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A discrete kinetic model admitting compression and expansion shock waves

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We propose a discrete kinetic model which has some properties typical for retrograde gases. The characteristic feature of the model is that the probabilities of direct and inverse collisions are not symmetric. We deduce the Euler and Navier–Stokes equations corresponding to the proposed model. The plane shock wave is studied by means of these three types of equations. We find that in some cases the number density must decrease in order for the shock to be stable. The transition line is shown to be the same for the Boltzmann and Navier–Stokes model equations and, in the case of weak shocks, it coincides with that found from the Euler model equations. The shock structure in the Boltzmann model equation is the same as that found by previous authors.

Proponowany jest dyskretny model kinetyczny, który ma pewne własności typowe dla gazów odwrotnych (ang. retrograde). Charakterystyczną cechą tego modelu jest to, że prawdopodobieństwa prostych i odwrotnych zderzeń nie są symetryczne. Wyprowadzamy równania Eulera i Naviera–Stokesa odpowiadające proponowanemu modelowi. Badamy fale uderzeniowe za pomocą tyh trzech typów równań. Stwierdzamy że w pewnych przypadkach gęstość liczbowa musi maleć po to, by fala była stabilna. Pokazujemy, że linia przejścia jest taka sama dla modelowych równań Boltzmanna i Naviera–Stokesa i w przypadku słabych fal uderzeniowych, pokrywa się z linią znalezioną z równań Eulera. Struktura fali uderzeniowej w modelowym równaniu Boltzmanna jest taka sama jak znaleziona przez wcześniejszych autorów.

Предлагается дискретная кинетическая модель, которая имеет некоторые свойства типичные для обратных газов (по-английскиr etrograde). Характеристическим свойством этой модели является то, что вероятности простых и обратных столкновений не являются симметричными. Выводим уравнения Эйлера и Навье–Стокса отвечающие предлагаемой модели. Исследуем ударные волны при помощи этих трех типов уравнений. Констатируем, что в некоторых случаях числовая плотность должна уменьшаться для того, чтобы волна была стабильной. Показываем, что линия перехода является такой самой для модельных уравнений Больцмана и Навье–Стокса и в случае слабых ударных волн совпадает с линией, найденной из уравнений Эйлера. Структура ударной волны, в модельном уравнении Больцмана, является такой самой как найденная авторами более ранних публикаций.

1. Introduction

A RETROGRADE fluid is a medium whose molecules are of high molecular weight and complex structure. This result in many vibrational degrees of freedom and high specific heats. Consequently, flows in such fluids can exhibit many very unusual properties [1].

Presumably, it could be possible to explain most of them by applying a kinetic theory approach. Unfortunately, to the present author's knowledge, no kinetic theory of retrograde fluids exists and, if it did, it would be of extreme complexity. Being, hoevever, convinced that even complicated phenomena can be better understood by means of simple models, in this paper we propose a discrete kinetic model resembling a retrograde gas. The encouragement was the great success of the discrete velocity models in describing many phenomena in single rarefied gases (cf. [2, 3, 4]) as well as in their binary mixtures [5, 6, 7, 8].

It is clear that no simple model can be used to describe properly all features in a medium like a retrograde gas. Therefore we limit ourselves to just one aspect of the problem, namely to the shock problem in a gaseous phase of a retrograde fluid [9, 10, 11].

In the next section we introduce the model, and discuss its physical sense. Also the Euler and Navier–Stokes equations corresponding to it are given.

In Sect. 3 we discuss the model Euler equations. They form a strictly hyperbolic system of two coupled quasilinear equations. However, the characteristic fields are neither genuinely nonlinear nor linearly degenerate.

In Sect. 4 the shock wave problem in these equations in considered. Using the generalized entropy condition [9, 10] we show that, similarly to the true Euler equations, both compression and expansion shocks are possible.

The shock profile in the model Navier–Stokes equations is studied in Sect. 5. The dissipation term involves now a term proportional to the density gradient, what is new compared to both the true Navier–Stokes equation and those corresponding to the original Broadwell model. As usual, however, the shock wave has a smooth profile, which is found exactly. Again the shock may be either a compression or an expansion wave. The transition line is however different from that predicted by the model Euler equations. The agreement is achieved for weak shock waves only. In this case our results agree with those by CRAMER and KLUWICK [11]

In Sect. 6 we study the shock profile in the model kinetic equations. Again, the shock can be either an expansion or a compression wave. The transition line is the same as in the model Navier–Stokes equations. However, the shock profiles agree for weak shock waves only. The shock wave thickness is discussed in detail. In particular, our results confirm the CRAMER and KLUWICK'S [11] conjecture about the weak shock thickness.

2. Construction of the kinetic model

The simplest and at the same time a sufficiently sophisticated model to be of physical interest is the celebrated Broadwell's model [12] (see also [2, 3, 4]). One of its one-dimensional versions is given by the equations

$$
\begin{aligned}
\frac{\partial N^1}{\partial t} + c\frac{\partial N^1}{\partial x} &= -\frac{1}{\varepsilon}Q,\\
\frac{\partial N^2}{\partial t} - c\frac{\partial N^2}{\partial x} &= -\frac{1}{\varepsilon}Q,\\
\frac{\partial N^3}{\partial t} &= \frac{1}{\varepsilon}Q,
\end{aligned}
\tag{2.1}
$$

where N^1, N^2, N^3 are scalar functions representing probability densities of particles moving in the positive x-direction, negative x-direction and perpendicularly to it, respectively, $x \in \mathrm{R}^1$, $t \in \mathrm{R}^1_+$, $c > 0$ is the particle velocity, $\varepsilon > 0$ can be identified as the Knudsen number. The collisional operator is of the form

$$
Q(N, N) = N^1 N^2 - (N^3)^2. \tag{2.2}
$$

The system of equations (2.1) has two collisional invariants

$$\psi_0 = (1, 1, 2) \tag{2.3}$$

and

$$\psi_1 = (1, -1, 0) \tag{2.4}$$

regardless of the form of Q.

The density ϱ and the mean velocity u are defined by

$$\varrho = \frac{1}{4}(N^1+N^2+2N^3), \tag{2.5}$$

and

$$\varrho u = \frac{1}{4}c(N^1-N^2), \tag{2.6}$$

respectively.

The transport equations are

$$\begin{aligned} &\frac{\partial}{\partial t}\varrho+\frac{\partial}{\partial x}(\varrho u) = 0,\\ &\frac{\partial}{\partial t}(\varrho u)+\frac{\partial}{\partial x}\left[\frac{1}{4}c^2(N^1+N^2)\right] = 0. \end{aligned} \tag{2.7}$$

The densities $\nu^i (i = 1, 2, 3)$ are said to be equilibrium densities if they all are positive and if $Q(\nu, \nu) = 0$. If the collisional operator Q is given by Eq. (2.2) then

$$\begin{aligned} \nu^1 &= \varrho\left(1+\frac{u}{c}\right)^2,\\ \nu^2 &= \varrho\left(1-\frac{u}{c}\right)^2,\\ \nu^3 &= \varrho\left(1-\left(\frac{u}{c}\right)^2\right). \end{aligned} \tag{2.8}$$

They are positive if and only if

$$\varrho > 0, \quad |u| < c. \tag{2.9}$$

Setting $N^i = \nu^i$ $(i = 1, 2, 3)$ is Eqs. (2.7) we obtain the Euler equations corresponding to the model (2.1), (2.2). They are

$$\begin{aligned} &\frac{\partial \varrho}{\partial t}+\frac{\partial}{\partial x}(\varrho u) = 0,\\ &\frac{\partial}{\partial t}(\varrho u)+\frac{\partial}{\partial x}(p+\varrho u^2) = 0. \end{aligned} \tag{2.10}$$

Here the pressure p is given by

$$p = \frac{1}{2}\varrho(c^2-u^2). \tag{2.11}$$

For the model kinetic equations (2.1) we can also write the Navier–Stokes equations [2, 13],

$$\tag{2.12} \begin{gathered} \frac{\partial \varrho}{\partial t}\varrho+\frac{\partial}{\partial x}(\varrho u)=0, \\ \frac{\partial}{\partial t}(\varrho u)+\frac{\partial}{\partial x}(p+\varrho u^2)=\frac{\partial}{\partial x}\left[\mu(u)\frac{\partial}{\partial x}u\right], \end{gathered}$$

where p is given by Eq. (2.11), and the viscosity coefficient $\mu(u)$ takes the form

$$\tag{2.13} \mu(u)=\frac{\varepsilon}{8}(c^2-u^2).$$

The Broadwell model or its versions was considered by many authors (cf. [3] and references therein). In particular, the relations between shock waves in the model Euler, Navier–Stokes and Boltzmann equations were studied by CAFLISH [14]. Recently the shock stability was proved by KAWASHIMA and MATSUMURA [15], and CAFLISH and LIU [16]. CORNILLE in [17] and in other papers (cf. [17] for references) constructed many interesting exact solutions for the Broadwell model.

Hence, both the theory and applications of the Broadwell model are so well developed that it can be a good reference for our modification.

We proceed to construct a model resembling retrograde gases. We want our model to be of the form Eqs. (2.1) with the collisional term Q different from Eq. (2.2).

As it is known [18], the simplest model of a retrograde fluid is that given by van der Waals. In this model, the relation between the presure p, the density ϱ and the temperature T is of the form

$$\tag{2.14} p=\frac{R\varrho T}{1-b\varrho}-a\varrho^2,$$

where R is the gas constant, $a \geqslant 0$ is a quantity proportional to the forces of molecular attraction, and $b \geqslant 0$ is a constant characterizing the maximal concentration.

For the sake of simplicity we set $b = 0$, i.e. we admit the gas to be infinitely compressible, what is in accordance with the Boltzmann concept of gas. Hence, we take

$$\tag{2.15} p=R\varrho T-a\varrho^2,$$

instead of Eq. (2.14). It can be checked easily that a gas with the constitutive relation (2.15) is still of the retrograde type.

The first term at the right-hand side of Eq. (2.15) is the usual ideal gas expression for the pressure, which in the Broadwell model is replaced by Eq. (2.11). Thus, inserting $\frac{1}{2}\varrho(c^2-u^2)$ in place of $R\varrho T$, we obtain from Eq. (2.15)

$$\tag{2.16} p=\frac{1}{2}\varrho(c^2-u^2)-a\varrho^2.$$

Hence, our model should be of the form (2.1) with such expression for the collisional operator Q that the pressure term is of the form (2.16) instead of Eq. (2.11). Consequently, the model Euler equations should be of the form (2.10) with p given by Eq. (2.16).

One can obtain the Euler equations from the general transport equations (2.7) by setting $N_i^2 = \nu^i (i = 1, 2, 3)$, where ν^i are equilibrium densities. Therefore

$$\frac{1}{4}4c^2(\nu^1+\nu^2) = p+\varrho u^2 = \frac{1}{2}\varrho(c^2+u^2)-a\varrho^2.$$

This relation along with Eqs. (2.5), (2.6) yields a system of three linear equations for three unknowns ν^1, ν^2, ν^3. The solution is

$$\nu^1 = \varrho\left(1+\frac{u}{c}\right)^2-\frac{2a}{c^2}\varrho^2,$$

$$\nu^2 = \varrho\left(1-\frac{u}{c}\right)^2-\frac{2a}{c^2}\varrho^2, \tag{2.17}$$

$$\nu^3 = \varrho\left(1-\frac{u^2}{c^2}\right)+\frac{2a}{c^2}\varrho^2.$$

Thus, we know the equilibrium densities of an unknown collisional operator Q.

We have

PROPOSITION 2.1. Let $P(N)$ be a third degree polynomial of $N = (N^1, N^2, N^3)$. The vector $\nu = (\nu^1, \nu^2, \nu^3)$ with components given by Eqs. (2.17) nullifies $P(N)$ if and only if $P(N)$ is of the form

$$P(N) = A\left[\frac{1}{8}\frac{a}{c^2}(N^1+N^2+2N^2)^3+N^1N^2-(N^3)^2\right], \tag{2.18}$$

where A is an arbitrary constant.

To prove the assertion of Proposition 2.1 we take an arbitrary third degree polynomial $P(N)$. It has 20 coefficients. Next we substitute $N = \nu$, where ν is given by Eqs. (2.17), and demand the obtained expression to be identically equal to zero for any values of ϱ and u. After some calculations we obtain Eq. (2.18). We take

$$Q(N) = P(N) = A\left[8\frac{a}{c^2}\varrho^3+N^1N^2-(N^3)^2\right] \tag{2.19}$$

$$= A\left[\left(N^1+\frac{2a}{c^2}\varrho^2\right)\left(N^2+\frac{2a}{c^2}\varrho^2\right)-\left(N^3-\frac{2a}{c^2}\varrho^2\right)^2\right].$$

We define the H-function by

$$H = \left(N^1+\frac{2a}{c^2}\varrho^2\right)\ln\left(N^1+\frac{2a}{c^2}\varrho^2\right)+\left(N^2+\frac{2a}{c^2}\varrho^2\right)\ln\left(N^2+\frac{2a}{c^2}\varrho^2\right)$$
$$+\left(N^3-\frac{2a}{c^2}\varrho^2\right)\ln\left(N^3-\frac{2a}{c^2}\varrho^2\right)^2.$$

For a uniform gas, we have $d\varrho/dt = 0$. Therefore, using that and Eq. (2.19) we obtain

$$(2.20)\qquad \frac{dH}{dt} = +\frac{A}{\varepsilon}\left(N^3-\frac{2a}{c^2}\varrho^2\right)^2\left[1-\frac{\left(N^1+\frac{2a}{c^2}\varrho^2\right)\left(N^2+\frac{2a}{c^2}\varrho^2\right)}{\left(N^3-\frac{2a}{c^2}\varrho^2\right)^2}\times\ln\frac{\left(N^1+\frac{2a}{c^2}\varrho^2\right)\left(N^2+2\frac{a}{c^2}\varrho^2\right)}{\left(N^3-\frac{2a}{c^2}\varrho^2\right)^2}\right].$$

From the above it follows that

$$\frac{dH}{dt}\leqslant 0 \quad \text{if and only if} \quad A\geqslant 0.$$

The case $A = 0$ is not interesting, therefore it must be $A > 0$. Without any loss of generality we can assume $A = 1$, since in Eqs. (2.1) Q is multiplied by an arbitrary constant $1/\varepsilon$. Hence, we take finally

$$(2.21)\qquad Q = 8\frac{a}{c^2}\varrho^3+N^1N^2-(N^3)^2 = \left(N^1+2\frac{a}{c^2}\varrho^2\right)\left(N^2+2\frac{a}{c^2}\varrho^2\right)-\left(N^3-2\frac{a}{c^2}\varrho^2\right)^2.$$

Again, from Eq. (2.20), in the case of $A = 1$, it follows that $dH/dt = 0$ if and only if N is given by (2.17).

Thus, an analogue of the Boltzmann H-theorem is proved.

Applying the Chapman-Enskog procedure to the model (2.1), (2.18) (see [2]) we derive again the Euler equations (2.10), (2.16) as the zero order approximation in ε, and the Navier–Stokes equations

$$(2.22)\qquad \begin{aligned}&\frac{\partial}{\partial t}\varrho+\frac{\partial}{\partial x}(\varrho u) = 0,\\ &\frac{\partial}{\partial t}(\varrho u)+\frac{\partial}{\partial x}(p+\varrho u^2) = \frac{\partial}{\partial x}\left[\mu\frac{\partial u}{\partial x}+\varkappa\frac{\partial}{\partial x}(\varrho u)\right],\end{aligned}$$

where p is given by (2.16), and

$$(2.23)\qquad \mu = \frac{\varepsilon}{8}(c^2-u^2-4a\varrho),$$

$$(2.24)\qquad \varkappa = \varepsilon a$$

as the first order approximation.

There is a difference between the Navier–Stokes equations (2.12) and (2.13) corresponding to the usual Broadwell model (2.1) and (2.2) and those given by Eqs. (2.22)–(2.24). Namely, a quantity $\frac{\partial}{\partial x}(\varrho u)$ is added to the dissipation term.

At least, we impose the following conditions on ϱ and u in order to have the equilibrium densities positive

$$(2.25)\qquad \varrho > 0,\quad |u| < c,\quad a\varrho < \frac{1}{2}(c-|u|)^2.$$

3. The model Euler equations

We consider the initial value problem for the Euler equations (2.10) and (2.16) subject to the conditions

$$\varrho(0, x) = \varrho_0(x), \quad u(0, x) = u_0(x), \quad x \in \mathrm{R}^1. \tag{3.1}$$

We write Eqs. (2.10) and (2.16) in the matrix form

$$\frac{\partial}{\partial t}\begin{pmatrix}\varrho\\ u\end{pmatrix} + M\frac{\partial}{\partial x}\begin{pmatrix}\varrho\\ u\end{pmatrix} = 0, \tag{3.2}$$

where

$$M = \begin{bmatrix} u & \varrho \\ \frac{1}{2\varrho}(c^2 - u^2 - 4a\varrho) & 0 \end{bmatrix}. \tag{3.3}$$

The eigenvalues of M are solutions of

$$\lambda^2 - \lambda u - \frac{1}{2}(c^2 - u^2 - 4a\varrho) = 0. \tag{3.4}$$

Hence

$$\lambda_\pm(u, \varrho) = \frac{1}{2}\left(u \pm \sqrt{2c^2 - u^2 - 8a\varrho}\right). \tag{3.5}$$

The eigenvalues are real and distinct provided that

$$a\varrho < \frac{1}{2}(2c^2 - u^2). \tag{3.6}$$

From Eq. (3.5) the following inequalities follow

$$\lambda_+ \leqslant 0 \quad \text{for} \quad u \in \left(-c, -\sqrt{c^2 - 4a\varrho}\right),$$

$$\lambda_+ > 0 \quad \text{for} \quad u \in \left(-\sqrt{c^2 - 4a\varrho}, c\right)$$

and

$$\lambda_- < 0 \quad \text{for} \quad u \in \left(-c, \sqrt{c^2 - 4a\varrho}\right),$$

$$\lambda_- \geqslant 0 \quad \text{for} \quad u \in \left< \sqrt{c^2 - 4a\varrho}, c\right).$$

Thus, if

$$\sqrt{c^2 - 4a\varrho} < |u| < c$$

then both characteristics are of the same sign. Because it is not a situation met in the true gas-dynamics we reject it as non-physical and assume

$$|u| < \sqrt{c^2 - 4a\varrho}$$

or

$$a\varrho < \frac{1}{4}(c^2 - u^2). \tag{3.7}$$

Combining Eqs. (2.25), (3.6) and (3.7), we obtain

$$(3.8)\qquad \varrho > 0, \quad |u| < c, \quad 0 < a\varrho < \frac{1}{2}\min\left\{(c-|u|)^2, \frac{1}{2}(c^2-u^2)\right\}.$$

The set of ϱ, u satisfying conditions (3.8) is denoted by U. In what follows we assume that ϱ, u belong to this set.

If $(\varrho, u) \in U$, then we have also

$$(3.9)\qquad -c < \lambda_- < 0 < \lambda_+ < c$$

and

$$(3.10)\qquad \lambda_- < u < \lambda_+.$$

The right and left eigenvectors of M are

$$(3.11)\qquad r_\pm = (\varrho, -\lambda_\mp),$$

$$(3.12)\qquad l_\pm = (\lambda_\pm, \varrho).$$

Thus we have proved

LEMMA 3.1. If $(\varrho, u) \in U$, then the system of the Euler equations (2.10), (2.16) is strictly hyperbolic.

We define

$$(3.13)\qquad \Gamma_\pm = r_\pm \operatorname{grad}\lambda_\pm,$$

where $\operatorname{grad} \equiv \left(\frac{\partial}{\partial\varrho}, \frac{\partial}{\partial u}\right)$.

After some calculations we obtain

$$(3.14)\qquad \Gamma_\pm = \pm\frac{\lambda_\mp^2 - 2a\varrho}{\sqrt{2c^2-u^2-8a\varrho}} = \mp\frac{c^2-6a\varrho-\lambda_\pm^2}{\sqrt{2c^2-u^2-8a\varrho}}.$$

From Eq. (3.14) it is seen that the characteristics are neither genuinely nonlinear nor linearly degenerate in the sense of LAX [23]. Indeed, $\Gamma_\pm$ vanish on the curves $\gamma_\pm$, respectively, given by

$$(3.15)\qquad \gamma_\pm = \{(\varrho, u) \in U : u = \operatorname{sgn}[\pm(c^2-8a\varrho)]\left[c^2-4a\varrho+2\sqrt{2a(c^2-6a)}\,\right]^{1/2}$$

or

$$u = \operatorname{sgn}[\pm(c^2-8a\varrho)]\left[c^2-4a\varrho-2\sqrt{2a\varrho(c^2-6a\varrho)}\right]^{1/2}\},$$

where

$$\operatorname{sgn} x = \begin{cases} -1 & \text{for} \quad x < 0, \\ 0 & \text{for} \quad x = 0, \\ 1 & \text{for} \quad x > 0. \end{cases}$$

The normal vectors $n_\pm$ to $\gamma_\pm$ at $(\varrho, u) \in \gamma_\pm$ are given by

$$n_\pm = \left(2\lambda_\pm \frac{\partial\lambda_\pm}{\partial\varrho} + 6a,\ 2\lambda_\pm \frac{\partial\lambda_\pm}{\partial u}\right)$$

what follows from the equation $\Gamma_\pm = 0$.

Therefore for $(\varrho, u) \in \gamma_\pm$

$$r_\pm \cdot n_\pm = 2\lambda_\pm \Gamma_\pm + 6a\varrho = 6a\varrho > 0.$$

Thus the right eigenvectors $r_\pm$ are transversal to $\gamma_\pm$. It means that none of the characteristics is linearly degenerate in any subregion of U.

Thus we have proved

LEMMA 3.2. The system of the Euler equations (2.10), (2.16) is neither genuinely nonlinear nor linearly degenerate in U. The linear degeneracy does not take place in any subregion of U.

From Lemmas 3.1 and 3.2 it follows that the general theory of such hyperbolic systems of conservation laws as developed by LIU in [9, 10] is applicable to the model Euler equations (2.10) and (2.16) at least in U.

4. Shocks in the Euler equations

Following LAX [20] we take solutions to (2.10), (2.16) and (3.1) in a weak sense. Such solutions consist of their domains of continuity separated by lines of discontinuity. Let $x = x(t)$ be the equation of a line of discontinuity, and let $s = x'(t)$. Then across it the Rankine–Hugoniot relations hold

$$\begin{gathered} s(\varrho_r - \varrho_l) = \varrho_r u_r - \varrho_l u_l, \\ s(\varrho_r u_r - \varrho_l u_l) = \frac{1}{2}[\varrho_r(c^2 + u_r^2 - 2a\varrho_r)] - \frac{1}{2}[\varrho_l(c^2 + u_l^2 - 2a\varrho_l)], \end{gathered} \tag{4.1}$$

where for any quantity $Q(x, t)$

$$\begin{aligned} Q_r &= Q(x(t)+0, t), \\ Q_l &= Q(x(t)-0, t). \end{aligned}$$

We define the R-H curves $S(\varrho_0, u_0)$ through a given state $(\varrho_0, u_0) \in U$ as follows [9, 10]

$$S(\varrho_0, u_0) = \Big\{(\varrho, u) \in U\colon \sigma(\varrho - \varrho_0) = \varrho u - \varrho_0 u_0 \quad \text{and} \quad (\varrho u - \varrho_0 u_0) \tag{4.2}$$

$$= \frac{1}{2}\varrho(c^2 + u^2 - 2a\varrho) - \frac{1}{2}\varrho_0(c^2 + u_0^2 - 2a\varrho_0) \quad \text{for some scalar} \quad \sigma = \sigma(\varrho_0, u_0; \varrho, u)\Big\}.$$

With this notion the Rankine–Hugoniot conditions become

$$(\varrho_l, u_l) \in S(\varrho_r, u_r), \qquad x'(t) = \sigma(\varrho_r, u_r; \varrho_l, u_l).$$

From the jump conditions (4.1) the following useful relations follow

$$\varrho_0 \varrho (u_0 - u)^2 = (\varrho_0 - \varrho)^2 [c^2 - \sigma^2 - 2a(\varrho_0 + \varrho)], \tag{4.3}$$

and

$$2a\varrho^2 - \varrho(c^2 - \sigma^2 - 2a\varrho_0) + \varrho_0(u_0 - \sigma)^2 = 0. \tag{4.4}$$

If $\sigma^2 > c^2 - 2a\varrho_0$, then all coefficients in Eq. (4.4) are positive and therefore it cannot have positive solutions. Hence, the necessary condition of positive ϱ is

(4.5) $$\sigma^2 < c^2 - 2a\varrho_0 < c^2.$$

We have

PROPOSITION 4.1. If condition (4.5) is satisfied and a is sufficiently small, then Eq. (4.4) has two positive solutions corresponding to the state (ϱ_0, u_0) and the shock speed σ.

Proof. Indeed, setting $a = 0$ in Eq. (4.4) we obtain

$$\varrho = \varrho_0 \frac{(u_0 - \sigma)^2}{c^2 - \sigma^2} > 0.$$

Therefore, by the Inverse Function Theorem, we obtain the thesis.

If we eliminate σ from Eq. (4.2), and denote

(4.6) $$\varrho = \varrho_0(1+w), \quad w > -1$$

then we obtain on $S(\varrho_0, u_0)$

(4.7) $$u = \frac{2}{w+2} \pm \frac{w}{w+2}\left[\frac{2c^2 - u_0^2 - 8a\varrho_0 + w(c^2 - 8a\varrho_0) - 2w^2 a\varrho_0}{1+w}\right]^{1/2}.$$

Therefore

(4.8) $$\sigma_\pm = \frac{1}{w+2}\{u_0 \pm [2c^2 - u_0^2 - 8a\varrho_0 + w(3c^2 - u_0^2 - 16a\varrho_0) + w^2(c^2 - 10a\varrho_0) - 2w^3 a\varrho_0]^{1/2}\}.$$

Thus we have proved (cf. [23])

LEMMA 4.1. The R-H curve $S(\varrho_0, u_0)$ consists of two smooth curves $S_\pm(\varrho_0, u_0)$.

The following expansions follow from Eq. (4.6) and (4.7)

(4.9) $$(\varrho, u) = (\varrho_0, u_0) + r_\pm(\varrho_0, u_0)w + O(w^2),$$

(4.10) $$\sigma_\pm(\varrho_0, u_0; \varrho, u) = \lambda_\pm(\varrho_0, u_0) + \frac{1}{2}\Gamma_\pm(\varrho_0, u_0)w + O(w^2),$$

and additionally from Eq. (3.5)

(4.11) $$\lambda_\pm(\varrho, u) = \lambda_\pm(\varrho_0, u_0) + \Gamma_\pm(\varrho_0, u_0)\, w + O(w^2)$$

(cf. [9, 10, 23]).

Extending the OLEINIK'S celebrated entropy condition [21] LIU in [9, 10] introduced the following one:

a discontinuity $(\varrho_l, u_l;\ \varrho_r, u_r)$ is an admissible discontinuity if $(\varrho_r, u_r) \in S_+(\varrho_l, u_l)$ (or $S_-(\varrho_l, u_l)$, respectively) and if it satisfies the following entropy condition

(E) $$\sigma(\varrho_l, u_l;\ \varrho_r, u_r) \leqslant \sigma(\varrho_l, u_l;\ \varrho, u)$$

for any $(\varrho, u) \in S_+(\varrho_l, u_l)$ ($S_-(\varrho_l, u_l)$, respectively) between (ϱ_l, u_l) and (ϱ_r, u_r).

The above entropy condition (E) is equivalent to the following:

(E) $$\sigma(\varrho_l, u_l;\ \varrho_r, u_r) \geqslant \sigma(\varrho_r, u_r;\ \varrho, u)$$

for any $(\varrho, u) \in S_+(\varrho_r, u_r)$ between (ϱ_l, u_l) and (ϱ_r, u_r), and similarly in the case of $S_-(\varrho_r, u_r)$.

Liu proves in [9, 10] that if a discontinuity is admissible and weak, i.e. if (ϱ_r, u_r) is close to (ϱ_l, u_l), then the following stability condition holds

$$\lambda_+(\varrho_r, u_r) \leqslant \sigma(\varrho_l, u_l; \varrho_r, u_r) \leqslant \lambda_+(\varrho_l, u_l),$$

and similarly in the case of $S_-(\varrho_l, u_l)$.

From Eqs. (4.9) and (4.10) and the entropy condition (E) we obtain (at least for weak shock waves)

i) for $(\varrho_l, u_l) \in S_+(\varrho_r, u_r)$:

(4.12)
$$\begin{aligned} &\text{if} \quad \Gamma_+(\varrho_r, u_r) > 0 \quad \text{then} \quad \varrho_l > \varrho_r \quad \text{and} \quad u_l > u_r, \\ &\text{if} \quad \Gamma_+(\varrho_r, u_r) < 0 \quad \text{then} \quad \varrho_l < \varrho_r \quad \text{and} \quad u_l < u_r; \end{aligned}$$

ii) for $(\varrho_r, u_r) \in S_+(\varrho_l, u_l)$

(4.13)
$$\begin{aligned} &\text{if} \quad \Gamma_-(\varrho_l, u_l) < 0 \quad \text{then} \quad \varrho_r > \varrho_l \quad \text{and} \quad u_r < u_l, \\ &\text{if} \quad \Gamma_-(\varrho_l, u_l) > 0 \quad \text{then} \quad \varrho_r < \varrho_l \quad \text{and} \quad u_l > u_r. \end{aligned}$$

From these results and the Rankine–Hugoniot jump conditions we have

(4.14)
$$\sigma(\varrho_r, u_r;\ \varrho_l, u_l) - u_r = \varrho_l \frac{u_r - u_l}{\varrho_r - \varrho_l} > 0$$

for $(\varrho_l, u_l) \in S_+(\varrho_r, u_r)$;

(4.15)
$$\sigma(\varrho_l, u_l;\ \varrho_r, u_r) - u_l = \varrho_r \frac{u_r - u_l}{\varrho_r - \varrho_l} < 0$$

for $(\varrho_r, u_r) \in S_-(\varrho_l, u_l)$.

Inequality (4.14) means that the gas leaks through the discontinuity to the left, therefore the shock $S_+(\varrho_r, u_r)$ moves to the right with respect to the gas. Because of that we call $S_+(\varrho_r, u_r)$ the forward shock wave. Next, from Eq. (4.5) we see that the *R-H* curve $S_-(\varrho_l, u_l)$ moves to the left with respect to the gas, and we call it the backward shock wave.

The first results in (4.12) and (4.13) can be interpreted as statements saying that the shock wave is a compression wave. This is a typical result concerning the shock waves both for the classical Broadwell model and for the true normal gases. The second statements in (4.12), (4.13) resemble retrograde gases, since the density is smaller behind the shock wave then that before it. We call such waves expansion waves.

In this way we have arrived at one of our main results, namely we have proved

THEOREM 4.1. *The shock waves in the model Euler equations* (2.10) *and* (2.16) *can be either compression waves or expansion waves.*

Finally, let us notice that the results of the previous section and those of the present one make it possible to apply the existence and uniqueness theorems proved by Liu [9, 10] to the initial value problem (2.10), (2.16) and (3.1).

5. The Navier-Stokes shock profile

We smooth out the Euler shock wave by using the Navier–Stokes equations (2.22)–(2.24) along with Eq. (2.16).

We look for solutions of this system of equations in the form

$$\varrho(x, t) = \varrho(x-st), \quad u(x, t) = u(x-st), \tag{5.1}$$

where s is a constant, subject to the limiting values

$$\begin{aligned}(\varrho, u)(y = -\infty) &= (\varrho_l, u_l),\\ (\varrho, u)(y = \infty) &= (\varrho_r, u_r),\end{aligned} \tag{5.2}$$

where $y = x-st$.

Solutions of the above form are called a Navier–Stokes shock profiles.

Substituting Eqs. (5.1) into the model Navier–Stokes equations we get

$$\varrho(u-s) = m, \tag{5.3}$$

$$\mu\frac{du}{dy}+\varkappa\frac{d}{dy}(\varrho u)-\frac{1}{2}\varrho(c^2+u^2-2a\varrho)+s\varrho u = j, \tag{5.4}$$

where m and j are constants.

We assume additionally that

$$\lim_{y\to\pm\infty}\left(\frac{d\varrho}{dy}, \frac{du}{dy}\right) = (0, 0). \tag{5.5}$$

From Eqs. (5.2)–(5.5) it follows that

$$\begin{aligned}m &= \varrho_l(u_l-s) = \varrho_r(u_r-s),\\ j &= -\frac{1}{2}\varrho_l(c^2+u_l^2-2a\varrho_l)+s\varrho_l u_l = -\frac{1}{2}\varrho_r(c^2+u_r^2-2a\varrho_r)+s\varrho_r u_r.\end{aligned} \tag{5.6}$$

Comparing (5.6) and (4.1) we see that relations (5.6) are nothing else but the Rankine–Hugoniot relations.

We make the substitution

$$\varrho = \frac{\varrho_l+\varrho_r}{2}+G\frac{\varrho_l-\varrho_r}{2} \tag{5.7}$$

and

$$\varrho u = \frac{\varrho_l u_l+\varrho_r u_r}{2}+G\frac{\varrho_l u_l-\varrho_r u_r}{2} \tag{5.8}$$

with

$$G(y = -\infty) = 1, \quad G(y = \infty) = -1. \tag{5.9}$$

With this substitution we satisfy Eq. (5.3) identically, and from Eq. (5.4) along with (5.6) we get

$$\begin{aligned}(\alpha_4 w^4G^4+\alpha_3 w^3G^3+\alpha_2 w^2G^2+\alpha_1 wG+\alpha_0)\frac{dG}{dy}\\ = -\frac{4a(\varrho_l-\varrho_r)^2}{\varepsilon}(2+w+wG)^3(G^2-1)(G+B),\end{aligned} \tag{5.10}$$

where

$$w = \frac{\varrho_l-\varrho_r}{\varrho_r}, \tag{5.11}$$

and

$$\alpha_4 = \frac{1}{2} a\varrho_r,$$

$$\alpha_3 = \frac{1}{2} a\varrho_r[u_r+s(3+2w)],$$

(5.12) $$\alpha_2 = -(u_r-s)\,[c^2-s^2-6a\varrho_r(2+w)],$$

$$\alpha_1 = 4(u_r-s)^2 s-2(u_r-s)\,(c^2+s^2)\,(2+w)+2a\varrho_r(2+w)^3[3u_r-s(7+4w)],$$

$$\alpha_0 = 4(u_r-s)^3-4s(u_r-s)^2(2+w)-(u_r-s)\,(c^2-s^2)\,(2+w)^2 + 2a\varrho_r(2+w)^3[u_r-s(3+w)],$$

and finally

(5.13) $$B = 1-2\,\frac{c^2-s^2-2a(\varrho_l+2\varrho_r)}{2a(\varrho_l-\varrho_r)}.$$

Integrating Eq. (5.10) one gets, if $|B| \neq 1$,

(5.14) $$\frac{A_1}{w}\ln(2+w+wG)-\frac{A_2}{w}\,\frac{1}{2+w+wG}-\frac{A_3}{2w}\,\frac{1}{(2+w+wG)^2} - A_4\ln(1-G)+A_5\ln(1+G)+A_6\ln(G+B) = \frac{4a(\varrho_l-\varrho_r)^2}{\varepsilon}\,(y-y_0),$$

where y_0 is the constant of integration and the constants A_1 to A_6 satisfy the following system of algebraic equations:

(5.15) $$A_1-wA_4+wA_5+wA_6 = 0,$$

(5.16) $$(2+wB)A_1+A_2-w[w(B+1)+3(2+w)]A_4 + w[w(B-1)+3(2+w)]A_5+3w(2+w)A_6 = -w^3\alpha_4,$$

(5.17) $$[(2+w)^2+2wB-w^2]A_1+[2+w(B+1)]A_2+A_3 - w[3(2+w)^2+3w(2+w)\,(B+1)+w^2B]A_4+w[3(2+w)^2+3w\,(2+w)\,(B-1) - w^2B]\,A_5+w[3(2+w)^2-w^2]A_6 = -w^3\alpha_3,$$

(5.18) $$[(2+w)^2B-2w-w^2B]\,A_1+[B(2+w)-w]A_2+BA_3 - [(2+w)^3+3w^2(2+w)^2(B+1)+3w^2B(2+w)A_4 + [(2+w)^3+3w^2(2+w)^2(B-1)-3w^2B(2+w)]A_5 + [(2+w)^3-3w^2(2+w)]A_6 = -w^2\alpha_2,$$

(5.19) $$[(2+w)^2+2wB]A_1+[2+w(B+1)]A_2+A_3 + [(2+w)^3(B+1)+3w(2+w)^2]A_4-[(2+w)^3(B-1)-3wB(2+w)^2]A_5 + 3w(2+w)^2A_6 = w\alpha_1,$$

(5.20) $$(2+w)^2BA_1+(2+w)BA_2+BA_3+(2+w)^3BA_4+(2+w)^3BA_5+(2+w)^3A_6 = \alpha_0.$$

It is evident from Eq. (5.14) that G can satisfy the limiting conditions (5.9) only if $|B| > 1$, since otherwise it would be: $|y| \to \infty$ for $G \to -B$, $-1 < B < 1$, contrary to Eq. (5.9). However, condition $|B| > 1$ does not guarantee that Eq. (5.13) satisfies Eq. (5.9). The final answer to this question depends on the sign of A_4 and A_5, which is difficult to be established. Later, we solve this problem for weak shock waves.

We proceed to the case when $B = 1$. We consider only the case of $B = 1$ since the case of $B = -1$ can be treated in a similar way. Setting $B = 1$ in Eq. (5.10) and integrating we obtain

$$(5.21)\qquad \frac{A_1}{w}\ln[2+w(G+1)]-\frac{A_2}{w}\frac{1}{2+w(G+1)}-\frac{A_3}{2w}\frac{1}{[2+w(G+1)]^2} -A_4\ln(1-G)+A_5\ln(1+G)-\frac{A_6}{1+G} = \frac{4a(\varrho_l-\varrho_r)^2}{\varepsilon}(y-y_0),$$

where y_0 is the constant of integration, and the constants A_1 to A_6 are solutions of the following system of linear algebraic equations

$$(5.22)\qquad A_1-wA_4+wA_5 = 0,$$

$$(5.23)\qquad (2+w)A_1+A_2-w(6+5w)A_4+3w(2+w)A_5+w^2A_6 = -w^3\alpha_4,$$

$$(5.24)\qquad 2(2+3w)A_1+2(1+w)A_2+A_3-3w(4+6w+3w^2)A_4+2w(6+6w+w^2)A_5 +2w^2(3+w)A_6 = -w^3\alpha_3,$$

$$(5.25)\qquad 2(2+3w)A_1+2A_2+A_3-2(2+w)(2+8w+5w^2)A_4+2(2+w)(2+2w-w^2)A_5 +3w(2+w)(1-2w-w^2)A_6 = -w^2\alpha_2,$$

$$(5.26)\qquad (4+6w+w^2)A_1+2(1+w)A_2+A_3+(2+w)^2(4+5w)A_4+3w(2+w)^2A_5 -2(2+w)^2(1-w)A_6 = w\alpha_1,$$

$$(5.27)\qquad (2+w)^2A_1+(2+w)A_2+A_3+(2+w)^3A_4+(2+w)^3A_5+(2+w)^3A_6 = \alpha_0.$$

Let us analyze in more detail the condition $|B| \geqslant 1$. We have from Eq. (5.13)

$B > 1$ if and only if one of the following relations takes place

i) either

$$(5.28)\qquad s^2 > c^2-2a\varrho_r(3+2w) \quad \text{and} \quad \varrho_l > \varrho_r,$$

ii) or

$$(5.29)\qquad s^2 < c^2-2a\varrho_r(3+2w) \quad \text{and} \quad \varrho_l < \varrho_r.$$

The equation $B = 1$ defines the transition line

$$(5.30)\qquad s^2 = c^2-2a\varrho_r(3+2w).$$

Similar results can be obtained if $B \leqslant -1$.

Qualitatively, these results agree with those of Theorem 4.1, i.e., the shock wave can be either a compression wave or an expansion wave, but there are some differences as well.

First, in Eqs. (5.28)–(5.30) the shock speed s appears instead of the characteristic speed $\lambda_\pm$ appearing in $\Gamma_\pm$ and, consequently, in the equation of the transition line. Secondly now the equation of the transition line depends not only on the state before the shock but also on the state after it. Hence, the criterion for a shock wave to be a compression or expansion wave is much more complicated in the case of the model Navier–Stokes equation than in the case of the model Euler equations. We cannot compare our model results with the corresponding ones obtained for the strong shock waves in the full hydrodynamic Navier–Stokes equation, because such results are missing at present.

An agreement between the Navier–Stokes and Euler equations is achieved for weak shock waves.

For weak shock waves, i.e. for small values of w, we obtain from Eqs. (5.15)–(5.20)

(5.31) $$A_i = O(w^2), \qquad i = 1, 2, 3,$$

and

(5.32)
$$A_4 = \frac{\alpha_0}{2(B+1)} + O(w),$$
$$A_5 = \frac{\alpha_0}{2(B-1)} + O(w),$$
$$A_6 = -\frac{\alpha_0}{B^2-1} + O(w),$$

where α_0 is given by

(5.33) $$\alpha_0 = -8\lambda_\pm(\varrho_r, u_r)\left(c^2 - \lambda_\pm^2(\varrho_r, u_r)\right) + O(w).$$

In deriving Eq. (5.33) we took into account Eqs. (5.12) and (4.9), (4.10).

Owing to Eqs. (5.31)–(5.33), the exact solutions (5.14) and (5.21) reduce to

(5.34) $$\frac{1}{B^2-1}\left\{2\ln\left(\frac{G}{B}+1\right) + B\ln\frac{1-G}{1+G} - \ln(1-G^2)\right\} = \frac{a(\varrho_l-\varrho_r)^2}{\varepsilon\lambda_\pm(\varrho_r, u_r)\left[c^2-\lambda_\pm^2(\varrho_r, u_r)\right]}(y-y_0)$$

for $|B| > 1$, and

(5.35) $$\frac{1}{2}\ln\frac{1-G}{1+G} - \frac{G}{1+G} = \frac{8a(\varrho_l-\varrho_r)^2}{\varepsilon\lambda_+(\varrho_r, u_r)\left[c^2-\lambda_+^2(\varrho_r, u_r)\right]}(y-y_0)$$

for $B = 1$. We took λ_+ in Eq. (5.35) to satisfy the limiting condition (5.9). In Eq. (5.34) we use the asymptotic formula for $B > 1$ as $w \to 0$,

(5.36) $$B = 1 - 2\,\frac{c^2 - \lambda_+^2(\varrho_r, u_r) - 6a\varrho_r + O(w)}{2a(\varrho_l - \varrho_r)},$$

where Eq. (4.10) has been taken into account.

The asymptotic solutions (5.34) satisfy the limiting conditions (5.9) if an only if we put there

(5.37)
the $(+)$ subscript (i.e. λ_+) and $B \geqslant 1$ or
the $(-)$ subscript (i.e. λ_-) and $B \leqslant -1$.

The first case corresponds to the forward shock waves, whereas the second case to the backward ones.

In what follows we limit ourselves to the forward shock waves, since the backward shocks can be analyzed in a similar way.

From Eq. (5.36) as well as from Eq. (3.14) we obtain (cf. also Eqs. (5.28) and (5.29)):

$B > 1$ if and only if one of the conditions is satisfied:

i) either

$$\Gamma_+(\varrho_r, u_r) > 0 \quad \text{and} \quad \varrho_l > \varrho_r, \tag{5.38}$$

ii) or

$$\Gamma_+(\varrho_r, u_r) < 0 \quad \text{and} \quad \varrho_l < \varrho_r. \tag{5.39}$$

Relations (5.38), (5.39) are identical with (4.12). Hence, for the weak shock waves we obtain full agreement with the Euler predictions of the character of the shock.

Let us notice also that the asymptotic form of the solutions is exactly the same as that obtained for the weak shock waves by CRAMER and KLUWICK [11] within the framework of the true Navier–Stokes equations (cf. formulae (5.7) and (5.8) of the cited paper). They also give graphs of the shock profile for a few values of $B \geqslant 1$.

6. The Boltzmann shock profile

The Boltzmann shock profile is a density vector $N(y)$, $y \in \mathrm{R}^1$, such that

$$N(x, t) = N(x-st) \tag{6.1}$$

is a solution of the model Boltzmann equation (2.1), (2.21) with the limiting values

$$\begin{aligned} N(y = -\infty) &= \nu_l, \\ N(y = \infty) &= \nu_r, \end{aligned} \tag{6.2}$$

where ν_l, ν_r are the equilibrium density vectors with hydrodynamical moments (ϱ_l, u_l) and (ϱ_r, u_r), respectively. We assume that both (ϱ_l, u_l) and (ϱ_r, u_r) are elements of the set U defined in Eq. (3.8).

The Boltzmann shock profile equations are found by substituting the assumed form of solutions (6.1) into the model Boltzmann equations (2.1)

$$\begin{aligned} (c-s)\frac{dN^1}{dy} &= -\frac{1}{\varepsilon}Q(N, N), \\ -(c+s)\frac{dN^2}{dy} &= -\frac{1}{\varepsilon}Q(N, N), \\ -s\frac{dN^3}{dy} &= \frac{1}{\varepsilon}Q(N, N). \end{aligned} \tag{6.3}$$

Equations (6.3) lead to two conservation laws, which are obtained by multiplying (6.1) by ψ_0 and ψ_1 given by Eqs. (2.3) and (2.4), respectively. The constants of integration are found by using the limiting conditions (6.2) and the Rankine–Hugoniot relations (5.6)

$$\begin{aligned} c(N^1-N^2)-s(N^1+N^2+2N^3) &= 4m, \\ c^2(N^1+N^2)-sc(N^1-N^2) &= 4j. \end{aligned} \tag{6.4}$$

Multiplying Eqs. (6.3) by the vector $\psi_2 = (1, 1, -1)$ we obtain

$$\frac{d}{dy}[(c-s)N^1-(c+s)N^2+sN^3] = -\frac{3}{\varepsilon}Q(N, N). \tag{6.5}$$

We make the substitution

$$N^i = \frac{1}{2}[(\nu_l^i+\nu_r^i)+G(y)(\nu_l^i-\nu_r^i)], \quad i = 1, 2, 3, \tag{6.6}$$

where $G(y)$ is an unknown function such that

$$G(y = -\infty) = 1, \quad G(y = \infty) = -1 \tag{6.7}$$

(cf. Eq. (5.9)).

With this substitution Eqs. (6.4) are satisfied identically, and Eq. (6.5) becomes

$$\frac{dG}{dy} = -\frac{a(\varrho_l-\varrho_r)^2}{\varepsilon s(c^2-s^2)}(G^2-1)(G+B), \tag{6.8}$$

where B is given by Eq. (5.13).

In deducing Eq. (6.8) we have used the Rankine–Hugoniot conditions and the identity (4.3).

Equation (6.8) is the same as that obtained by CRAMER and KLUWICK in [11] for weak shock waves in the genuine Navier–Stokes equation. However, now it is an exact consequence of the model Boltzmann equation and describes not only the weak but also the strong shock waves.

The solution of Eq. (6.8) for the forward ($s > 0$) shock waves is:
for $B > 1$

$$\frac{1}{B^2-1}\left\{2\ln\left(\frac{G}{B}+1\right)+B\ln\frac{1-G}{1+G}-\ln(1-G^2)\right\} = -\frac{a(\varrho_l-\varrho_r)^2}{\varepsilon s(c^2-s^2)}(y-y_0), \tag{6.9}$$

and for $B = 1$

$$\frac{1}{2}\ln\frac{1-G}{1+G}-\frac{G}{1+G} = -\frac{a(\varrho_l-\varrho_r)^2}{\varepsilon s(c^2-s^2)}(y-y_0), \tag{6.10}$$

where y_0 is the constant of integration.

These solutions have the same form as those found by CRAMER and KLUWICK [11 for weak shock waves in the true Navier–Stokes equations. They agree with the model Navier–Stokes equations considered in the previous section for weak shock waves only, what follows from Eqs. (6.9) (6.10) and the expansion (4.10).

Since the parameter B is the same for the model Navier–Stokes and Boltzmann equations, the transition line is also the same for the two descriptions and does not agree in general with the Euler description. Full agreement between the three types of equations is attained for weak shock waves only.

The last problem we consider is the shock wave thickness. As its definition we take

$$L = \frac{|\varrho_l-\varrho_r|}{\sup\limits_y\left|\dfrac{d\varrho}{dy}\right|}. \tag{6.11}$$

From Eq. (6.6) and definition (2.5) of the density we have

$$\frac{d\varrho}{dy} = \frac{1}{2}(\varrho_l-\varrho_r)\frac{dG}{dy}.$$

Inserting that into Eq. (6.11) we obtain

$$L = \frac{2}{\sup\limits_y \left|\frac{dG}{dy}\right|}. \tag{6.12}$$

The point of inflection of $G(y)$ is the point where the derivative dG/dy attains its maximal value. We have

$$G_{\inf} = \frac{1}{B+\sqrt{B^2+3}}. \tag{6.13}$$

Thus the point of inflection is shifted downstream of the point where ϱ attains its average value and, hence, the shock profile is no longer symmetric as it was in the case of the classical Broadwell model and true regular gases (cf. [11]). From (6.13) and Eq. (6.8) we obtain

$$\sup_y \left|\frac{dG}{dy}\right| = \frac{2a(\varrho_l-\varrho_r)^2}{9\varepsilon s(c^2-s^2)} \frac{(2B+\sqrt{B^2+3})^2}{B+\sqrt{B^2+3}}.$$

Using this in Eq. (6.12) we get

$$L = \frac{9\varepsilon s(c^2-s^2)}{a(\varrho_l-\varrho_r)^2} \frac{B+\sqrt{B^2+3}}{(2B+\sqrt{B^2+3})^2}. \tag{6.14}$$

Two cases must be considered.

i) $B = O(1)$, i.e. the downstream and upstream states are close to the transition line. Then, as it is seen from Eq. (6.14), for the case of weak shock waves

$$L \approx \frac{1}{(\varrho_l-\varrho_r)^2}. \tag{6.15}$$

Hence, the weak shock wave thickness is much larger now than in the case of the classical Broadwell shock wave, where (cf. [14])

$$L \approx \frac{1}{|\varrho_l-\varrho_r|}.$$

Cramer and Kluwick in their considerations had to assume the estimate (6.15) in order to have their asymptotic theory self-consistent (see [11]). Thus our rigorous result (6.15) confirms Cramer–Kluwick's conjecture.

ii) $B \to \infty$, what corresponds to the case $a \to 0$. For $a \to 0$ we get from Eqs. (6.14) and (5.13)

$$L \approx \frac{2\varepsilon s(c^2-s^2)}{a(\varrho_l-\varrho_r)^2 B} \approx \frac{2\varepsilon s}{|\varrho_l-\varrho_r|}\left(1+\frac{3a(\varrho_l+\varrho_r)}{c^2-s^2}\right). \tag{6.16}$$

It follows from (6.16) that, in this case, the shock wave thickness is much smaller than in the previous case. Also from the graphs of G given by CRAMER and KLUWICK [11] it is seen that the growth of B steepens the shock profile and makes it symmetric. From Eq. (6.9) we obtain as $B \to \infty$

$$\ln\frac{1-G}{1+G} = -\frac{Ba(\varrho_l-\varrho_r)^2}{\varepsilon s(c^2-s^2)}(y-y_0). \tag{6.17}$$

Thus, in this limit, the shock profile becomes the classical hyperbolic tangent. If, additionally, the shock is such that

$$\frac{\varrho_l+\varrho_r}{|\varrho_l-\varrho_r|} \approx 1,$$

then (6.16) gives

$$L \approx \frac{2\varepsilon s}{|\varrho_l-\varrho_r|}+\frac{6as\varepsilon}{c^2-s^2}. \tag{6.18}$$

This expression reminds very much the following formula for the shock thickness

$$L = \alpha f(M_s)+\beta, \tag{6.19}$$

obtained recently experimentally in [22]. Here α is a constant, $f(M_s)$ represents the dependence of shock thickness on the Mach number, and β seems to be a constant characterizing the gas.

The qualitative agreement between (6.18) and (6.19) is not full, however, since (6.18) was obtained for sufficiently strong shock waves, and (6.19) is to be true for shocks of any strength. This disagreement can be ascribed to the extreme simplicity of the proposed model.

7. Final remarks

A simple discrete velocity model resembling retrograde gases has been proposed. The resemblance has been fully shown. In particular we have shown that at all three levels of description: i.e. the Boltzmann, the Navier–Stokes and Euler description of the gas, the shock waves can be either a compression wave or an expansion wave. For each of the levels of description the transition line has been found. In the Euler equations the transition line was the curve along which $\Gamma_\pm(\varrho, u)$ vanished. This agrres completely with previous results obtained by other authors from the full hydrodynamic Euler equations. However, if the dissipative effects are included, a shift of the transition line arises, particularly for strong waves. On the other hand, in the case of weak shock waves an agreement between the three levels of description was obtained.

The second problem we would like to discuss is the presence of space gradient of the density in the model Navier–Stokes equations (2.19).

This form can be a distorted representation of the second viscosity, and such its form can be a result of the simplicity of the model. (Another travesty is the dependence of the press on the velocity in the Broadwell model).

The third and the most difficult problem is that of the kinetic equations even for the case of one-phase retrograde gases. The proposed discrete model has the characteristic feature the probabilities of direct and inverse collisions are different.

At present there exist general true kinetic equations in which both the ternary collisions and many physical and chemical phenomena occurring in real gases are taken into account (see for example [19]). However, it is very difficult to make practical use of them since we do not know the exact molecular structure and composition (for instance, the number of the internal degrees of freedom of the moleculas) of a retrograde gas, of the law of molecular interactions. If, additionally, we resign of the symmetry between the direct and inverse collisions, then it becomes extremely difficult (if possible at all) to determine all states of equilibrium. Owing to that, it seems that discrete velocity models, more sophisticated than the one proposed in this paper can be very helpful in the qualitative understanding of many phenomena occurring not only in the retrograde gases but also in the general real gases.

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