Arch. Mech., 42, 3, pp. 267–277, Warszawa 1990

Structural tensors for anisotropic solids

J. M. ZHANG (SHANGHAI) and J. RYCHLEWSKI (WARSZAWA)

A SIMPLER PROOF is given for Rychlewski's theorem that clarifies the idea of extending a g-invariant function into a function which is invariant under a larger group. For an anisotropic solid, the theorem ensures the possibility of transforming the problem of representation of an anisotropic constitutive function into that of an isotropic function through some tensors which characterize the symmetry group. The structural tensors for all 32 crystal groups are presented. The structural tensors for all the orthogonal subgroups of non-crystal symmetries are also investigated.

Podano prostszy dowód twierdzenia Rychlewskiego, wyjaśniający ideę polegającą na rozszerzeniu funkcji g-niezmienniczej na funkcję niezmienniczą względem obszerniejszej grupy przekształceń. W przypadku ciał anizotropowych twierdzenie to zapewnia możliwość przetransformowania problemu reprezentacji anizotropowej funkcji konstytutywnej na problem funkcji izotropowej za pomocą pewnych tensorów charakteryzujących grupę symetrii. Podano takie tensory strukturalne dla wszystkich trzydziestu dwóch grup symetrii kryształów. Zbadano również problem tensorów strukturalnych w przypadku wszystkich ortogonalnych podgrup symetrii niekrystalicznych.

Приведено более простое доказательство теоремы Рыхлевского, выясняющей идею, заключающуюся в расширении g-инвариантной функции на инвариантную функцию по отношению к более обширной группы преобразований. В случае анизотропных тел эта теорема обеспечивает возможность преобразования задачи представления анизотропной определяющей функции в задачу изотропной функции при помощи некоторых тензоров, характеризующих группу симметрии. Приведены такие структурные тензоры для всех тридцати двух групп симметрии кристаллов. Исследована тоже задача структурных тензоров в случае всех ортогональных подгрупп некристаллических симметрий.

1. Introduction

REPRESENTATIONS of the constitutive functions of isotropic materials have been extensively investigated. Thus, engineering materials often present mechanical anisotropy. The material symmetries characterizing the anisotropy impose definite restrictions on the form of the constitutive function of the material. Therefore, representations of anisotropic functions, which specify all possible forms of the functions meeting the restrictions, are indispensable in obtaining constitutive equations for anisotropic materials.

An approach presented by BOEHLER [1] was to work with the representation problem of anisotropic functions using the well-known results for isotropic functions. LIU [2] precised the idea and represented it in a theorem form which establishes a relation between isotropic functions and anisotropic ones through some vectors and second order tensors which characterize the symmetry groups. Only transverse isotropy, orthotropy and some special crystal symmetries are considered because many subgroups of the orthogonal transformation group of the three-dimensional Euclidean space cannot be characterized only by vectors and second order tensor.

http://rcin.org.pl

If the tensors which characterize the symmetry groups, called structural tensors here, are found, the representations of the constitutive functions with such symmetry groups are transformed into representations of some isotropic constitutive functions. Therefore, it must be very meaningful to find the structural tensors for all the subgroups of the orthogonal group $O(\mathbf{V})$. LOHIN and SEDOV [3] have obtained suitable results.

In the present study a simpler proof is given first for RYCHLEWSKI [4] theorem on invariant extensions of anisotropic functions. The theorem clarifies the idea of extending a g-invariant function into a function which is invariant under a larger group. For an anisotropic solid our theorem ensures the possibility of transforming the problem of representation of an anisotropic constitutive function into that of an isotropic function through some tensors which characterize the symmetry group. A detailed analysis of the structural tensor for the group D_{nh} is given, and on this basis an exhaustive presentations of the structural tensors for all orthogonal subgroups, both of crystal symmetry types and of non-crystal symmetry types, are given.

2. General considerations for invariant extension

Let A be a set and g a group. The group operation will be denoted by $(\alpha, \beta) \to \alpha\beta$. We say that the group acts on the set A if for every $\mathbf{A} \in \mathsf{A}$ and every $\alpha \in g$ there exists a unique element $\mathbf{A} \in \mathsf{A}$ such that

1) for $\forall\alpha, \beta \in g$ and $\forall\mathbf{A} \in \mathsf{A}$, $(\alpha\beta) \circ \mathbf{A} = \alpha \circ (\beta \circ \mathbf{A})$,

2) for the identity $\mathbf{i}$ of the group g and any an $\mathbf{A} \in \mathsf{A}$, we have $\mathbf{i} \circ \mathbf{A} = \mathbf{A}$,

where "$\circ$" represents the action approach of the group g on the set A.

An element $\mathbf{A}$ in A is said to be invariant under the group element α if

$$\alpha \circ \mathbf{A} = \mathbf{A}. \tag{2.1}$$

All the group elements in g which satisfy Eq. (2.1) form a subgroup of g and is called the symmetry group of element $\mathbf{A}$.

Let $\mathsf{F}: \mathsf{A} \to \mathsf{B}$ be a function and the group g acts on set A and set B simultaneously. We distinguish the group action on the sets A and B through denoting them, respectively, by $(g, \circ)$ and $(g, \times)$. A function $\mathsf{F}: \mathsf{A} \to \mathsf{B}$ is said to be invariant under group action $(g, \circ, \times)$ if for every group element $\alpha \in g$ and every $\mathbf{A} \in \mathsf{A}$

$$\mathsf{F}(\mathbf{A}) = \alpha^{-1} \times \mathsf{F}(\alpha \circ \mathsf{A}), \tag{2.2}$$

where α^{-1} is the inverse of α.

EXAMPLE. $\mathbf{L}(\mathbf{V}, n_i)$ is the space of n_i-th order tensors on the three-dimensional Euclidean space $\mathbf{V}$. Let F be a function

$$\mathsf{F}: \mathbf{L}(\mathbf{V}, n_1) \times, \ldots, \times \mathbf{L}(\mathbf{V}, n_P) \to \mathbf{L}(\mathbf{V}, m). \tag{2.3}$$

The action of the orthogonal transformation group $O(\mathbf{V})$ on the product space is defined as follows:

$$\mathbf{Q} \circ \mathbf{A} = (\mathbf{Q} * \mathbf{A}_1, \mathbf{Q} * \mathbf{A}_2, \ldots, \mathbf{Q} * \mathbf{A}_p), \tag{2.4}$$

http://rcin.org.pl

where $\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_p)$, $\mathbf{A}_i \in \mathbf{L}(\mathbf{V}, n_i)$ and $\mathbf{Q}*$ is a linear operator defined by the equation

$$\mathbf{Q}*(\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \ldots \otimes \mathbf{v}_P) = \mathbf{Q}\mathbf{v}_1 \otimes \mathbf{Q}\mathbf{v}_2 \otimes \ldots \otimes \mathbf{Q}\mathbf{v}_p \tag{2.5}$$

where $\mathbf{v}_i$ are vectors in space $\mathbf{V}$.

It is easy to check that $(O(\mathbf{V}), \circ)$ defined by Eq. (2.4) is a group action indeed. Thus the symmetry group of the function F consists of all the orthogonal transformations satisfying the following equation:

$$\mathsf{F}(\mathbf{A}_1, \ldots, \mathbf{A}_p) = \mathbf{Q}^T * \mathsf{F}(\mathbf{Q}*\mathbf{A}_1, \ldots, \mathbf{Q}*\mathbf{A}_p). \tag{2.6}$$

In continuum mechanics we call all the orthogonal transformations satisfying the above equation the symmetry group of the material the constitutive function F describes.

Let $\mathscr{G}$ be a group acting on set A, B and C simultaneously and let g be its subgroup. Assume that $\mathbf{S}$ is such an element in set C that g is its symmetry group, i.e.,

$$g = \{\alpha \in \mathscr{G} | \alpha \odot \mathbf{S} = \mathbf{S}\}, \tag{2.7}$$

where "$\odot$" denotes the action of $\mathscr{G}$ on set C. Let $\mathsf{M} = \{\alpha \odot \mathbf{S} | \alpha \in \mathscr{G}\}$. Now we can present the main theorem as follows:

THEOREM 2.1 (RYCHLEWSKI [4]). *A function* $\mathsf{F}\colon \mathsf{A} \to \mathsf{B}$ *is invariant under group action* $(g, 0, x)$ *iff there exists a function* $\hat{\mathsf{F}}\colon \mathsf{A} \times \mathsf{M} \to \mathsf{B}$ *such that*

$$\mathsf{F}(\mathbf{A}) = \hat{\mathsf{F}}(\mathbf{A}, \mathbf{S}), \quad \forall \mathbf{A} \in \mathsf{A} \tag{2.8}$$

and for every $\alpha \in \mathscr{G}$ and every $\mathbf{A} \in \mathsf{A}$

$$\alpha^{-1} \times \hat{\mathsf{F}}(\alpha \circ \mathbf{A}, \alpha \odot \mathbf{S}) = \hat{\mathsf{F}}(\mathbf{A}, \mathbf{S}) \tag{2.9}$$

i.e., $\hat{\mathsf{F}}$ *is invariant under group* $\mathscr{G}$.

P r o o f. Necessity. Define

$$\hat{\mathsf{F}}(\mathbf{A}, \mathbf{T}) = \alpha^{-1} \times \mathsf{F}(\alpha \circ \mathbf{A}), \tag{2.10}$$

where $\alpha \in \mathscr{G}$ is such that $\alpha \odot \mathbf{T} = \mathbf{S}$. The function $\hat{\mathsf{F}}$ is well-defined. In fact, if there exists another element α' such that $\alpha' \circ \mathbf{T} = \mathrm{S}$, then

$$(\alpha'\alpha^{-1}) \odot \mathbf{S} = \mathbf{S}, \quad \text{i.e.,} \quad \alpha'\alpha^{-1} \in g \tag{2.11}$$

and

$$\alpha^{-1} \times \mathsf{F}(\alpha \circ \mathbf{A}) = \alpha^{-1} \times (\alpha'\alpha^{-1}) \times \mathsf{F}\big((\alpha'\alpha^{-1}) \circ (\alpha \circ \mathbf{A})\big) = \alpha'^{-1} \times \mathsf{F}(\alpha' \circ \mathbf{A}). \tag{2.12}$$

We can show that $\hat{\mathsf{F}}$ defined by Eq. (2.10) is invariant under $\mathscr{G}$. For every $\alpha \in \mathscr{G}$, let $(\alpha'\alpha) \odot \mathbf{T} = \mathrm{S}$, then

$$\begin{aligned} \alpha^{-1} \times \hat{\mathsf{F}}(\alpha \circ \mathbf{A}, \alpha \odot \mathbf{T}) &= \alpha^{-1} \times \mathsf{F}\big(\alpha'^{-1} \times (\alpha' \circ (\alpha \circ A))\big) \\ &= (\alpha'\alpha)^{-1} \mathsf{F}\big((\alpha'\alpha) \circ \mathbf{A}\big) = \hat{\mathsf{F}}(\mathbf{A}, \mathbf{T}). \end{aligned} \tag{2.13}$$

Sufficiency. For every $\alpha \in g$, $\mathbf{A} \in \mathsf{A}$

$$\alpha^{-1} \times \mathsf{F}(\alpha \circ \mathbf{A}) = \alpha^{-1} \times \hat{\mathsf{F}}(\alpha \circ \mathbf{A}, \mathbf{S}) = \alpha^{-1} \times \hat{\mathsf{F}}(\alpha \circ \mathbf{A}, \alpha \odot \mathbf{S}) = \hat{\mathbf{F}}(\mathbf{A}, \mathbf{S}) = \hat{\mathsf{F}}(\mathbf{A}). \tag{2.14}$$

Hence F is invariant under group g. QED.

http://rcin.org.pl

For solid materials, the constitutive equation takes the form (2.3). When an undistorted reference frame is taken, the function F is invariant under some orthogonal subgroup. Therefore, if there exists a series of tensors such that the series is symmetric under the orthogonal subgroup, then the mechanical behaviour described by the anisotropic function F can be described by some isotropic function:

$$\mathsf{F}^{\mathrm{I}}: \mathbf{L}(\mathbf{V}, n_1)\times \dots \times\mathbf{L}(\mathbf{V}, n_p)\times \mathsf{M} \to \mathbf{L}(\mathbf{V}, m_i). \tag{2.15}$$

Thus it is very meaningful to find the tensors $\mathbf{S}$ that characterize the symmetry groups of solid materials.

3. Basic properties of structural tensors

Now we restrict ourselves to an investigation of the structural tensors for various anisotropic solids whose symmetry groups, by definition, are subgroups of the orthogonal group $O(\mathbf{V})$ provided suitable reference frameworks are taken. Therefore all the groups we will discuss here are supposed to be orthogonal subgroups. There are so many orthogonal subgroups to be considered that it will be of great benefit to investigate the basic properties of the structural tensors for different subgroups first. The general result we obtain in this section shows that we need only to find the structural tensors for several typical orthogonal subgroups in order to gain a clear idea of the structural tensors for all orthogonal subgroups.

Let $\mathbf{S}$ be a vector, a tensor of arbitrary order or a tensor series. $\mathbf{S}$ is called a structural tensor series for the orthogonal subgroup g if its symmetry group is g, i.e.,

$$g = \mathbf{\Gamma}(\mathbf{S}) \equiv \{\mathbf{Q} \in O(\mathbf{V}) | \mathbf{Q}{*}\mathbf{S} = \mathbf{S}\}, \tag{3.1}$$

where, when $\mathbf{S} = (\mathbf{s}_1, \dots, \mathbf{s}_p)$ is a tensor series, it is meant by $\mathbf{Q}{*}\mathrm{S} = \mathrm{S}$ that $\mathbf{Q}{*}\mathbf{s}_i = \mathbf{s}_i$, $i = 1, \dots, p$.

THEOREM 3.1. *Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two structural tensor series for group g_1 and g_2, respectively. Then $\{\mathbf{S}_1, \mathbf{S}_2\}$ is a structural tensor series for the subgroup $g_1 \cap g_2$.*

The proof of this theorem is quite obvious because for any element $Q \in O(\mathbf{V})$, $\mathbf{Q}{*}\{\mathbf{S}_1, \mathbf{S}_2\} = \{\mathbf{S}_1, \mathbf{S}_2\}$ if both $\mathbf{Q}{*}\mathbf{S}_1 = \mathbf{S}_1$ and $\mathbf{Q}{*}\mathbf{S}_2 = \mathbf{S}_2$.

Let $\mathbf{S} = \{\mathbf{s}_1, \dots, \mathbf{s}_n\}$ be a structural tensor series for some group g, where $\mathbf{s}_1, \dots, \mathbf{s}_n$ are n tensors of arbitrary orders. We say that $\mathbf{S}$ is a non-superfluous structural tensor series for the group if no one among the series $\tilde{\mathbf{S}}_1, \dots, \tilde{\mathbf{S}}_n$ possesses the symmetry group g, where the series $\tilde{\mathbf{S}}_i$ is defined by

$$\tilde{\mathbf{S}}_i = \{\mathbf{s}_1, \dots \mathbf{s}_{i-1}, \mathbf{s}_{i+1}, \dots, \mathbf{s}_n\}. \tag{3.2}$$

It can be easily seen that any orthogonal group is uniquely determined by one of its structural tensor series. It is enough for representing the constitutive function of a solid with such symmetries to find only one structural tensor series for the symmetry group of the solid. Of course, non-superfluous ones are best for such a purpose because in such cases the extended isotropic functions have fewer arguments.

http://rcin.org.pl

4. Structural tensors for crystals

The orthogonal group $O(\mathbf{V})$ has 32 subgroups which correspond to 32 different crystal classes. It can be shown that all space lattices will be characterized by one of the seven typical groups, namely, C_i, C_{2h}, D_{2h}, D_{4h}, D_{3d}, D_{6h} and O_h (For the meanings of these Schonflies notations for point groups the readers are referred to [5] or any advanced text-books on crystallography). These seven groups are called the holohedral groups and it is easily seen that the 32 point groups are subgroups of these 7 holohedral groups. According to Theorem 3.1, the main problem to be solved is to find the non-superfluous structural tensors for several typical subgroups.

Let us discuss the structural tensors for the subgroup D_{nh} first.

Assume that the rotation axis of the group D_{nh} is $\mathbf{e}_3$. The following n vectors are on the plane $\mathscr{P}$ perpendicular to the axis $\mathbf{e}_3$.

$$\mathbf{a}_k = \mathbf{e}_1\cos(k\gamma)+\mathbf{e}_2\sin(k\gamma), \quad k = 1, \ldots, n, \quad \gamma = \frac{2\pi}{n}. \tag{4.1}$$

Obviously, for any integer m, the tensor

$$\mathbf{T}(n, m) = \sum_{k=1}^{n} (\mathbf{a}_k)^m \equiv \sum_{k=1}^{n} (\mathbf{a}_k\otimes \ldots \otimes\mathbf{a}_k) \tag{4.2}$$

is invariant under the group D_{nh}.

Taking into consideration the formulae

$$\begin{aligned} \cos\varkappa &= \frac{1}{2}(e^{i\varkappa}+e^{-i\varkappa}),\\ \sin\varkappa &= -\frac{i}{2}(e^{i\varkappa}-e^{-i\varkappa}) \end{aligned} \tag{4.3}$$

(where i is the pure imaginary number unit), we can rewrite Eq. (4.1) into the form

$$\mathbf{a}_k = \frac{1}{2}(\mathbf{e}_1-i\mathbf{e}_2)e^{k\gamma i}+\frac{1}{2}(\mathbf{e}_1+i\mathbf{e}_2)\mathbf{e}^{-k\gamma i}. \tag{4.4}$$

Substituting Eq. (4.4) into Eq. (4.2), we get another form of the tensor $T(n, m)$ as follows:

$$\begin{aligned} \mathbf{T}(n, m) &= \sum_{k=1}^{n} (\mathbf{k} \; e^{k\gamma i}+\mathbf{k}_2 e^{-k\gamma i})^m \\ &= \sum_{k=1}^{n} n(\mathbf{k}_1^m e^{km\gamma i}+[\mathbf{k}_1^p\otimes\mathbf{k}_2^{m-p}]e^{-k(m-2p)\gamma i}+\mathbf{k}_2^m e^{-km\gamma i}), \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} \mathbf{k}_1 &= \frac{1}{2}(\mathbf{e}_1-i\mathbf{e}_2),\\ \mathbf{k}_2 &= \frac{1}{2}(\mathbf{e}_1+i\mathbf{e}_2) \end{aligned} \tag{4.6}$$

http://rcin.org.pl

and $[\mathbf{k}_1^p \otimes \mathbf{k}_2^{m-p}]$ denotes the sum of all the possible permutations of the tensor $\mathbf{k}_1 \otimes \ldots \ldots \otimes \mathbf{k}_1 \otimes \mathbf{k}_2 \otimes \ldots \otimes \mathbf{k}_2$.

After taking into consideration the formula

$$\sum_{k=1}^{n} e^{-k(m-2p)\gamma i} = \frac{e^{-(m-2p)n\gamma i}-1}{e^{-(m-2p)\gamma i}-1} = \begin{cases} 0, & \text{if} \quad m-2p \neq Nn, \\ n & \text{if} \quad m-2p = Nn, \end{cases}$$

where N is an arbitrary integer, we will get the following important conclusions:

$$\mathbf{T}(n,m) = 0 \qquad n > m, \quad m \text{ odd}, \tag{4.8}$$

$$\mathbf{T}(n,m) = n\left[\mathbf{k}_1^{\frac{m}{2}} \otimes \mathbf{k}_2^{\frac{m}{2}}\right] \qquad n > m, \quad m \text{ even}, \tag{4.9}$$

$$\mathbf{T}(n,m) = n(\mathbf{k}_1^n + \mathbf{k}_2^n) \qquad n = m, \quad n \text{ odd}, \tag{4.10}$$

$$\mathbf{T}(n,m) = n\left(\mathbf{k}_1^n + \mathbf{k}_2^n + \frac{1}{2n}\mathbf{T}(2n,n)\right) \qquad n = m, \quad n \text{ even}. \tag{4.11}$$

Equations (4.10) and (4.11) simply imply that the real part of $\mathbf{k}_1^n$ is invariant under D_{nh}, because $\mathbf{k}_2^n$ is conjugate to $\mathbf{k}_2^n$. On the other hand, because the tensor $\mathbf{T}_{n,m}$, $n > m$, keeps unchanged under any rotation about axis $\mathbf{e}_3$, $\mathbf{T}_{n,m}$ is transversely isotropic. That is to say, all tensors $\mathbf{T}_{n,m}$, $m < n$, can not be structural tensor for the group D_{nh}.

The following expressions are very useful for later investigation:

$$\begin{gathered} \mathrm{Im}(\mathbf{k}_1^n) = \frac{i}{2}(\mathbf{k}_2^n - \mathbf{k}_1^n) = \frac{1}{2}(\mathbf{N}_3\mathbf{k}_2^n + \mathbf{N}_3\mathbf{k}_1^n) = \mathbf{N}_3\mathbf{T}_{n,n}, \\ \mathbf{k}_1^n = \mathbf{T}_{n,n} + i\mathbf{N}_3\mathbf{T}_{n,n}, \qquad \mathbf{k}_2^n = \mathbf{T}_{n,n} - i\mathbf{N}_3\mathbf{T}_{n,n}. \end{gathered} \tag{4.12}$$

THEOREM 4.1. *The symmetry group of the tensor* $\mathbf{T}(n,n)$ *is exactly equal to the group* D_{nh}.

P r o o f. Let us rotate the plane $\mathscr{P}$ about the axis $\mathbf{e}_3$ by an angle α. Then the vector $\mathbf{a}_k$ will be rotated to the position

$$\mathbf{a}'_k = \mathbf{k}_1 e^{(k\gamma+\alpha)i} + \mathbf{k}_2 e^{-(k\gamma+\alpha)i}. \tag{4.13}$$

Therefore the tensor $\mathbf{T}(n,n)$ is changed into

$$\begin{aligned} \mathbf{T}'(n,n) &= \sum_{k=1}^{n} (\mathbf{a}'_k)^n \\ &= \begin{cases} n(\mathbf{k}_1^n e^{n\alpha i} + \mathbf{k}_2^n e^{-n\alpha i}) & \text{if } n \text{ is odd}, \\ n(\mathbf{k}_1^n e^{n\alpha i} + \mathbf{k}_2^n e^{-n\alpha i} + \frac{1}{2n}\mathbf{T}(2n,n) & \text{if } n \text{ is even}. \end{cases} \end{aligned} \tag{4.14}$$

For both cases of odd n and of even n, the invariance requirement $\mathbf{T}'(n,n) = \mathbf{T}(n,n)$ will lead to the equation for angle α as follows:

$$e^{\pm n\alpha i} = 1 \tag{4.15}$$

which has solutions

$$\alpha = 0, \ \pm\frac{2\pi}{n}, \ \ldots, \ \pm\frac{(2n-1)\pi}{n}, \ldots. \tag{4.16}$$

http://rcin.org.pl

On the other hand, the other symmetry rotation axes of $\mathbf{T}(n, n)$ which are different from $\mathbf{e}_3$, if any, can be only two-fold and the axes must be on the plane $\mathscr{P}$. Taking into account the fact that $\mathbf{T}(n, n)$ is also invariant under reflection of plane $\mathscr{P}$, we have the conclusion that $\mathbf{T}(n, n)$ is a structural tensor series for group D_{nh}. QED.

We can further prove that the tensor $\mathbf{T}(n, m)$ is invariant under any rotation defined by Eq. (4.13) if $m < n$, that is, $\mathbf{T}(n, m)$ is invariant under group $D_{\infty h}$.

It is derived from Eq. (4.14) that if n is odd, the tensor $\mathbf{T}(n, n)$ changes its sign when

$$n\alpha = \pm 1\times\pi, \pm 3\times\pi, \ldots, \pm(2k-1)\times\pi, \ldots. \tag{4.17}$$

Therefore $\mathbf{T}(n, n)\otimes\mathbf{e}_3$ is invariant under D_{nd}. If n is even, the difference between the tensor $\mathbf{T}'(n, n)$ and $1/2\mathbf{T}'(2n, n)$ also changes its sign when α satisfies Eq. (4.17). Thus $\mathbf{S}(n, n)\otimes\mathbf{e}_3 \equiv (\mathbf{T}(n, n)-1/2\mathbf{T}(2n, n))\otimes\mathbf{e}_3$ is invariant under D_{nd}.

Table 1. Basic tensors and their invariant transformations

No	Basic tensors	Invariant transformation
1	$\mathbf{e}_3$	C_∞
2	$\mathbf{e}_3\otimes\mathbf{e}_3$	D_∞, I, σ_h
3	$\mathbf{e}_1\otimes\mathbf{e}_2\otimes\mathbf{e}_3$	S_4
4	$\mathbf{E}$	$SO(\mathbf{V})$
5	$\mathbf{N}_3 = \mathbf{e}_1\otimes\mathbf{e}_2-\mathbf{e}_2\otimes\mathbf{e}_1$	C_∞, I, σ_h
6	$\mathbf{T}(n, n)$	D_n, D_{nh}, S_n, σ_h
7	$\mathbf{O}_h$	O_h, S_4
8	$\mathbf{T}_d$	T_d, S_4
9	$\mathbf{T}_h$	T_h, S_4

Table 1 provides the basic tensors and the transformations under the action of which the basic tensors are invariant. This table may be used to construct the structural tensors for all point groups. In Table 1, σ_h is the reflection in the plane $\mathscr{P}$ and I the inversion, $SO(\mathbf{V})$ is the proper orthogonal group and

$$\mathbf{E} = \mathbf{e}_1\otimes\mathbf{e}_2\otimes\mathbf{e}_3-\mathbf{e}_2\otimes\mathbf{e}_1\otimes\mathbf{e}_3+\mathbf{e}_2\otimes\mathbf{e}_3\otimes\mathbf{e}_1-\mathbf{e}_3\otimes\mathbf{e}_2\otimes\mathbf{e}_1 +\mathbf{e}_3\otimes\mathbf{e}_1\otimes\mathbf{e}_2-\mathbf{e}_1\otimes\mathbf{e}_3\otimes\mathbf{e}_2, \tag{4.18}$$

$$\mathbf{O}_h = \mathbf{e}_1\otimes\mathbf{e}_1\otimes\mathbf{e}_1\otimes\mathbf{e}_1+\mathbf{e}_2\otimes\mathbf{e}_2\otimes\mathbf{e}_2\otimes\mathbf{e}_2+\mathbf{e}_3\otimes\mathbf{e}_3\otimes\mathbf{e}_3\otimes\mathbf{e}_3, \tag{4.19}$$

$$\mathbf{T}_h = \mathbf{e}_1\otimes\mathbf{e}_1\otimes\mathbf{e}_2\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_2\otimes\mathbf{e}_3\otimes\mathbf{e}_3+\mathbf{e}_3\otimes\mathbf{e}_3\otimes\mathbf{e}_1\otimes\mathbf{e}_1, \tag{4.20}$$

$$\mathbf{T}_d = \mathbf{e}_1\otimes\mathbf{e}_2\otimes\mathbf{e}_3+\mathbf{e}_2\otimes\mathbf{e}_1\otimes\mathbf{e}_3+\mathbf{e}_2\otimes\mathbf{e}_3\times\otimes\mathbf{e}_1+\mathbf{e}_3\otimes\mathbf{e}_2\otimes\mathbf{e}_1 +\mathbf{e}_3\otimes\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_1\otimes\mathbf{e}_3\otimes\mathbf{e}_2. \tag{4.21}$$

http://rcin.org.pl

Table 2. Structural tensors for 32 point groups

Crystal system	No	Schonflies notations	Structural sets
Triclinic	1	C_1	$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$
	2	C_i	$\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$
Monoclinic	3	C_s	$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3 \otimes \mathbf{e}_3$
	4	C_2	$\mathbf{T}(2, 2)$, $\mathbf{e}_3$, $\mathbf{E}$
	5	C_{2h}	$\mathbf{T}(2, 2)$, $\mathbf{N}_3$, $\mathbf{e}_3 \otimes \mathbf{e}_3$
Orthorhombic	6	C_{2v}	$\mathbf{T}(2, 2)$, $\mathbf{e}_3$
	7	D_2	$\mathbf{T}(2, 2)$, $\mathbf{E}$, $\mathbf{e}_3 \otimes \mathbf{e}_3$
	8	D_{2h}	$\mathbf{T}(2, 2)$, $\mathbf{e}_3 \otimes \mathbf{e}_3$
Tetragonal	9	C_4	$\mathbf{T}(4, 4)$, $\mathbf{e}_3$, $\mathbf{E}$
	10	S_4	$\mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_3$, $\mathbf{N}_3$
	11	C_{4h}	$\mathbf{T}(4, 4)$, $\mathbf{N}_3$
	12	C_{4v}	$\mathbf{T}(4, 4)$, $\mathbf{e}_3$
	13	D_{2d}	$\mathbf{T}_d$, $\mathbf{e}_3 \otimes \mathbf{e}_3$
	14	D_4	$\mathbf{T}(4, 4)$, $\mathbf{E}$
	15	D_{4h}	$\mathbf{T}(4, 4)$
Rhombohedral	16	C_3	$\mathbf{T}(3, 3)$, $\mathbf{e}_3$, $\mathbf{E}$
	17	C_{3i}	$\mathbf{T}(3, 3) \otimes \mathbf{e}_3$, $\mathbf{N}_3$
	18	C_{3v}	$\mathbf{T}(3, 3)$, $\mathbf{e}_3$
	19	D_3	$\mathbf{T}(3, 3)$, $\mathbf{E}$
	20	D_{3d}	$\mathbf{T}(3, 3) \otimes \mathbf{e}_3$
Hexagonal	21	C_{3h}	$\mathbf{T}(3, 3)$, $\mathbf{N}_3$
	22	C_6	$\mathbf{T}(6, 6)$, $\mathbf{e}_3$, $\mathbf{E}$
	23	C_{6h}	$\mathbf{T}(6, 6)$, $\mathbf{N}_3$
	24	D_{3h}	$\mathbf{T}(3, 3)$
	25	C_{6v}	$\mathbf{T}(6, 6)$, $\mathbf{e}_3$
	26	D_6	$\mathbf{T}(6, 6)$, $\mathbf{E}$
	27	D_{6h}	$\mathbf{T}(6, 6)$
Cubic	28	T	$\mathbf{T}_h$, $\mathbf{E}$
	29	T_h	$\mathbf{T}_h$
	30	T_d	$\mathbf{T}_d$
	31	O	$\mathbf{O}_h$, $\mathbf{E}$
	32	O_h	$\mathbf{O}_h$

http://rcin.org.pl

By analyzing carefully the relationship of all the 32 point groups with those invariant groups determined by the basic tensors presented in the above table, we will get the structural tensors for all 32 point groups as presented in Table 2. In the table the tensors $\mathbf{N}_1$ and $\mathbf{N}_2$ are defined by

$$\mathbf{N}_1 = \mathbf{e}_2 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_2, \qquad \mathbf{N}_2 = \mathbf{e}_1 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_1. \tag{4.22}$$

It is worthwhile pointing out that LOHIN and SEDOV [3] found the structural tensors for all 32 point groups in 1963. Our presentations are equivalent to Lohin and Sedov's in as far as they determine the same groups. But, basing on the analysis for the symmetry properties of the tensor $\mathbf{T}(n, n)$, we have struck out the tensor $\mathbf{e}_3 \otimes \mathbf{e}_3$ from the structural tensor series and replaced Lohin and Sedov's $\mathbf{D}_{nh}$ by our tensor $\mathbf{T}(n, n)$, which makes our presentation remain non-superfluous.

5. Structural tensors for non-crystal solids

There are infinite orthogonal subgroups of the non-crystal type. Fortunately, it is possible for us to present the structural tensors for all non-crystal solids according to some general formulae. All orthogonal subgroups of the non-crystal type can be classified into four classes: isotropic, transversely isotropic, icosahedral and non-crystal dihedrals. By isotropic we mean either isotropic or hemitropic. Transverse isotropy is characterized by a preferred direction $\mathbf{e}_3$. Its symmetry groups can be classified into the following five classes: C_∞, $C_{\infty v}$, $C_{\infty h}$, D_∞ and $D_{\infty h}$. The icosahedral group Y_h presents the complete symmetry group of a regular icosahedron. It contains the proper subgroup Y. We classify all non-crystal dihedral groups into the same class which contains the subgroups C_n, C_{nh}, C_{nv}, S_{2n}, D_{nv}, D_{nd} and D_{nh}. In Table 3 we present the structural tensors for all these groups. The proofs of most of them are almost obvious from Table 1 and Theorem 3.1. Here we pay little attention to the analysis of the structural tensor Y_h.

The icosahedral group is the complete symmetry of a regular icosahedron whose all possible symmetry axes are: 6 fivefold axes through pairs of opposite vertices, 10 threefold axes through the midpoints of opposite faces, and 15 twofold axes through the midpoints of opposite edges. The 6 fivefold axes are as follows:

$$\begin{aligned} &\mathbf{a}_1 = \mathbf{e}_3, \\ &\mathbf{a}_k = \mathbf{e}_3 \cos\phi + (\mathbf{e}_1 \cos k\xi + \mathbf{e}_2 \sin k\xi)\sin\phi, \qquad k = 2, \dots 6, \end{aligned} \tag{5.1}$$

where

$$\xi = \frac{2\pi}{5} \qquad \cos\phi = \frac{1}{\sqrt{5}}, \qquad \sin\phi = \frac{2}{\sqrt{5}}.$$

Obviously the following tensor of rank six is invariant under the icosahedral group Y_h.

$$\mathbf{Y}_h = \sum_{k=1}^{6} (\mathbf{a}_k \otimes \mathbf{a}_k \otimes \mathbf{a}_k \otimes \mathbf{a}_k \otimes \mathbf{a}_k \otimes \mathbf{a}_k) \tag{5.2}$$

http://rcin.org.pl

(5.2) [cont.]

$$= \frac{2}{25}\{2(5\mathbf{e}_1^6+[\mathbf{e}_1^2\otimes\mathbf{e}_2^2]+[\mathbf{e}_1^2\otimes\mathbf{e}_2^4]+5\mathbf{e}_2^6)+[\mathbf{e}_1^5\otimes\mathbf{e}_3]$$

$$-[\mathbf{e}_1^3\otimes\mathbf{e}_2^2\otimes\mathbf{e}_3]+[\mathbf{e}_1\otimes\mathbf{e}_2^4\otimes\mathbf{e}_3]+3[\mathbf{e}_1^4\otimes\mathbf{e}_3^2]+[\mathbf{e}_1^2\otimes\mathbf{e}_2^2\otimes\mathbf{e}_3^2]-$$

$$+3[\mathbf{e}_2^4\otimes\mathbf{e}_3^2]+[\mathbf{e}_1^2\otimes\mathbf{e}_3^4]+[\mathbf{e}_2^2\otimes\mathbf{e}_3^4]+13\mathbf{e}_3^6\}.$$

Table 3. Structural tensor for non-crystal solids

Class	No	Schonlies notations	Structural tensors
Isotropic	1	isotropic	$\mathbf{I}=\mathbf{e}_1\otimes\mathbf{e}_1+\mathbf{e}_2\otimes\mathbf{e}_2+\mathbf{e}_3\otimes\mathbf{e}_3$
	2	hemitropic	$\mathbf{I}, \mathbf{E}$
Transversely Isotropic	3	C_∞	$\mathbf{e}_3$, $\mathbf{E}$
	4	$C_{\infty v}$	$\mathbf{e}_3$
	5	$C_{\infty h}$	$\mathbf{e}_3\otimes\mathbf{e}_3$, $\mathbf{N}_3$
	6	D_∞	$\mathbf{e}_3\otimes\mathbf{e}_3$, $\mathbf{E}$
	7	$D_{\infty h}$	$\mathbf{e}_3\otimes\mathbf{e}_3$
Icosahedral	8	Y	$\mathbf{Y}_h$, $\mathbf{E}$
	9	Y_h	$\mathbf{Y}_h$
Non-crystal dihedrals	10	C_n	$\mathbf{T}(n, n)$, $\mathbf{E}$, $\mathbf{e}_3$
	11	C_{nh}	$\mathbf{T}(n, n)$, $\mathbf{N}_3$
	12	C_{nv}	$\mathbf{T}(n, n)$, $\mathbf{e}_3$
	13	D_n	$\mathbf{T}(n, n)$, $\mathbf{E}$
	14	S_{2n}	$\mathbf{T}(n, n)\otimes\mathbf{e}_3$, $\mathbf{N}_3$
	15	D_{nd}(n odd)	$\mathbf{T}(n, n)\otimes\mathbf{e}_3$,
	16	D_{nd}(n even)	$\mathbf{S}(n, n)\otimes\mathbf{e}_3$
	17	D_{nh}	$\mathbf{T}(n, n)$

Tedious but trivial calculation will show that $\mathbf{Y}_h$ indeed is the structural tensor for group Y_h.

http://rcin.org.pl

6. Concluding remarks

It is possible to transform the problems of representations of the constitutive functions of anisotropic solids into those of isotropic functions according to our present research. However, up to now, almost all the results about representations of isotropic functions are confined to functions of vectors and second order tensors [6, 7]. Relatively little is known about the representations of isotropic functions whose arguments contain tensors of a rank higher than two, see ([8, 9] and [10]). It is necessary to investigate representations of such functions.

Acknowledgement

The first author is indebted to the Institute of Fundamental Technological Research of the Polish Academy of Sciences for a fellowship during the completion of this work. His research also was supported in part by the Chinese National Foundation of Science. The second author wishes to express his appreciation of discussions on the problem of extension of the anisotropic functions with Dr V. V. LOHIN.

References

1. J. P. BOEHLER, *A simple derivation of representations for non-polynomial constitutive equations in some cases of anisotropy*, ZAMM, **59**, 7–167, 1979.
2. I. LIU SHIH, *On representations of anisotropic invariants*, Int. J. Engng. Sci., **20**, 1099–1109, 1982.
3. V. V. LOHIN and L. I. SEDOV, *Non-linear tensorial functions of some tensorial arguments*, Prikl. Math. Mech., **27**, 393–417, 1963.
4. J. RYCHLEWSKI, *Symmetry of causes and effects*, Shanghai Institute of Applied Mathematics and Mechanics, Research Report SM8706, 1987.
5. S. BHAGAVANTAM, *Crystal symmetry and physical properties*, Academic Press, 1966.
6. C. C. WANG, *A new representation theorem for isotropic functions, An answer to Professor G. F. Smith's criticism of my papers on representations for isotropic functions, Part I and II*, Arch. Ratl. Mech. Anal., **36**, pp. 166–223, 1970.
7. A. J. M. SPENCER, *Theory of invariants*, in: Continuum Physics [ed.] A. C. ERINGEN, pp. 239–353, Academic Press 1971.
8. J. BETTEN, *Integrity basis for a second-order and a fourth-order tensor*, Int. J. Math. and Math. Sci., 5, pp. 87–96, 1982.
9. J. RYCHLEWSKI, *On a general form of isotropic transformation of tensor space*, Bull. Acad. Polon., Série Sci. Techn., **7**, 283–287, 1970.
10. J. RYCHLEWSKI, *Symmetry of tensor functions and spectral theorem*, Uspiechy Mech., **11**, 77–124, 1988.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH

Received June 28, 1989

http://rcin.org.pl