Eq. (3.1) we can easily see that any of the four elementary flow discontinuities is strictly connected with the discontinuous jump in energy of absolute value equal to $4\pi\beta$, namely:

Discontinuity type	Energy change
A-rupture (Fig 2)	$4\pi\beta$
A-puncture (Fig. 3)	$-4\pi\beta$
B-rupture	$4\pi\beta$
B-puncture	$-4\pi\beta$

Let us notice that no matter how we transform the A-sphere into the A-torus either by "drilling a hole" (A-puncture), or elongating it with subsequent bending and joining its ends (B-puncture Fig. 4), the net result remains invariant — energy change equal to $4\pi\beta$.

5. Final remarks

At the present time the authors can scarcely point out any significant physical considerations or experimental results which can point out that the introduction of the two additional constants α and β has any practical significance. On the other hand, however, as it has already been shown, the detection of the influence of these constants requires quite special experiments which at the authors' best knowledge have not yet been conducted.

Appendix

For any smooth surface specified by the vector function $\mathbf{x} = \mathbf{x}(u^\alpha)$, $(\alpha = 1, 2)$ the following notions can be introduced: ([8, 12, 4])

(A.1) $\mathbf{e}_\alpha = \dfrac{\partial \mathbf{x}}{\partial u^\alpha}$ base vector,

(A.2) $a_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta$ covariant representation of the metric tensor,

(A.3) $a^{\alpha\beta}$ contravariant representation of the metric tensor defined by the relation

$a^{\alpha\gamma} a_{\gamma\beta} = \delta^\alpha_\beta$,

(A.4) $\boldsymbol{\epsilon}$ — unit skew-symmetric tensor which can be defined by its representation in orthonormal basis in which $\varepsilon^{11} = \varepsilon^{22} = 0$, $\varepsilon^{12} = 1$, $\varepsilon^{21} = -1$,

(A.5) $n_k \equiv \dfrac{1}{2} t^i_\alpha t^j_\beta \varepsilon^{\alpha\beta} \varepsilon_{ijk}$ components of the unit vector, normal to the surface,

where

$$t^i_\alpha = \mathbf{e}^i \cdot \mathbf{e}_\alpha \quad (i = 1, 2, 3), \tag{A.6}$$

$\mathbf{e}^i$ being a contravariant space basis vector, the set of t^i_α quantities form the representation of some tensor (shifter) in the mixed basis $(\mathbf{e}_i \otimes \mathbf{e}^\alpha)$.

Surface covariant differentiation is introduced the same way as for the three-dimensional space e.g.

$$a^i_{,\alpha} = \frac{\partial a^i}{\partial u^\alpha} + \begin{Bmatrix} i \\ lm \end{Bmatrix} a^l t^m_\alpha, \tag{A.7}$$

$$a_{\alpha,\beta} = \frac{\partial a_\alpha}{\partial u^\beta} - \begin{Bmatrix} \gamma \\ \alpha\beta \end{Bmatrix} a_\gamma, \tag{A.8}$$

$$A^i_{\alpha,\beta} = \frac{\partial A^i_\alpha}{\partial u^\beta} - \begin{Bmatrix} \gamma \\ \alpha\beta \end{Bmatrix} A^i_\gamma + \begin{Bmatrix} i \\ jk \end{Bmatrix} A^j_\alpha t^k_\beta, \tag{A.9}$$

where

(A.10) $$\begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} = \frac{\partial \mathbf{e}_\beta}{\partial u^\gamma} \cdot \mathbf{e}^\alpha \quad \text{surface Christoffel symbols.}$$

At the arbitrary coordinate system the following relations are valid:

(A.11) $$a_{\alpha\beta,\gamma} = 0,$$

(A.12) $$t^i_{\alpha,\beta} = n^i b_{\alpha\beta},$$

where $b_{\alpha\beta}$ — representation of the second fundamental form of the surface.

(A.13) $$n^i_{,\alpha} = -b_{\alpha\beta} a^{\beta\gamma} t^i_\gamma,$$

(A.14) $$b_{\alpha\beta,\gamma} = b_{\alpha\gamma,\beta} \quad \text{(Codazzi–Patterson theorem),}$$

(A.15) $$\int_S a^\alpha_{,\alpha} dS = \int_{\partial S} a^\alpha \nu_\alpha dl,$$

where $\mathbf{a}(u^\alpha)$ — arbitrary tangent veetor field, $\boldsymbol{\nu}$ — unit tangent vector normal to ∂S (Gauss–Ostrogradskii–Green formula).

Mean curvature H and Gauss curvature K obey the following relations:

(A.16) $$H = \frac{1}{2} a^{\alpha\beta} b_{\alpha\beta},$$

(A.17) $$K = \frac{1}{2} [(b^{\alpha\beta} a_{\alpha\beta})^2 - b^{\alpha\beta} b_{\alpha\beta}],$$

(A.18) $$b_{\alpha\gamma} a^{\gamma\delta} b_{\delta\beta} - 2H b_{\alpha\beta} + K a_{\alpha\beta} = 0$$

(Cayley–Hamilton formula).

If the u^α coordinates are material convective coordinates, and the dot over the symbol denotes the material time derivative, then the following relations are valid ([4, 13]) (if $\mathbf{e}_i$ vectors are fixed):

(A.19) $$\dot{\overline{a_{\alpha\beta}}} = t^i_\alpha v_{i,\beta} + t^i_\beta v_{i,\alpha},$$

where $\mathbf{v}$ stands for the velocity vector, and $\dot{\overline{a_{\alpha\beta}}}$ is the material derivative of the representation of the metric tensor (it is not the representation of the material derivative of the metric tensor)

(A.20) $$\dot{\overline{a^{\alpha\delta}}} = -a^{\alpha\beta} a^{\gamma\delta} \dot{\overline{a_{\beta\gamma}}},$$

(A.21) $$\dot{\overline{t^i_\alpha}} = v^i_{,\alpha},$$

(A.22) $$\dot{\overline{b_{\alpha\beta}}} = n^i v_{i,\alpha\beta},$$

(A.23) $$\dot{n}^k = \dot{\overline{n^k}} = -n_j v^j_{,\gamma} t^k_\delta a^{\gamma\delta},$$

(A.24) $$\dot{\overline{\int_S f dS}} = \int_S \left(\dot{f} + \frac{1}{2} a^{\alpha\beta} \dot{\overline{a_{\alpha\beta}}} \right) dS,$$

where S denotes the material domain of the surface and f is any density function (with respect to the surface measure).

The relations (A.16) and (A.22) yield the following formulae:

(A.25) $$\dot{H} = -b^{\alpha\beta}t^i_\beta v_{i,\alpha} + \frac{1}{2}a^{\alpha\beta}n^i v_{i,\alpha\beta};$$

(A.26) $$\dot{K} = (2Ha^{\alpha\beta} - b^{\alpha\beta})n^i v_{i,\alpha\beta} - 2Ka^{\alpha\beta}t^i_\beta v_{i,\alpha}.$$

From (A.5) and (A.23) taken together with well known relation

$$\varepsilon_{ijk}\varepsilon_{lmn}g^{il} = g_{jm}g_{kn} - g_{jn}g_{km},$$

we get the following relation:

(A.27) $$n^k \dot{n}^l \varepsilon_{ikl} = t^m_\beta \varepsilon^{\beta\gamma} g_{mi} n^p v_{p,\gamma}.$$

Finally we inspect the two particular differential expressions

$$(2Ha^{\alpha\beta} - b^{\alpha\beta})_{,\beta} \quad \text{and} \quad (f\varepsilon^{\alpha\beta}b_{\gamma\beta}a^{\gamma\delta}t^k_\delta - f_{,\beta}\varepsilon^{\alpha\beta}n^k)_{,\alpha},$$

where $f = f(u^\alpha)$ is an arbitrary scalar function. For the first of them we have

$$(2Ha^{\alpha\beta} - b^{\alpha\beta})_{,\beta} = (b^{\mu\gamma}a_{\mu\gamma}a^{\alpha\beta} - b^{\alpha\beta})_{,\beta} = (b_{\varrho\pi}a^{\varrho\pi}a^{\alpha\beta} - b_{\varrho\pi}a^{\varrho\alpha}a^{\pi\beta})_{,\beta} = b_{\varrho\pi,\beta}(a^{\varrho\pi}a^{\alpha\beta} - a^{\varrho\alpha}a^{\pi\beta})$$

however, by virtue of the Codazzi identity (A.14) we have

$$b_{\pi\varrho,\beta}a^{\varrho\pi}a^{\alpha\beta} = b_{\pi\varrho,\beta}a^{\pi\beta}a^{\alpha\varrho}$$

hence

(A.28) $$(2Ha^{\alpha\beta} - b^{\alpha\beta})_{,\beta} \equiv 0.$$

For the second one we write

$$
\begin{aligned}
&(f\varepsilon^{\alpha\beta}b_{\gamma\beta}a^{\gamma\delta}t^k_\delta - f_{,\beta}\varepsilon^{\alpha\beta}n^k)_{,\alpha} \\
\mathrm{I} \quad &= f_{,\alpha}\varepsilon^{\alpha\beta}b_{\gamma\beta}a^{\gamma\delta}t^k_\delta \\
\mathrm{II} \quad &+ f\varepsilon^{\alpha\beta}b_{\gamma\beta,\alpha}a^{\gamma\delta}t^k_\delta \\
\mathrm{III} \quad &+ f\varepsilon^{\alpha\beta}b_{\gamma\beta}a^{\gamma\delta}b_{\delta\alpha}n^k \\
\mathrm{IV} \quad &- f_{,\beta\alpha}\varepsilon^{\alpha\beta}n^k \\
\mathrm{V} \quad &+ f_{,\beta}\varepsilon^{\alpha\beta}b_{\alpha\gamma}a^{\gamma\delta}t^k_\delta.
\end{aligned}
$$

We have used here Eqs. (A.12) and (A.13) and the fact that $\varepsilon^{\alpha\beta}_{,\gamma} \equiv 0$.

The term (II) vanishes in virtue of the Codazzi identity, the term (III) is also equal to zero in view of the symmetry of $C_{\alpha\beta} \overset{\mathrm{df}}{=} b_{\gamma\beta}a^{\gamma\delta}b_{\delta\alpha}$, the next term (IV) vanishes in view of the symmetry of $f_{,\alpha\beta}$. The sum of the (I) and (V) terms can be rewritten in the form

$$a^{\gamma\delta}t^k_\delta \varepsilon^{\alpha\beta}(f_{,\alpha}b_{\gamma\beta} + f_{,\beta}b_{\gamma\alpha})$$

but the term in parentheses is symmetric with respect to α and β indices, thus the whole product also vanishes. Hence

(A.28) $$(f\varepsilon^{\alpha\beta}b_{\gamma\beta}a^{\gamma\delta}t^k_\delta - f_{,\beta}\varepsilon^{\alpha\beta}n^k)_{,\alpha} = 0.$$

References

1. J. Bodziony, *Charakterystyka przestrzennej struktury materiałów krystalicznych na podstawie pomiarów na przekrojach w świetle metod geometrii całkowej*, w: Metody geometryczne w fizyce i technice, WNT, Warszawa 1968.

2. A. TRZĘSOWSKI, *On the liquid-like response of the size effect body*, Bull. Acad. Polon. Sci., Série Sci. techn., **28**, 2, 1980.
3. A. TRZĘSOWSKI, *On the ideal elastometric body with the liquid-like response*, Bull. Acad. Polon. Sci., Série Sci. techn., **28**, 2, 1980.
4. A. BLINOWSKI, *On curvature dependent surface tension*, Arch. Mech., **23**, 2, 213–222, 1971.
5. H. HADWIGER, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer Verlag, Berlin—Göttingen—Heidelberg 1957 (Russian translation, Nauka, 1966).
6. Z. WESOŁOWSKI, *On the surface tension and the edge tension*, Arch. Mech., **20**, 2, 243–258, 1968.
7. A. E. GREEN, P. M. NAGHDI, W. L. WAINWRIGHT, *A general theory of a Cosserat surface*, Arch. Rat. Mech. Anal., **20**, 287–308, 1965.
8. L. E. SCRIVEN, *Dynamics of fluid interface*, Chem. Engn. Sci., **12**, 98–108, 1960.
9. A. GOETZ, *Geometria różniczkowa*, PWN, Warszawa 1965.
10. A. K. CHESTERS (private communication).
11. A. H. WALLACE, *Differential topology*, W. A. Benjamin, New York, Amsterdam 1968 (Russian translation Mir 1972).
12. R. SULANKE, P. WINTGEN, *Geometria różniczkowa i teoria wiązek*, PWN, Warszawa 1977.
13. R. ARIS, *Vectors, tensors and basic equations of fluid mechanics*, chapt. 9, 10, Prentice Hall.

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