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Selfgravitational instability analysis of a gas core liquid jet by using energy principle

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THE STABILITY of a self-gravitating gas jet surrounded by a self-gravitating liquid is discussed analytically, and the results are confirmed numerically. A general eigenvalue problem describing the characteristics of the gas-core liquid jet, based on the linear perturbation techniques, is derived by employing the energy principle. It is found that the fluids densities ratio S plays an important role in (de-)stabilizing of the present model. If $0 \leqslant S < 1$ ($S = s_2/s_1$, where s_2 is the liquid density and s_1 is the gas density), the model is unstable for certain values of the longitudinal wavenumber x (mainly $0 \leqslant x < 1.0668$), and stable for other values of x. However with increasing values of S (provided $0 < S < 1$), the instability domain decreases but never vanishes. As $S > 1$, unexpected results have been obtained: the model is gravitationally unstable not only for long wavelengths but also for very short wavelengths. These analytical results are interpreted physically and confirmed numerically, and the disturbance wave-numbers at which stability as well as instability occurs are tabulated. For $S = 0$ the results known from literature are obtained.

Problem stateczności grawitacyjnej strumienia gazu otoczonego cieczą przedyskutowano metodami analitycznymi, a uzyskane wyniki potwierdzono obliczeniami numerycznymi. Problem wartości własnych opisujący zachowanie się strumienia cieczy z rdzeniem gazowym sformułowano posługując się metodami perturbacji liniowych z wykorzystaniem zasad energetycznych. Stwierdzono, że stosunek gęstości płynów S spełnia istotną rolę w (de-) stabilizacji omawianego modelu. Jeśli $0 \leqslant S < 1$ ($S = s_2/s_1$, gdzie s_2 jest gęstością cieczy, a s_1 — gazu), model staje się niestateczny dla pewnych wartości liczby falowej x (przeważnie dla $0 \leqslant x < 1.0668$). Jednak przy wzrastających wartościach S (jeśli tylko $0 < S < 1$) obszar niestateczności szybko maleje, choć nigdy nie znika. Przy $S > 1$ otrzymano niespodziewany wniosek, że model staje się grawitacyjnie niestateczny nawet dla fal bardzo krótkich. Wyniki analityczne poparto analizą fizyczną i numeryczną zjawiska i stabelaryzowano wartości parametrów fal w chwili utraty stateczności. W przypadku $S = 0$ uzyskano potwierdzenie wyników znanych z literatury.

Проблема гравитационной устойчивости потока газа, окруженного жидкостью, обсуждена аналитическими методами, а полученные результаты подтверждены численными расчетами. Задача на собственные значения, описывающая поведение потока жидкости с газовым сердечником, сформулирована, послуживаясь методами линейных пертурбаций, опираясь на энергетические принципы. Констатировано, что отношение плотности жидкостей S играет существенную роль в (де-)стабилизации обсуждаемой модели. Если $0 \leqslant S < 1$ ($S = s_2/s_1$, где s_2 — плотность жидкости, s_1 — плотность газа), модель становится неустойчивой для некоторого значения волнового числа x (в большинстве случаев для $0 \leqslant x < 1{,}0668$). Однако при возрастающих значениях S (если только $0 < S < 1$) область неустойчивости быстро убывает, хотя никогда неисчезает. При $S > 1$ получено неожиданное следствие, что модель становится гравитационно неустойчивой даже дла очень коротких волн. Аналитические результаты подкреплены физическим и численным анализом явления и табулированы значения параметров волн в момент потери устойчивости. В случае $S = 0$ получено подтверждение результатов, известных из литературы.

1. Introduction

THE STABILITY of a full liquid jet has been studied since a long time ago, owing to its important applications in several domains of physics. It was PLATEAU [1] who for the first time obtained the critical capillary wavelength, both experimentally and theoretically. RAYLEIGH [2] derived the dispersion relation and developed the important concept of maximum mode of instability based on the linear theory. By extending Rayleigh's theory, WEBER [3] studied the capillary instability of a viscous liquid jet. These and other extensions were summarized by RAYLEIGH [4]; see also CHANDRASEKHAR [5].

The effect of nonlinearities on the capillary instability of a full liquid jet was considered by YUEN [6], WANG [7], NAYFEH [8], NAYFEH and HASSAN [9] and a complete analysis was given by KAKUTANI *et al.* [10].

The response of a self-gravitating incompressible cylinder to small axisymmetric disturbance was investigated by CHANDRASEKHAR and FERMI [11] by means of the energy principle. Soon afterwards, OGANESIAN [12] was the first to perform a detailed normal mode analysis for both axisymmetric and non-axisymmetric perturbations; see also CHANDRASEKHAR [5] (p. 516). Their pioneering analysis demonstrated that for dimensionless wavenumbers x which are less than the cut-off wavenumber $x_c = 1.0668$, the rotationally axisymmetric perturbations render the configuration gravitationally unstable, thus leading to the break-up of the fluid jet. This problem is of considerable interest in describing the appearance of condensation within celestial bodies. The effect of finite amplitude disturbances in a self-gravitating medium (fluid column) was first examined by TASSOUL and AUBIN [13], see also MALIK and SINGH [14]. The latter authors, moreover, investigated the modulation instability in a self-gravitating fluid column [15], and later on its nonlinear break-up [16].

The problem of stability of an annular liquid jet is also attractive owing to its important applications in physics. The capillary instability of an annular liquid jet (a liquid jet having a gas-core jet) has recently been investigated experimentally by KENDALL [17]. The last author explained clearly the importance and possible applications of the annular jet in astronomy. Moreover, he [17] drew the attention to the problem of stability and studied of that model analytically. The capillary instability of a gas jet surrounded by liquid (such that the liquid inertia force is greater than that of the gas) subject to different forces has recently been investigated [18, 19]. Indeed, the principle and basic physics of the new type of liquid-in-air jet are described by HERTZ and HERMANRUD [20]. The capillary instability of a liquid jet with a thin shell is studied by PETRYANOV and SHUTOV [21], see also SHUTOV [22]. More recently MAYER and WEIHS [23] developed an analytical investigation of the stability of an annular jet moving in an inviscid medium.

The main purpose of the present work is to investigate the self-gravitating instability of a gas-core liquid jet by employing the energy principle. The present results reduce to those of refs. [11, 12], if the inertia force of the gas is assumed to be greater than that of the liquid.

2. Formulation and eigenvalue relation

We shall consider an inviscid, incompressible self-gravitating gas-core liquid jet (with a gas jet of radius R and density s_1 and the liquid jet density s_2). The model is acted on by the gas inertia force, liquid inertia force and the variable gravitating force corresponding to each fluid.

To carry out the present theoretical approach based on the energy principle, one has to compute the change in the total kinetic energy E and that of the gravitational potential Ω in order to write down the Lagrangian function L. It may be noted that the Lagrangian function L is constructed as

$$L = E - \Omega \tag{2.1}$$

and the equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varepsilon}}\right) - \frac{\partial L}{\partial \varepsilon} = 0, \tag{2.2}$$

where ε is the Lagrangian variable and the dot over ε means the derivative of ε with respect to time. In a cylindrical coordinate (r, φ, z) system (with the z-axis coinciding with the axis of the annular liquid jet), the deformation of the (gas-liquid) interface can be described by

$$r = R + \varepsilon R \cos(kz + m\varphi). \tag{2.3}$$

The second term of the right-hand side of equation (2.3) is the distortion of the surface wave normalized with respect to R and measured from the unperturbed level, where k (any real number) is the longitudinal wavenumber, m (an integer) is the azimuthal wavenumber and ε is the deformation amplitude at time t

$$\varepsilon = \varepsilon_0 \exp(nt). \tag{2.4}$$

Here ε_0 is the initial amplitude and n is the growth rate of the perturbation; if n is imaginary, $n = i\omega$, then $\omega/2\pi$ is the oscillation frequency.

The basic equations which govern the gravitational potentials V_1 and V_2 are

$$\nabla^2 V_j = 4\pi G s_j, \quad j = 1, 2, \tag{2.5}$$

where G is the gravitational constant; from now on the quantities with subscript 1 mark the variables of the gas-core jet and those with 2 characterize the variables of the liquid. Solution of these equations referred to the deformed interface (2.3) is

$$V_1 = -\pi G s_1 r^2 + \varepsilon A_1 I_m(kr) \cos(kz + m\varphi), \tag{2.6}$$

$$V_2 = -\pi G s_2 r^2 + 2\pi G R^2 (s_1 - s_2) \ln\frac{R}{r} + \varepsilon A_2 K_m(kr) \cos(kz + m\varphi), \tag{2.7}$$

where I_m and K_m are, respectively, the modified first and second kind Bessel functions of order m; A_1 and A_2 are arbitrary constants. A_1 and A_2 are determined from the condition that the gravitational potential (under a suitably selected reference frame), and its derivatives are continuous at the interface (2.3);

$$A_1 = 4\pi G (s_1 - s_2) R^2 K_m(kR) \tag{2.8}$$

and

$$A_2 = 4\pi G (s_1 - s_2) R^2 I_m(kR). \tag{2.9}$$

It is worth noting that the solution obtained here for V_1 and V_2 reduces to that of reference [11] if we put $s_2 = 0$ and $m = 0$.

Now suppose that the amplitude of deformation ε is increased by $\delta\varepsilon$; then, due to this infinitesimal increment in the amplitude of deformation, the change $\delta\Omega$ in the gravitational potential energy can be determined by evaluating the work done during the displacement of the matter required to produce the change in ε. To evaluate this work it is necessary to specify quantitatively the redistribution which does take place.

Arbitrary deformation of an incompressible fluid can be thought of as resulting from the Lagrangian displacement $\boldsymbol{\xi}_j$ ($j = 1$ and 2) applied to each point of the fluids. Following Kendall, we assume that the perturbed motion is irrotational; this is the result of the irrotational motion of an inviscid fluid, see DRAZIN and REID [24]. Therefore, the Lagrangian displacements of the gas and liquid can be expressed as

$$\boldsymbol{\xi}_j = \operatorname{grad}\phi_j, \quad j = 1, 2. \tag{2.10}$$

From Eq. (2.10) and the incompressibility condition it follows that the displacement potentials ϕ_j satisfy Laplace's equation

$$\nabla^2\phi_j = 0, \quad j = 1, 2. \tag{2.11}$$

In view of equation (2.3), equation (2.11) takes the form of an ordinary differential equation, its solution being given in terms of Bessel functions of purely imaginary arguments. Therefore the non-singular solutions of ϕ_j under the present circumstances must be

$$\phi_1 = B_1 I_m(kr)\cos(kz+m\varphi), \tag{2.12}$$

$$\phi_2 = B_2 K_m(kr)\cos(kz+m\varphi). \tag{2.13}$$

The constants B_1 and B_2 are determined by applying the condition that the radial components of $\boldsymbol{\xi}_j$ are equal and reduce to $R\cos(kz+m\varphi)$ at $r = R$;

$$B_1 = R/(kI'_m(kR)) \quad \text{and} \quad B_2 = R/\big(k\,K'_m(kR)\big). \tag{2.14}$$

Hence

$$\boldsymbol{\xi}_1 = \big(\varepsilon R/k\big(I'_m(kR)\big)\big)\operatorname{grad}\big(\big(I_m(kr)\cos/(kz+m\varphi)\big), \tag{2.15}$$

$$\boldsymbol{\xi}_2 = \big(\varepsilon R/(k\big(K'_m(kR)\big)\big)\operatorname{grad}(K_m(kr)\cos(kz+m\varphi)) \tag{2.16}$$

and therefore the corresponding displacements $\delta\boldsymbol{\xi}_j$, which must be applied to each point of the fluids in order to increase the amplitude of deformation by $\delta\varepsilon$, are given by

$$\delta\boldsymbol{\xi}_1 = \big(R\delta\varepsilon/\big(kI'_m(kR)\big)\big)\operatorname{grad}\big(I_m(kr)\cos(kz+m\varphi)\big), \tag{2.17}$$

$$\delta\boldsymbol{\xi}_2 = \big(R\delta\varepsilon/\big(kK'_m(kR)\big)\big)\operatorname{grad}\big(K_m(kr)\cos(kz+m\varphi)\big). \tag{2.18}$$

Now, due to that additional deformation $\delta\varepsilon$, the change in the total gravitational potential energies $\delta\Omega_j$ (per unit length) can be obtained by integrating the work done by the displacements $\delta\boldsymbol{\xi}_j$ in the gravitational potentials V_j.

Thus for the gas-core jet we have

$$\delta\Omega_1 = 2\pi s_1\left\langle\!\!\left\langle \int_0^{R(1+\varepsilon\cos(kz+m\varphi))} (\delta\boldsymbol{\xi}_1\cdot\operatorname{grad}V_1)\,r\,dr\right\rangle\!\!\right\rangle \tag{2.19}$$

where the angular brackets signify that the quantity enclosed should be averaged over φ and z. Combining equations (2.6), (2.8), (2.17) and (2.19), we find

$$\delta\Omega_1 = 2\varepsilon\,\delta\varepsilon\,\pi^2 G R^4 s_1 [s_1 - 2(s_1 - s_2)\big(K_m(kR)/\big(kRI'_m(kR)\big)J_m(y)], \tag{2.20}$$

where $y = kr$ and

$$J_m(y) = \int_0^{kR} [\big(I'_m(y)\big)^2 + (1 + m^2 y^{-2}) I_m^2(y)] y\,dy.$$

Using the identity (which follows from Bessel's equations)

$$\frac{d}{dy}\big(yQ_m(y)Q'_m(y)\big) = y[\big(Q'_m(y)\big)^2 + (1 + m^2 y^{-2}) Q_m^2(y)], \tag{2.21}$$

where Q_m stands for the modified Bessel functions I_m and K_m; hence equation (2.20) yields

$$\delta\Omega_1 = 4\pi^2 G s_1 R^4 [\tfrac{1}{2}s_1 - (s_1 - s_2)\, I_m(kR)\, K_m(kR)]. \tag{2.22}$$

By integrating equation (2.22) from zero to ε we get

$$\Omega_1 = -2\pi^2 G s_1 R^4 [(s_1 - s_2) I_m(kR) K_m(kR) - \tfrac{1}{2}s_1]\varepsilon^2. \tag{2.23}$$

In a similar manner, the change in the total gravitational potentional energy Ω_2 of the liquid (per unit length) is obtained,

$$\Omega_2 = 2\pi^2 G s_2 R^4 [(s_1 - s_2) I_m(kR) K_m(kR) - \tfrac{1}{2}s_1]\varepsilon^2. \tag{2.24}$$

Henceforth the change in the total gravitational potential energy (per unit length) of the gas-core liquid jet is given by

$$\begin{aligned} \Omega &= \Omega_1 + \Omega_2 \\ &= -2\pi^2 G s_1 (s_1 - s_2) R^4 [(s_1 - s_2) I_m(kR) K_m(kR) - \tfrac{1}{2}s_1]\varepsilon^2. \end{aligned} \tag{2.25}$$

Now we have to evaluate the change in the total kinetic energy of the gas-core liquid jet. Since the Lagrangian coordinate ε is a function of time, each element of the fluids will move. This can be derived from the Lagrangian displacements

$$\mathbf{u}_j = \partial\boldsymbol{\xi}/\partial t, \quad j = 1, 2,$$

so that the velocity vectors of the gas-core and liquid jet, respectively, are

$$\mathbf{u}_1 = \big(R/\big(kI'_m(kR)\big)\big)\frac{d\varepsilon}{dt}\,\mathrm{grad}\big(I_m(kr)\cos(kz + m\varphi)\big) \tag{2.26}$$

and

$$\mathbf{u}_2 = \big(R/\big(kK'_m(kR)\big)\big)\frac{d\varepsilon}{dt}\,\mathrm{grad}\big(K_m(kr)\cos(kz + m\varphi)\big). \tag{2.27}$$

The change in the total kinetic energy E_1 (per unit length) of the gas-core jet associated with the motions specified by (2.26) is

$$\begin{aligned} E_1 &= \frac{1}{2}s_1 \int_0^{2\pi}\int_0^{kz=2\pi}\int_0^{R} u_1^2 r\,dr\,\frac{dkz}{2\pi}\,d\varphi \\ &= \pi s_1 R^2/\big(2k^2 I'^2_m(kR)\big)\left(\frac{d\varepsilon}{dt}\right)^2 J_m(y) = \pi s_1 R^3\big(I_m(kR)/(2kI'_m(kR)\big)\left(\frac{d\varepsilon}{dt}\right)^2, \end{aligned} \tag{2.28}$$

where the identity (2.21) has been used.

In a similar way the change in the total kinetic energy E_2 (per unit length) of the liquid associated with the motions specified by (2.27) is obtained and given by

$$E_2 = -\pi s_2 R^3\big(K_m(kR)/\big(2kK'_m(kR)\big)\big)\left(\frac{d\varepsilon}{dt}\right)^2. \tag{2.29}$$

Therefore the change in the total kinetic energy E of the gas-core liquid jet is

$$E = E_1+E_2 = \big(\pi R^3/(2k)\big)\left(\frac{d\varepsilon}{dt}\right)^2 [\big(s_1 I_m(kR)/I'_m(kR)\big)-\big(s_1 K_m(kR)/K'_m(kR)\big)]. \tag{2.30}$$

By applying relations (2.30), (2.25) and (2.1), Eq. (2.2) is transformed into the equation of motion for ε and, hence, the use of Eq. (2.4) yields the following relation:

$$n^2 = 4\pi G[(s_1-s_2)^2 I_m(x) K_m(x)-\tfrac{1}{2}s_1(s_1-s_2)]N_m(x), \tag{2.31}$$

where

$$N_m(x) = \big(xI'_m(x)K'_m(x)\big)/[s_1 I_m(x)K'_m(x)-s_2 I'_m(x)K_m(x)] \tag{2.31'}$$

and where $x(=kR)$ is the longitudinal dimensionless wavenumber.

3. Discussions of the results

Equation (2.31) is the eigenvalue relation of a gravitating liquid having a gravitating gas-core jet. By means of this relation the characteristics of the present model can be determined: one can identify the regions of instability (in particular their critical wavenumbers, maximum growth rate values and the corresponding wavenumbers) and those of stability as well.

The eigenvalue relation (2.31) ralates the growth rate n (or rather the oscillation frequency ω) with the densities s_1 and s_2 of the two fluids, the value of $(4\pi Gs_j)^{-\frac{1}{2}}$ as unit of time, the characteristic length R, the azimuthal and dimensionless longitudinal wavenumbers m, x, and the cylindrical functions appropriate to the problem at hand.

Since this problem is somewhat more general, one can recover other dispersion relations as limiting cases from the present relation (2.31) with suitable assumptions.

If we assume $s_2 = 0$, Eq. (2.31) gives

$$n^2 = 4\pi Gs_1[xI'_m(x)/I_m(x)]\big(I_m(x)K_m(x)-\tfrac{1}{2}\big). \tag{3.1}$$

The dispersion relation (3.1) was established by OGANESIAN [12], see also reference [11] as $m = 0$.

If we set $s_2 = 0$ and at the same time $m = 0$; Eq. (2.31) reduces (since $I'_0 = I_1$) to

$$n^2 = 4\pi Gs_1\big(xI_1(x)/I_0(x)\big)\big(I_0(x)K_0(x)-\tfrac{1}{2}\big). \tag{3.2}$$

Equation (3.2) is the dispersion relation of a gravitating full fluid cylinder in vacuum for the rotationally axisymmetric perturbations $m = 0$ (nowadays this kind of perturbations is called "sausage mode"). It was CHANDRASEKHAR and FERMI [11] who first derived that relation by means of the energy principle. For the stability discussions of Eqs. (3.1) and (3.2) we may refer to OGANESIAN [12].

If we impose $s_1 = 0$, Eq. (2.31) yields

$$n^2 = 4\pi G s_2\big(-xI_m(x)K'_m(x)\big). \tag{3.3}$$

This is the eigenvalue relation of a hollow jet (i.e. a liquid jet having a vacuum-core cylinder which is a mirror case of the full liquid jet) subjected to the gravitation force. One can show (see the recurrence relation (3.6)) that the right-hand side of Eq. (3.3) is always positive for each non-zero real value of x. This means that the gravitating hollow jet model, if it exists, is unstable for all (axisymmetric $m = 0$ and non-axisymmetric $m \geqslant 1$) modes of perturbation.

Now, for investigating the (in-)stability of the present model, it is convenient to rewrite the eigenvalue relation (2.31) in a dimensionless form,

$$n^2/4\pi G s_1 = (1-S)[(1-S)I_m(x)K_m(x)-\tfrac{1}{2}]F_m(x), \tag{3.4$_1$}$$

where S and $F_m(x)$ are defined as

$$F_m(x) = [xI'_m(x)K'_m(x)]\big(I_m(x)K'_m(x)-SI'_m(x)K_m(x)\big)^{-1} \tag{3.4$_2$}$$

and

$$S = s_2/s_1. \tag{3.4$_3$}$$

This eigenvalue relation is valid for all modes of perturbations: sausage mode $m = 0$ and non-axisymmetric modes $m \geqslant 1$.

Consider now the recurrence relations (see ABRAMOWITZ and STEGUN [25])

$$2I'_m(x) = I_{m-1}(x)+I_{m+1}(x), \tag{3.5}$$

$$2K'_m(x) = -K_{m-1}(x)-K_{m+1}(x). \tag{3.6}$$

It is known that $I_m(x)$ is always positive and monotonic increasing and that $K_m(x)$ is monotonic decreasing but never negative for each non-zero real value of x; hence one can observe that $I'_m(x)$ is positive while $K'_m(x)$ is always negative. On the basis of these arguments, one can show that

$$F_m(x) > 0 \tag{3.7}$$

for each non-zero real value of x, all S values and all modes of perturbations $m \geqslant 0$; and that $F_m(x)$ never changes its sign. Now we have to distinguish between the two different kinds of perturbations: the sausage mode $m = 0$ and the non-axisymmetric modes $m \geqslant 1$.

3.1. Non-axisymmetric perturbations $m \geqslant 1$

It is worthwhile to mention here that, due the properties of the modified Bessel functions

$$I_m(x)\,K_m(x) < \frac{1}{2}, \quad \text{for all} \quad m \geqslant 1 \tag{3.8}$$

and is never negative.

Now to find out whether the problem at hand is gravitationally stable or not we should consider the different cases when S is greater than, equal to, and less than unity, the inequality (3.7) being taken into account.

If $S > 1$, Eq. $(3.4)_1$ shows that the dimensionless growth rate $n/(4\pi G s_1)^{\frac{1}{2}}$ is real. This means that a liquid having a gas-core jet is gravitationally unstable in the non-axisymmetric modes $m \geqslant 1$ if the liquid is more dense than the gas-core jet.

If $S = 1$, Eq. $(3.4)_1$ shows that the growth rate is zero. This means that we have neutral stability and there is no dispersion. This is intuitively clear since in such a case we have a gravitational homogeneous medium of uniform density.

If $0 < S < 1$, Eq. $(3.4)_1$ shows (taking into account Eq. (3.8)) that $n/(4\pi G s_1)^{\frac{1}{2}}$ is purely imaginary. This means that the model is gravitationally stable in the non-axisymmetric modes $m \geqslant 1$ as long as the gas-core jet is more dense than the liquid.

Let us mention here (as a special case) that if $S = 0$, Eq. $(3.4)_1$ shows that the model is stable for all purely-axisymmetric perturbations $m \geqslant 1$. This coincides with the previously reported results (see OGANESIAN [12] and also CHANDRASEKHAR and FERMI [11]).

3.2. Sausage perturbations $m = 0$

For such a case the inequality (3.8) does not hold for all x. Equations $(3.4)_1$ and $(3.4)_2$ yield (since $I_0(x) = I_1(x)$ and $K_0'(x) = -K_1(x)$)

$$(3.9)_1 \qquad n^2/4\pi G s_1 = (1-S)[(1-S)I_0(x)K_0(x)-\tfrac{1}{2}]F_0(x)$$

and

$$(3.9)_2 \qquad F_0(x) = [xI_1(x)K_1(x)]\big(I_0(x)K_1(x)+SI_1(x)K_0(x)\big)^{-1}.$$

In a similar manner as for the non-axisymmetric modes $m \geqslant 1$, Eq. $(3.9)_1$ has been studied analytically and the obtained results are exactly the same as those of the non-axisymmetric perturbations. In order to be sure about the correctness of these results (since for $m = 0$, the inequality (3.8) does not hold for all x), the dispersion relation $(3.9)_1$ has been studied numerically by computer simulation and then the numerical results are illustrated in Figs. (1–3). It is seen, what confirms the analytical results, that the values of the dimensionless growth rate $n/(4\pi G s_1)^{\frac{1}{2}}$ and those of the critical wavenumber x decrease

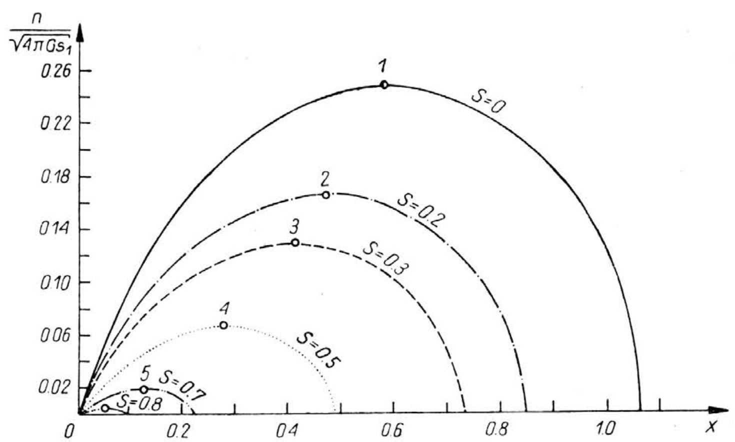

FIG. 1.

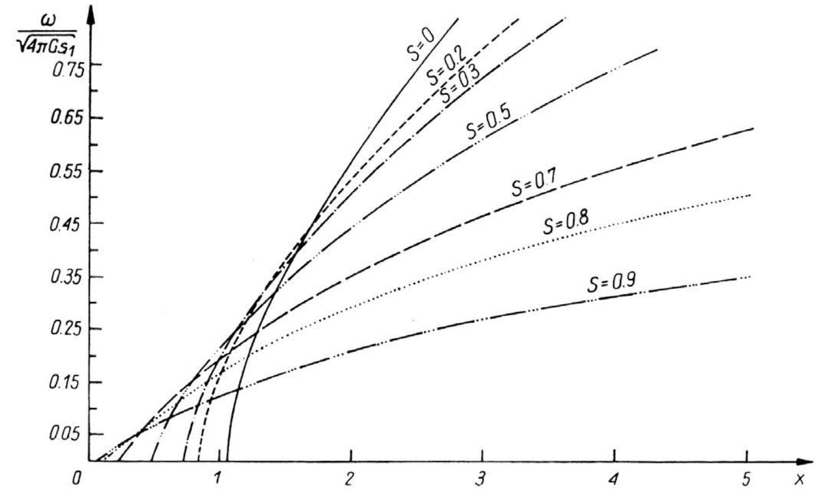

FIG. 2.

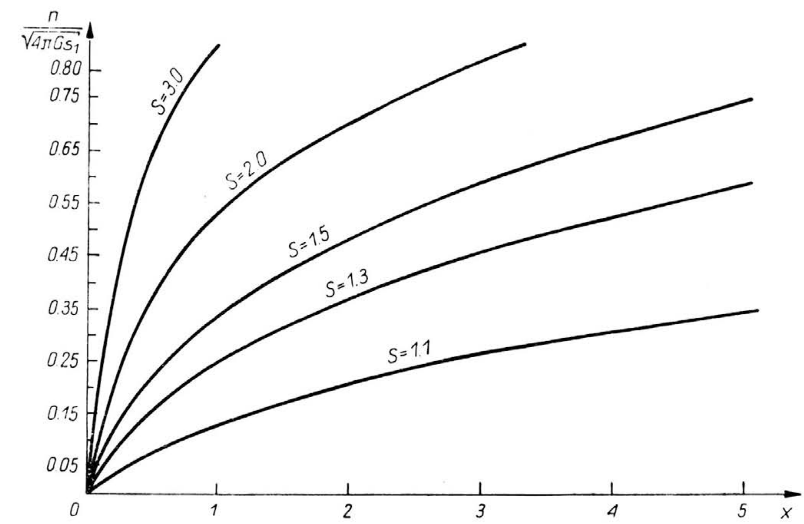

FIG. 3.

with increasing density ratio $0 < S < 1$. For S equal to 0, 0.2, 0.3, 0.5, 0.7, 0.8 and 0.9 we get 0.2455, 0.1650, 0.1289, 0.06614, 0.0196, 0.0056 and 0.0002 for $n/(4\pi G s_1)^{\frac{1}{2}}$ at $x =$ $= 0.580$, 0.469, 0.411, 0.282, 0.133, 0.057 and 0.005, respectively; and the corresponding values of the critical wavenumbers x are 1.066, 0.847, 0.732, 0.489, 0.223, 0.093 and 0.007, respectively. Indeed, this indicates how fast the domain of instability shrinks with increasing values of S ($0 \leqslant S < 1$). In the case when the model is subjected to the gas and liquid inertia forces and acted upon by pressure, RADWAN [19] and experimentally KENDALL [17] proved that the model is stable in the sausage mode $m = 0$ with wavelength longer than the circumference of the gas-core jet and is also stable in the nonaxisymmetric modes $m \geqslant 1$ for all wavelengths. It could be expected that the model is stable as S is greater than unity. But it is found that the model is absolutely unstable for S greater than unity (see also the numerical results for $S = 1.1$, 1.3, 1.5, 2.0, and 3.0) not only for very long wavelengths but also for short wavelengths. This maybe is logical, since in such a case the liquid is much more dense than the gas and the acting force is self-gravitational; and it is known that the gravitational force is a long range force in contrast to the capillary force.

4. Conclusions

The (in-)stability of a self-gravitating gas cylinder surrounded by a self-gravitating liquid is investigated on the basis of the energy principle. It is found that the densities of the liquid to the gas ratio S plays an essential role in identifying the (in-)stability features or the gas-core liquid cylinder. That is true not only for the symmetric mode $m = 0$ but also for asymmetric modes $m \neq 0$.

i) When the gas cylinder is more dense than the surrounding liquid, the model is unstable if $2SI_m(x)K_m(x)$ is greater than unity (and vice versa).

ii) When the density of the gas cylinder is equal to the density of the surrounding liquid, the model is stable. Note also that in the case if $S < 1$ and simultaneously $SI_m(x)K_m(x) = 1/2$ in all modes $m \geqslant 0$ of perturbations for all wavelengths.

iii) When the gas cylinder is less dense than surrounding liquid, it is found that the gas-core liquid cylinder is unstable in all symmetric and asymmetric modes. This instability is true not only for long wavelengths but also for short wavelengths, what is surprising. However, it has a good interpretation in the process of destruction of interstellar clouds and also in the break-up or spiral arms of galaxies (cf. reg. [11]). These analytical results are confirmed numerically, see Figs. 1–3.

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