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Long time tails of hydrodynamic friction coefficients of rigid spheres

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NONSTATIONARY hydrodynamic interactions of a finite number of rigid spheres, immersed in an incompressible fluid, are considered. The main point of interest is the impact of the hydrodynamic interactions on the drag force, and torque exerted by the fluid on the spheres. The properties of the friction coefficients at long time are discussed.

W pracy tej rozpatrujemy niestacjonarne Stokes'owskie oddziaływania hydrodynamiczne między skończoną liczbą sztywnych kul, umieszczonych w cieczy nieściśliwej. W szczególności, praca dotyczy wpływu, jaki oddziaływania hydrodynamiczne wywierają na siłę i moment siły oporu hydrodynamicznego kul. Analizowane są własności współczynników tarcia w granicy długich czasów.

В этой работе рассматриваем нестационарные стоксовские гидродинамические взаимодействия между конечным количеством жестких шаров, помещенных в несжимаемой жидкости. В частности, работа касается влияния, какое гидродинамические взаимодействия вызывают на силу и момент силы гидродинамического сопротивления шаров. Анализируются свойства коэффициентов трения в пределе больших времен.

1. Introduction

THE PAPER CONCERNS the transient effects arising in the hydrodynamic interactions of a finite number of rigid spheres immersed in an incompressible, unbounded fluid. Attention is confined to the case of hydrodynamic interactions which can be described within the fully linearized scheme, both with respect to the velocities of the spheres and the velocity field of the fluid. Hence the velocity and pressure fields of the fluid are governed by the nonstationary Stokes equations. This range of hydrodynamic interactions has been regarded recently by VAN SAARLOOS and MAZUR [1], in relation to the hydrodynamic mobilities of the spheres, and by WEINBAUM [2], in relation to the trajectories of the sedimenting spheres (comp. [5, 8] for further literature).

In this paper the main point of interest is to discuss the appearance of the long time tails of the hydrodynamic drag exerted on the spheres by the fluid. To this aim the interactions at long time $t \gg l^2/\nu$ are considered, where l denotes the characteristic distance of the interactions, ν—the kinematic viscosity of the fluid, l^2/ν—the viscous relaxation time. This means the range of the hydrodynamic interactions involved is close to the range of the quasi-stationary hydrodynamic interactions described within the frame-work of the quasi-stationary Stokes equations. The appearance of the long time tails of the hydrodynamic drag gives rise to "fluid memory" effects which are absent at quasi-stationary conditions.

Restricting our attention to the spherical particles, we note that the geometry of the particles immersed in the fluid is an important factor in establishing hydrodynamic interactions [3, 4].

2. Multiple scattering representation of hydrodynamic interactions

To account for the presence of the spheres in the flow, we use the idea of time-dependent induced forces $\mathbf{f}_j(t)$, $j = 1, \ldots, N$ (N—the number of spheres) distributed on the surfaces of the spheres [1]. Taking advantage of that idea, the dependence of the induced forces $\mathbf{f}_j(t)$ on the relative velocities $\mathbf{V}_j(t)$ of the spheres with respect to the fluid can be expressed in terms of a set of integral equations [5]. For the particular case of the hydrodynamic interactions of N spheres specified by:

(i) the hydrodynamic conditions close to the quasi-stationary ones,

(ii) the non-slip boundary conditions on the surfaces of spheres.

(iii) the zero velocity field at time $t = 0$, that set of integral equations assumes the form

$$\begin{aligned} \mathbf{V}_j(\Omega_j(0), t) = \int_0^t dt' \int d\Omega'_j \mathbf{G}[\mathbf{R}_j(\Omega_j(0), t) - \mathbf{R}'_j(\Omega'_j(0), t'), t-t'] \\ \cdot \mathbf{f}_j(\Omega'_j(0), t') + \sum_{k\neq j}^{N} \int_0^t \mathrm{d}t' \int d\Omega'_k \mathbf{G}[\mathbf{R}_j(\Omega_j(0), t) \\ -\mathbf{R}'_k(\Omega'_k(0), t'), t-t'] \cdot \mathbf{f}_k(\Omega'_k(0), t'), \end{aligned} \tag{2.1}$$

$$\mathbf{V}_j(\Omega_j(0), t) = \dot{\mathbf{R}}_j(\Omega_j(0,) t) - \mathbf{v}^0(\Omega_j(0), t), \qquad t > 0,$$

where the initial distribution of spheres in a fixed laboratory reference frame is described by $\mathbf{R}_j^0(t)$—the position of the centre of the j-th sphere, $\mathbf{R}_j(t)$—the position of an arbitrary point on the surface of the j-th sphere, $j = 1, \ldots, N$.

The velocities of the spheres are given by

$\dot{\mathbf{R}}_j^0(t)$	the translational velocity of the j-th sphere,
$\boldsymbol{\omega}_j(t)$	the rotational velocity of the j-th sphere,
$\dot{\mathbf{R}}_j(t) = \dot{\mathbf{R}}_j^0(t) + \boldsymbol{\omega}_j(t) x \mathbf{r}_j$	the velocity of the j-th sphere, where $\mathbf{r}_j = \mathbf{R}_j - \mathbf{R}_j^0$,
$\mathbf{r}_j(a, \Omega_j) = \mathbf{r}_j(a, \theta_j, \phi_j)$	in the spherical polar coordinates connected to the j-th sphere,
a	the radius of the spheres.

The fluid velocity due to the external forces acting on the fluid is denoted by $\mathbf{v}^0(\mathbf{r}, t)$. $\mathbf{G}(\mathbf{r}, t)$ denotes the Green tensor [6] which reads in the form of the Fourier transform with the respect to the space variables

$$\begin{aligned} \mathbf{G}[\mathbf{R}_j(\Omega_j, t) - \mathbf{R}'_k(\Omega'_k, t'), t-t'] \\ = \frac{1}{\varrho} \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \exp[i\mathbf{k} \cdot (\mathbf{R}_j(\Omega_j, t) - \mathbf{R}'_k(\Omega'_k, t')) - \nu k^2 (t-t')] \left(\mathbf{1} - \frac{\mathbf{kk}}{k^2}\right), \end{aligned} \tag{2.2}$$

where $\mathbf{r}$ and $\mathbf{k}$ are the variables conjugated by the Fourier transformation, $\mathbf{k} = (k, \chi, \xi)$ in spherical polar coordinates, ϱ—the density of the fluid.

In the set of integral equations (2.1), the first term on the r.h.s. describes the interaction of a single sphere with the fluid, the second—the hydrodynamic interactions of N spheres.

After the following steps:

the expansions of $\mathbf{V}_j$, $\mathbf{f}_j$ in terms of the normalized surface spherical harmonics Y_l^m [10],

the integrations with respect to the angular variables Ω_j, using the orthogonality relations for Y_l^m,

the Laplace transformation with respect to time t, involving the convolution form of the equations,

we arrive at the set of algebraic equations

$$(2.3)\qquad \mathbf{V}_{j,l_1m_1}(p) = \sum_{l_2m_2}^{\infty} \mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{0}_j, p)\cdot \mathbf{f}_{j,l_2m_2}(p) + \sum_{k\neq j}^{N}\sum_{l_2m_2}^{\infty} \mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{R}_{kj}, p)\cdot \mathbf{f}_{k,l_2m_2}(p), \qquad \mathrm{Re}\sqrt{p} \geqslant 0, \quad l\geqslant 0, \quad |m|\leqslant l,$$

where p is the variable conjugate to t by the Laplace transformation, $\mathbf{V}_{j,lm}$, $\mathbf{f}_{j,lm}$ are the expansion coefficients of $\mathbf{V}_j$, $\mathbf{f}_j$ in terms of the surface harmonics, $\mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{0}_j, p)$ and $\mathbf{T}_{l_1m_1}^{l_2m_2}(R_{kj}, p)$ are so-called self- and mutual-hydrodynamic interaction tensors.

According to the results, obtained in the paper [5], the interaction tensors can be presented in the following form:

(i) the self-interaction tensors:

$$\mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{0}_j, p) = F_{l_1l_2,0}(0_j, p)\mathbf{K}_{l_1m_1,00}^{l_2m_2} Y_0^0,$$

where the functions $F_{l_1l_2,0}$ describe the p—dependence of the self-interactions of a single sphere with the fluid:

$$(2.4)\qquad \begin{aligned} F_{l_1l_1,0}(0_j, p) &= \frac{1}{\varrho a\nu} I_{l_1+\frac{1}{2}}\left(a\sqrt{\frac{p}{\nu}}\right) K_{l_1+\frac{1}{2}}\left(a\sqrt{\frac{p}{a}}\right),\\ F_{l_1l_1+2,0}(0j, p) &= -\frac{1}{\varrho a\nu} I_{l_1+\frac{5}{2}}\left(a\sqrt{\frac{p}{\nu}}\right) K_{l_1+\frac{1}{2}}\left(a\sqrt{\frac{p}{\nu}}\right), \end{aligned}$$

$I_{l_1+\frac{1}{2}}$, $K_{l_1+\frac{1}{2}}$ denote the modified Bessel functions,

the second order tensors, equal to:

$$\mathbf{K}_{l_1m_1,l_3m_3}^{l_2m_2} = i^{l_1-l_2+l_3}\int \sin\chi d\chi d\xi Y_{l_1}^{-m_1} Y_{l_2}^{m_2} Y_{l_3}^{-m_3}\left(\mathbf{1}-\frac{\mathbf{kk}}{k^2}\right)$$

describe the tensorial properties of the interaction tensors,

(ii) the mutual interaction tensors having a similar structure

$$\mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{R}_{kj}, p) = \sum_{2\beta=-2}^{\max} F_{l_1l_2l_1+l_2-2\beta}(R_{kj}, p)\sum_m \mathbf{K}_{l_1m_1,l_1+l_2-2\beta m}^{l_2m_2}\cdot Y_{l_1+l_2-2\beta}^{m}(\varkappa, \gamma),$$

where $2\beta = l_1+l_2-l_3$

$$\max = \begin{Bmatrix} l_1+l_2 & \text{if} & l_1+l_2 & \text{is} & \text{even}\\ l_1+l_2-1 & \text{if} & l_1+l_2 & \text{is} & \text{odd}\end{Bmatrix}, \qquad \mathbf{R}_{kj} = \mathbf{R}_j^0(t=0)-\mathbf{R}_k^0(t=0),$$

$\mathbf{R}_{kj} = (R_{kj}, \varkappa, \gamma)$ in spherical polar coordinates,

$$F_{l_1 l_2, l_3}(R_{kj}, p) = \frac{\sqrt{\pi}}{\sqrt{2}\varrho a\nu}\left(R_{kj}\sqrt{\frac{p}{\nu}}\right)^{-\frac{1}{2}} I_{l+\frac{1}{2}}\left(a\sqrt{\frac{p}{\nu}}\right) I_{l_2+\frac{1}{2}}\left(a\sqrt{\frac{p}{\nu}}\right) \cdot K_{l_3+\frac{1}{2}}\left(R_{kj}\sqrt{\frac{p}{\nu}}\right) \quad \text{for} \quad 2\beta = 0, 2, 4, \ldots,$$

$$F_{l_1 l_2, l_3}(R_{kj}, p) = \frac{\sqrt{\pi}}{\varrho a\nu}\left(\frac{a}{R_{kj}}\right)^{l_1+l_2+3} \frac{\Gamma\left(l_1+l_2+\frac{7}{2}\right)}{\Gamma\left(l_1+\frac{3}{2}\right)\Gamma\left(l_2+\frac{3}{2}\right)}\left(a\sqrt{\frac{p}{\nu}}\right)^{-2}$$

$$-\frac{\sqrt{\pi}}{2\varrho a\nu}\left(\frac{1}{2}\right)^{l_1+l_2+\frac{1}{2}}\left(\frac{a}{R_{kj}}\right)^{l_1+l_2+3}\left(R_{kj}\sqrt{\frac{p}{\nu}}\right)^{l_1+l_2+\frac{5}{2}} K_{l_1+l_2+\frac{5}{2}}\left(R_{kj}\sqrt{\frac{p}{\nu}}\right)\left(a\sqrt{\frac{p}{\nu}}\right)^{-2} \cdot\left[\frac{l_1+l_2+\frac{5}{2}}{\Gamma\left(l_1+\frac{3}{2}\right)\Gamma\left(l_2+\frac{3}{2}\right)}\right.$$

$$\left.+\sum_{n=1}^{\infty}\frac{\Gamma(l_1+l_2+2+2n)}{\Gamma(l_1+l_2+2+n)\Gamma\left(l_1+\frac{3}{2}+n\right)\Gamma\left(l_2+\frac{3}{2}+n\right)}\left(\frac{a}{2}\sqrt{\frac{p}{\nu}}\right)^{2n}\right], \quad \text{for} \quad 2\beta = -2,$$

where Γ is the gamma function.

We note that the dependence of $F_{l_1 l_2, l_3}$ functions on the distances between the centres of two spheres is involved at time $t = 0$.

Starting from Eq. (2.3) the expansion coefficients of the induced forces can be presented in the following form:

$$\text{(2.5)} \qquad \mathbf{f}_{j,l_1 m_1}(p) = \sum_{l_2 m_2}\tilde{\mathbf{T}}_{l_1 m_1}^{l_2 m_2}(\mathbf{0}_j, p)\cdot\Big[\mathbf{V}_{j,l_2 m_2}(p) - \sum_{k\neq j}^{N}\sum_{l_3 m_3}\sum_{l_4 m_4}\mathbf{T}_{l_2 m_2}^{l_3 m_3} \cdot(\mathbf{R}_{kj}, p)\tilde{\mathbf{T}}_{l_3 m_3}^{l_4 m_4}(\mathbf{0}_k, p)\cdot\mathbf{V}_{k,l_4 m_4} + \sum_{k\neq j}^{N}\sum_{k_1\neq k}^{N}\sum_{l_2 m_2}\mathbf{T}_{l_2 m_2}^{l_3 m_3}(\mathbf{R}_{kj}, p) \cdot\tilde{\mathbf{T}}_{l_3 m_3}^{l_4 m_4}(\mathbf{0}_k, p)\cdot\mathbf{T}_{l_4 m_4}^{l_5 m_5}(\mathbf{R}_{k_1 k}, p)\cdot\tilde{\mathbf{T}}_{s m_5}^{6 m_6}\ldots\Big], \qquad i = 3, 4, 5, 6,$$

where $\tilde{\mathbf{T}}_{l_1 m_1}^{l_2 m_2}(\mathbf{0}_j, p)$ denote the respective inverse tensor [10]:

$$\sum_{l_3 m_3}\tilde{\mathbf{T}}_{l_1 m_1}^{l_3 m_3}(\mathbf{0}_j, p)\cdot\mathbf{T}_{l_3 m_3}^{l_2 m_2}(\mathbf{0}_j, p) = \mathbf{1}\delta_{l_1 l_2}\delta_{m_1 m_2}.$$

The expression (2.5) gives the multiple scattering representation of the hydrodynamic interactions of N spheres. The first term on the r.h.s. describes the interaction of a single sphere with the fluid, the next ones—the interactions of two, three, and so on, spheres.

In view of the fact that the hydrodynamic drag force and torque can be expressed in terms of the coefficients $\mathbf{f}_{j,lm}(p)$ [5] the expression (2.5) is used (Sect. 4) to establish the form of the friction relations of N spheres.

3. Nonstationary hydrodynamic interaction tensors

Here we restrict ourselves only to the discussion of these properties of the hydrodynamic interaction tensors which concern the long time effects. Consideration of other properties of these tensors is presented in [5, 8].

The functions $F_{l_1 l_2, 0}(0\ , p)$ describing the p-dependence of the interaction of a single sphere with the surrounding fluid, for the considered case of the hydrodynamic interactions close to the quasi-stationary case (i.e., for $a\sqrt{p/\nu} \ll 1$) assume the form

$$F_{l_1 l_1, 0}(0_j, p) = \frac{1}{2\sqrt{\pi}\varrho a\nu} \cdot \frac{\pi}{2\sin\left[\left(l_1+\frac{1}{2}\right)\pi\right]} \left\{\alpha_0\beta_0 + \alpha_0\beta_1\left(\frac{a}{2}\sqrt{\frac{p}{\nu}}\right)^2 + \beta_0\alpha_1\left(\frac{a}{2}\sqrt{\frac{p}{\nu}}\right)^2 - \left(\frac{a}{2}\sqrt{\frac{p}{\nu}}\right)^{2l_1+1}\left[\alpha_0^2 + 2\alpha_0\alpha_1\left(\frac{a}{2}\sqrt{\frac{p}{\nu}}\right)^2\right] + \ldots\right\},$$

(3.1)

$$F_{l_1 l_1+2, 0}(0_j, p) = \frac{(-1)}{2\sqrt{\pi}\varrho a\nu} \cdot \frac{1}{8\left(l_1+\frac{5}{2}\right)\left(l_1+\frac{3}{2}\right)\left(l_1+\frac{1}{2}\right)} \left\{\left(a\sqrt{\frac{p}{\nu}}\right)^2 + \ldots\right\},$$

where

$$\alpha_0 = \frac{1}{\Gamma\left(l_1+\frac{3}{2}\right)}, \quad \alpha_1 = \frac{1}{\Gamma\left(l_1+\frac{5}{2}\right)},$$

$$\beta_0 = \frac{1}{\Gamma\left(-l_1+\frac{1}{2}\right)}, \quad \beta_1 = \frac{1}{\Gamma\left(-l_1+\frac{3}{2}\right)}.$$

We see that only the function $F_{00,0}(0_j, p)$ vanishes as $\sqrt{p}$ for $a\sqrt{p/\nu} \ll 1$. From Eq. (2.4) it follows that among the self-interaction tensors, only the tensor $\mathbf{T}^{00}_{00}(\mathbf{0}_j, p)$ is built up of this function. Hence only the tensors $\mathbf{T}^{00}_{00}(\mathbf{0}_j, p)$ and $\tilde{\mathbf{T}}^{00}_{00}(\mathbf{0}_j, p)$ possess the long time tails.

The functions $F_{l_1 l_2 l_3}\ (R, p)$ for the case of interest (i.e., for $R\sqrt{p/\nu} \ll 1$) read
(i) if $2\beta = 0, 2, 4, \ldots$:

$$F_{l_1 l_2, l_1+l_2-2\beta} = A\left(\frac{a}{R}\right)^{l_1+l_2+1}\left\{\left(\frac{R}{2}\sqrt{\frac{p}{\nu}}\right)^{2\beta}\left[a_0 b_0 + a_0 b_1\left(\frac{R}{2}\sqrt{\frac{p}{\nu}}\right)^2 + a_1 b_0\left(\frac{a}{2}\sqrt{\frac{p}{\nu}}\right)^2\right] - \left(\frac{R}{2}\sqrt{\frac{p}{\nu}}\right)^{2l_1+2l_2+1-2\beta}\left[a_0 d_0 + a_0 d_1\left(\frac{R}{2}\sqrt{\frac{p}{\nu}}\right)^2 + a_1 d_0\left(\frac{a}{2}\sqrt{\frac{p}{\nu}}\right)^2\right] + \ldots\right\},$$

(ii) if $2\beta = -2$

$$(3.2)\quad F_{l_1 l_2, l_1+l_2+2} = -\frac{\pi\sqrt{\pi}}{2\varrho a \nu}\cdot\frac{1}{\Gamma\left(l_1+\frac{3}{2}\right)\Gamma\left(l_2+\frac{3}{2}\right)\sin\left[\left(l_1+l_2+\frac{5}{2}\right)\pi\right]}\left(\frac{a}{R}\right)^{l_1+l_1+1}$$

$$\cdot\Bigg\{\left(l_1+l_2+\frac{5}{2}\right)\left[\frac{1}{2}b_1+\frac{1}{8}b_2\left(R\sqrt{\frac{p}{\nu}}\right)^2\right]-\left(l_1+l_2+\frac{5}{2}\right)\left(\frac{1}{2}\right)^{2l_1+2l_2+4}\left(R\sqrt{\frac{p}{\nu}}\right)^{2l_1+2l_2+3}$$

$$\cdot\left[d_0+d_1\left(\frac{R}{2}\sqrt{\frac{p}{\nu}}\right)^2\right]+\frac{1}{9}(l_1+l_2+3)\left(\frac{a}{R}\right)^2\left[2b_0+2b_1\left(\frac{R}{2}\sqrt{\frac{p}{\nu}}\right)^2\right]$$

$$-\frac{1}{9}(l_1+l_2+3)\left(\frac{a}{R}\right)^2\left(\frac{1}{2}\right)^{2l_1+2l_2+4}\left(R\sqrt{\frac{p}{\nu}}\right)^{2l_1+2l_2+5}\left[d_0+d_1\left(\frac{R}{2}\sqrt{\frac{p}{\nu}}\right)^2\right]+\ldots\Bigg\},$$

where

$$A = \frac{\pi\sqrt{\pi}}{4\varrho a\nu}\cdot\frac{1}{\sin\left[\left(l_1+l_2-2\beta+\frac{1}{2}\right)\pi\right]},$$

$$a_0 = \frac{1}{\Gamma\left(l_1+\frac{3}{2}\right)\Gamma\left(l_2+\frac{3}{2}\right)},\qquad a_1 = \frac{1}{\Gamma\left(l_1+\frac{5}{2}\right)\Gamma\left(l_2+\frac{3}{2}\right)}+\frac{1}{\Gamma\left(l_1+\frac{3}{2}\right)\Gamma\left(l_2+\frac{5}{2}\right)},$$

$$b_0 = \frac{1}{\Gamma\left(-l_3+\frac{1}{2}\right)},\qquad b_1 = \frac{1}{\Gamma\left(-l_3+\frac{3}{2}\right)},$$

$$d_0 = \frac{1}{\Gamma\left(l_3+\frac{3}{2}\right)},\qquad d_1 = \frac{1}{\Gamma\left(l_3+\frac{5}{2}\right)},$$

R—the typical distance between the centres of two spheres at time $t = 0$.

From Eq. (3.2) it follows that the long time effects $\left(\sim R\sqrt{p/\nu}\right)$ are described only by the function $F_{00.0}(R, p)$. In turn, this function appears only in the tensor $\mathbf{T}^{00}_{00}(\mathbf{R}, p)$. In the result, the slow decay at long time is characteristic of only the particular sequences of the hydrodynamic interactions described by Eq. (2.5). These sequences have to involve at least one of the following self-, and mutual interaction tensors:

$$\mathbf{T}^{00}_{00}(\mathbf{0}_j, p),\ \tilde{\mathbf{T}}^{00}_{00}(\mathbf{0}_j, p),\quad \text{or}\quad \mathbf{T}^{00}_{00}(\mathbf{R}, p).$$

4. Long time tails of friction coefficients

Taking advantage of the property that the hydrodynamic drag force $\mathbf{F}_j(p)$, and the torque $\mathbf{T}_j(p)$ can be expressed in terms of the expansion coefficients $\mathbf{f}_{j,00}(p)$ and $\mathbf{f}_{j,1m}(p)$ of the induced forces, and calculating the relevant expansion coefficients from the iterative expression (2.5), we obtain the nonstationary friction relations in the following form:

$$\mathbf{F}_j(p) = -\sum_{k=1}^{N} \boldsymbol{\xi}_{jk}^{TT}(p)\cdot\dot{\mathbf{R}}_k^0(p) - \sum_{k=1}^{N} \boldsymbol{\xi}_{jk}^{TR}(p)\cdot\boldsymbol{\omega}_k(p) + \mathbf{I}_j^T(p) - \sum_{k=1}^{N}\sum_{lm}{}' \boldsymbol{\xi}_{jk,lm}^{TV}(p)\cdot\mathbf{v}_{k,lm}^0(p),$$

(4.1)

$$\mathbf{T}_j(p) = -\sum_{k=1}^{N} \boldsymbol{\xi}_{jk}^{RT}(p)\cdot\dot{\mathbf{R}}_k^0(p) - \sum_{k=1}^{N} \boldsymbol{\xi}_{jk}^{RR}(p)\cdot\boldsymbol{\omega}_k(p) + \mathbf{I}_j^R(p) - \sum_{k=1}^{N}\sum_{lm}{}' \boldsymbol{\xi}_{jk,lm}^{RV}(p)\cdot\mathbf{v}_{k,lm}^0(p),$$

where

$$\mathbf{I}_j^T = \frac{4}{3}\pi a^3 \varrho p \dot{\mathbf{R}}_j^0, \qquad \mathbf{I}_j^R = \frac{8}{15}\pi a^5 \varrho p \boldsymbol{\omega}_j$$

describe the inertia of the fluid displaced by the sphere, $\boldsymbol{\xi}_{jk}^{(\dots)}$ denote the self ($j = k$), and mutual ($j \neq k$) friction coefficients acting on the velocities of the spheres, $\boldsymbol{\xi}_{jk,(\dots)}^{(\dots)}$ denote the self ($j = k$) and mutual ($j \neq k$) friction coefficients acting on the fluid velocity $\mathbf{v}^0(\mathbf{r}, p)$.

The summation $\sum_{lm}'$ is given by

$$\sum_{lm}{}' \boldsymbol{\xi}_{jk,lm}^{(\dots)}\cdot\mathbf{v}_{k,lm}^0 = \boldsymbol{\xi}_{jk,00}^{(\dots)}\cdot\mathbf{v}_{k,00}^0 + \boldsymbol{\xi}_{jk,W^0}^{(\dots)}:\mathbf{W}_k^0 + \sum_{l\geqslant 2m} \boldsymbol{\xi}_{jk,lm}^{(\dots)}\cdot\mathbf{v}_{k,lm}^0, \tag{4.2}$$

where $\mathbf{W}^0 = \sum_{m=-1}^{1} \boldsymbol{\alpha}_{-m}\mathbf{v}_{j,1m}^0$ is the tensor of the second order, i-th component of $\boldsymbol{\alpha}_m$ reads:

$$(\boldsymbol{\alpha}_m)_i = \delta_{i1}(\delta_{m1}+\delta_{m-1})\frac{1}{\sqrt{2}} + \delta_{i2}(-\delta_{m1}+\delta_{m-1})\frac{i}{\sqrt{2}} + \delta_{i3}\delta_{m0}.$$

The friction coefficients are the tensors of the second order with one exception—the coefficient $\boldsymbol{\xi}_{jk,W^0}^{(\dots)}$, being the tensor of third order.

Hence we have the linear relations (4.1) of the hydrodynamic drag, exerted on the spheres by the fluid, to the velocities of the spheres and to the fluid velocity $\mathbf{v}^0$. This feature is the common one for both cases of quasi-stationary hydrodynamic interactions (described within the framework of quasi-stationary Stokes equations [10]), and for the range of the nonstationary hydrodynamic interactions specified in this paper. The difference lies in the p-dependence of the quantities involved, which leads to the memory character of the friction relations. This kind of memory effects reflects the nonstationary character of the flow of the fluid. Here the properties of the friction coefficients are discussed from the point of view of the prediction of long time tails. Hence the influence of the nonstationary character of the flow of the fluid is accounted for up to terms proportional to $\sqrt{p}$. The dependence of the friction coefficients on the spatial distribution of spheres is retained up to the terms of the order of $(a/R)^3$, to account for the first contributions, due to the nonadditivity of the hydrodynamic interactions.

In what follows we restrict our attention to the case of the spheres immersed in the constant fluid velocity field $\mathbf{v}_{j,00}^0(\mathbf{v}_{j,lm}^0 \equiv 0$ for $l \geqslant 1$. Then we have to deal with six self-, and mutual friction tensors entering the friction relations (4.1). From the formulae for the friction coefficients, given in terms of the hydrodynamic interaction tensors in an earlier paper [8] it follows that only five self- and mutual friction coefficients depend on the tensors $\mathbf{T}_{00}^{00}(\mathbf{0}_j, p)$ and/or $\tilde{\mathbf{T}}_{00}^{00}(\mathbf{R}_{kj}, p)$, describing the long time tail effects. For the case of the force $\mathbf{F}_j(p)$ the respective friction tensors, being tensors of the second order, assume the following form:

translational–translational friction coefficients:

$$(4.3)\quad \frac{1}{6\pi\mu a}\boldsymbol{\xi}^{TT}_{jj} = \left(1+a\sqrt{\frac{p}{\nu}}\right)\mathbf{1} + \sum_{k\neq j}\left(\frac{a}{R_{kj}}\right)^2\left\{\frac{9}{16}\left(1+3a\sqrt{\frac{p}{\nu}}\right)(\mathbf{1}+3\mathbf{e}_{kj}\mathbf{e}_{kj}) - \frac{3}{2}\left(R_{kj}\sqrt{\frac{p}{\nu}}\right)(\mathbf{1}+\mathbf{e}_{kj}\mathbf{e}_{kj})\right\}$$
$$+\sum_{k_1\neq j}\sum_{\substack{k_2\neq k_1\\ k_2\neq j}}\frac{a^3}{R_{k_1j}R_{k_2k_1}R_{jk_2}}\left\{-\frac{27}{64}\left(1+4a\sqrt{\frac{p}{\nu}}\right)(\mathbf{1}+\mathbf{e}_{k_1j}\mathbf{e}_{k_1j})\cdot(\mathbf{1}+\mathbf{e}_{k_2k_1}\mathbf{e}_{k_2k_1})\right.$$
$$\cdot(\mathbf{1}+\mathbf{e}_{jk_2}\mathbf{e}_{jk_2}) + \frac{9}{16}\left(R_{k_1j}\sqrt{\frac{p}{\nu}}\right)(\mathbf{1}+\mathbf{e}_{k_2k_1}\mathbf{e}_{k_2k_1})\cdot(\mathbf{1}+\mathbf{e}_{jk_2}\mathbf{e}_{jk_2}) + \frac{9}{16}\left(R_{jk_2}\sqrt{\frac{p}{\nu}}\right)$$
$$\cdot(\mathbf{1}+\mathbf{e}_{k_1j}\mathbf{e}_{k_1j})\cdot(\mathbf{1}+\mathbf{e}_{k_2k_1}\mathbf{e}_{k_2k_1}) + \frac{9}{16}\left(R_{k_2k_1}\sqrt{\frac{p}{\nu}}\right)(\mathbf{1}+\mathbf{e}_{k_1j}\mathbf{e}_{k_1j})$$
$$\left.\cdot(\mathbf{1}+\mathbf{e}_{jk_2}\mathbf{e}_{jk_2})\right\},\qquad \mu=\nu\varrho,\qquad \mathbf{e}_{jk}=\frac{\mathbf{R}_{jk}}{|\mathbf{R}_{jk}|};$$

the first term on the r.h.s. describes the hydrodynamic interaction of a single sphere with the fluid, the second—the additive contributions due to two sphere interactions, the third—the nonadditive contributions due to three sphere interactions;

translational–translational mutual friction coefficients:

$$(4.4)\quad \frac{1}{6\pi\mu a}\boldsymbol{\xi}^{TT}_{jk} = -\frac{a}{R_{kj}}\left\{\frac{3}{4}\left(1+2a\sqrt{\frac{p}{\nu}}\right)(\mathbf{1}+\mathbf{e}_{kj}\mathbf{e}_{kj}) - \left(R_{kj}\sqrt{\frac{p}{\nu}}\right)\mathbf{1}\right\}$$
$$+\sum_{\substack{k_1\neq k\\ k_1\neq j}}\frac{a^2}{R_{k_1j}R_{kk_1}}\left\{\frac{9}{16}\left(1+3a\sqrt{\frac{p}{\nu}}\right)(\mathbf{1}+\mathbf{e}_{k_1j}\mathbf{e}_{k_1j})\cdot(\mathbf{1}+\mathbf{e}_{kk_1}\mathbf{e}_{kk_1})\right.$$
$$\left.-\frac{3}{4}\left(R_{kk_1}\sqrt{\frac{p}{\nu}}\right)(\mathbf{1}+\mathbf{e}_{k_1j}\mathbf{e}_{k_1j}) - \frac{3}{4}\left(R_{k_1j}\sqrt{\frac{p}{\nu}}\right)(\mathbf{1}+\mathbf{e}_{kk_1}\mathbf{e}_{kk_1})\right\} - \frac{a^3}{2R^3_{kj}}\left(1+2a\sqrt{\frac{p}{\nu}}\right)$$
$$\cdot(\mathbf{1}-3\mathbf{e}_{kj}\mathbf{e}_{kj}) + \sum_{k\neq k_1}\sum_{\substack{k_1\neq k_2\\ j\neq k_2}}\frac{a^3}{R_{k_2j}R_{k_1k_2}R_{kk_1}}\left\{-\frac{27}{64}\left(1+8a\sqrt{\frac{p}{\nu}}\right)\right.$$
$$\cdot(\mathbf{1}+\mathbf{e}_{k_2j}\mathbf{e}_{k_2j})\cdot(\mathbf{1}+\mathbf{e}_{k_1k_2}\mathbf{e}_{k_1k_2})\cdot(\mathbf{1}+\mathbf{e}_{kk_1}\mathbf{e}_{kk_1}) + \frac{9}{16}\left(R_{k_2j}\sqrt{\frac{p}{\nu}}\right)(\mathbf{1}+\mathbf{e}_{k_1k_2}\mathbf{e}_{k_1k_2})$$
$$\cdot(\mathbf{1}+\mathbf{e}_{kk_1}\mathbf{e}_{kk_1}) + \frac{9}{16}\left(R_{k_1k_2}\sqrt{\frac{p}{\nu}}\right)(\mathbf{1}+\mathbf{e}_{k_2j}\mathbf{e}_{k_2j})\cdot(\mathbf{1}+\mathbf{e}_{kk_1}\mathbf{e}_{kk_1})$$
$$\left.+\frac{9}{16}\left(R_{kk_1}\sqrt{\frac{p}{\nu}}\right)(\mathbf{1}+\mathbf{e}_{k_2j}\mathbf{e}_{k_2j})\cdot(\mathbf{1}+\mathbf{e}_{k_1k_2}\mathbf{e}_{k_1k_2})\right\},$$

where the first term on the r.h.s. is due to two sphere interactions, the second, and the third, respectively, due to nonadditive three, and four sphere interactions, translational–rotational mutual friction coefficient:

$$(4.5)\qquad \frac{1}{6\pi\mu a}\boldsymbol{\xi}_{jj}^{TR} = \sum_{k\neq j}\frac{a}{R_{kj}}\left\{\frac{3}{4}\left(1+2a\sqrt{\frac{p}{\nu}}\right)(1+\mathbf{e}_{kj}\mathbf{e}_{kj})-\left(R_{kj}\sqrt{\frac{p}{\nu}}\right)\mathbf{1}\right\}\cdot\boldsymbol{\Psi}_{jk}:\boldsymbol{\epsilon}$$

is generated by additive two sphere interactions,

translational–rotational mutual friction coefficient:

$$(4.6)\qquad \frac{1}{6\pi\mu a}\boldsymbol{\xi}_{jk}^{TR} = -\left(1+a\sqrt{\frac{p}{\nu}}\right)\mathbf{1}\cdot\boldsymbol{\Psi}_{jk}:\boldsymbol{\epsilon}$$

$$+\sum_{\substack{k_1\neq j\\ k_1\neq k}}\frac{a}{R_{k_1j}}\left\{\frac{3}{4}\left(1+2a\sqrt{\frac{p}{\nu}}\right)(1+\mathbf{e}_{k_1j}\mathbf{e}_{k_1j})-\left(R_{k_1j}\sqrt{\frac{p}{\nu}}\right)\mathbf{1}\right\}\cdot\boldsymbol{\Psi}_{kk_1}:\boldsymbol{\epsilon},$$

where $\boldsymbol{\epsilon}$ is the Levi–Civita tensor,

the tensors $\boldsymbol{\Psi}_{jk}$, which do not contain the long time tails, read

$$\boldsymbol{\Psi}_{jk} = \frac{a}{\sqrt{3}}\sum_{m_1=-1}^{1}\sum_{m_2=-1}^{1}\mathbf{T}_{00}^{1m_1}(\mathbf{R}_{jk},p)\cdot\tilde{\mathbf{T}}_{1m_1}^{1m_2}(\mathbf{0}_j,p)\alpha_{m_2},$$

$\mathbf{T}_{00}^{1m_1}$ and $\tilde{\mathbf{T}}_{1m_1}^{1m_2}$ are defined by (2.4) and (2.5); here we have two sphere contributions, and nonadditive three sphere contributions,

the friction tensors, acting on the velocity of the fluid

$$(4.7)\qquad \boldsymbol{\xi}_{jj,00}^{TV} = -\boldsymbol{\xi}_{jj}^{TT},\qquad \boldsymbol{\xi}_{jk,00}^{TV} = -\boldsymbol{\xi}_{jk}^{TT}.$$

The simple relations (4.7) reflect the property that the hydrodynamic drag depends in fact on the relative velocity of the sphere with respect to the fluid (comp. also [5, 10]).

In turn the friction coefficients describing the torque $\mathbf{T}_j(p)$ can be written down as follows:

rotational–translational friction coefficient:

$$(4.8)\qquad \frac{1}{6\pi\mu a}\boldsymbol{\xi}_{jj}^{RT} = -\boldsymbol{\epsilon}:\sum_{k\neq j}\boldsymbol{\Lambda}_{kj}\cdot\frac{a}{R_{jk}}\left\{\frac{3}{4}\left(1+2a\sqrt{\frac{p}{\nu}}\right)(1+\mathbf{e}_{jk}\mathbf{e}_{jk})-\left(R_{jk}\sqrt{\frac{p}{\nu}}\right)\mathbf{1}\right\},$$

this coefficient arises as a result of the additive interactions of two spheres.

rotational–translational mutual friction coefficient:

$$(4.9)\qquad \frac{1}{6\pi\mu a}\boldsymbol{\xi}_{jk}^{RT} = \boldsymbol{\epsilon}:\boldsymbol{\Lambda}_{kj}\cdot\left(1+a\sqrt{\frac{p}{\nu}}\right)\mathbf{1}-\boldsymbol{\epsilon}$$

$$:\sum_{\substack{k_1\neq j\\ k\neq k_1}}\boldsymbol{\Lambda}_{k_1j}\cdot\frac{a}{R_{kk_1}}\left\{\frac{3}{4}\left(1+2a\sqrt{\frac{p}{\nu}}\right)(1+\mathbf{e}_{kk_1}\mathbf{e}_{kk_1})-\left(R_{kk_1}\sqrt{\frac{p}{\nu}}\right)\mathbf{1}\right\},$$

where we have two sphere contributions, and nonadditive three sphere contributions.

In the above formula the following shorthand notation is used:

$$\boldsymbol{\Lambda}_{kj} = \frac{a}{\sqrt{3}}\sum_{m_1=-1}^{1}\sum_{m_2=-1}^{1}\boldsymbol{\alpha}_{-m_1}\tilde{\mathbf{T}}_{1m_1}^{1m_2}(\mathbf{0}_j,p)\cdot\mathbf{T}_{1m2}^{00}(\mathbf{R}_{kj},p),$$

and the tensor $\boldsymbol{\Lambda}_{kj}$ does not exhibit the slow decay $\sim\sqrt{p}$;

the friction tensors, acting on the velocity of the fluid:

$$\xi^{RV}_{jj,00} = -\xi^{RT}_{jj}, \quad \xi^{RV}_{jk,00} = -\xi^{RT}_{jk}. \tag{4.10}$$

We see that the contributions proportional to $\sqrt{p}$ arise either due to the nonstationary character of the interaction of a single sphere with the fluid (the terms $\sim a\sqrt{p/\nu}$), or due to the nonstationary mutual interactions of spheres (the terms $\sim R\sqrt{p/\nu}$). In comparison with the friction tensors, describing the quasi-stationary interactions, here the coefficients of the terms of all order with respect to a/R are changed due to slow decay effects. This concerns also the terms, describing the nonadditivity of the interactions (comp. Eqs. (4.3), (4.4), (4.6), (4.9)).

We note that in the approximation considered all the friction coefficients entering the expression (4.1) for the drag force $\mathbf{F}_j(p)$ contain the long time tails. In contrast, only four of the friction coefficients describing the torque $\mathbf{T}_j(p)$ decay as $\sqrt{p}$.

The effects of long time tails have been calculated by VAN SAARLOOS and MAZUR [1] for the case of the mobility relations for a finite number of rigid spheres. Qualitatively, the impact of the contributions proportional to $\sqrt{p}$ is similar for both the friction, and the mobility relations. The experimental results on the influence of the long time tails known to the author concern, however, different physical conditions. Namely, they have to do with a single Brownian sphere, immersed in a quiescent fluid [9], and confirm qualitatively the impact of long time tails of hydrodynamic friction.

Summarizing, from the relations (4.3)–(4.7) it follows that both the hydrodynamic interactions of a single sphere and the hydrodynamic interactions of N spheres with the fluid, impacting the drag force $\mathbf{F}(p)$, exhibit the long time tails $\sim\sqrt{p}$. In contrast, the torque exerted by the fluid on a single sphere does not exhibit the long time tail $\sim\sqrt{p}$ [11], whereas the torque in the presence of N spheres decays as $\sqrt{p}$, due to the influence of the friction coefficients (4.8)–(4.10).

References

1. W. VAN SAARLOOS, P. MAZUR, Physica A, **120**, 77, 1983.
2. S. WEINBAUM, Fluid Dynamics Trans., **11**, 229, 1983.
3. C. POZRIKIDIS, J. Fluid Mech., **202**, 17, 1989.
4. W. H. LIN, ZAMM, **67**, 487, 1987.
5. I. PIEŃKOWSKA, Arch. Mech., **36**, 749, 1984.
6. О. ЛАДЫЖЕНСКАЯ, *Математические вопросы динамики вязкой несжимаемой жидкости*, Наука, Москва 1970.
7. P. MAZUR, Physica A, **110**, 128, 1982.
8. I. PIEŃKOWSKA, IFTR Reports, 22, 1987.
9. G. L. PAUL, P. N. PUSEY, J. Phys., **14A**, 3301, 1981.
10. T. YOSHIZAKI, H. YAMAKAWA, J. Chem. Phys., **73**, 578, 1980.
11. F. FEUILLEBOIS, A. LASEK, Q. J. Mech. and Applied Mathem., **31**, 435, 1978.

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