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Body force effects on time-harmonic inhomogeneous waves

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THE EFFECTS of the body force on the propagation of time-harmonic waves in a viscoelastic (or viscous) fluid are examined. It turns out that inhomogeneous plane waves can exist subject to conditions on the complex-valued polarization and wavenumber. Such conditions are determined by having recourse to thermodynamic inequalities via a detailed analysis of transverse and longitudinal waves. Quantitative effects are also assessed.

Zbadano wpływ sił masowych na proces propagacji fal harmonicznych w płynie lepkosprężystym (lub lepkim). Okazuje się, że niejednorodne fale płaskie mogą istnieć przy założeniu zespolonych wartości polaryzacji i liczby falowej. Warunki takie określono opierając się na nierównościach termodynamicznych i analizując szczegółowo propagację fal poprzecznych i podłużnych. Przeprowadzono również ilościową analizę zjawiska.

Исследовано влияние массовых сил на процесс распространения гармонических волн в вязкоупругой (или в вязкой) жидкости. Оказывается, что неоднородные плоские волны могут существовать при предположении комплексных значений поляризации и волнового числа. Такие условия определены, опираясь на термодинамические неравенства и анализируя подробно распространение поперечных и продольных волн. Проведен тоже количественный анализ явления.

1. Introduction

USUALLY WAVE propagation problems, in solids and fluids, are investigated through-homogeneous systems of equations which result from disregarding the body force in mechanical contexts or charge and current sources in electromagnetic contexts [1-6]. Meanwhile the literature shows that, when dealing with scattering problems, the body force is explicitly taken into account [7]. Apart from consistency requirements, it seems of interest to investigate the effect of the body force on wave propagation.

Wave propagation induced (or affected) by the body force is considered in [8] in the case of discontinuity waves in elastic materials. A corresponding procedure would lead to analogous conclusions for time-harmonic waves. Motivated by the typical framework underlying scattering problems, we investigate in this paper time-harmonic wave propagation in dissipative fluids, namely viscoelastic or viscous fluids. Of course we examine in detail the effect of the body force and, eventually, we discuss whether and how the usual approximation of a vanishing body force is plausible.

Time-harmonic waves in lossy media are usually inhomogeneous. Accordingly we let the waves be inhomogeneous and develop the analysis by taking into account the thermodynamic restrictions on the material parameters. It is shown that transverse and longitudinal waves may occur. While transverse waves merely have a polarization orthogonal to the body force, longitudinal waves show more involved effects. Really, to obtain detailed results we consider longitudinal waves with a wave-number orthogonal to the body force.

As a general comment on body force effects, we say that the material symmetry associated with the stress tensor is broken, and then reduced, by the body force. This in turn amounts to a reduction of solutions (admissible waves). As we should have expected, though, quantitative effects prove to be negligible in standard conditions.

2. Linearized form of the equation of motion

Let $\mathscr{E}^3$ be the three-dimensional Euclidean space and $\mathscr{R}^* \subset \mathscr{E}^3$ a domain which is regarded as reference placement. The motion of the body is described by a function $\mathbf{x}' = \mathbf{x}'(\mathbf{X}, t)$, $\mathbf{X} \in \mathscr{R}^*$, which is a diffeomorphism for any time $t \in \mathsf{R}$. The motion $\mathbf{x}'$ maps $\mathscr{R}^*$ into the time-dependent region $\mathscr{R}'(t)$. Letting ϱ, $\mathbf{T}$, $\mathbf{b}$ be the mass density, the Cauchy stress tensor and the body force (per unit mass) in $\mathscr{R}'$, we can write the balance of mass and linear momentum in the local forms

$$\dot{\varrho}+\varrho\nabla'\cdot\dot{\mathbf{x}}' = 0, \tag{2.1}$$

$$\varrho\ddot{\mathbf{x}}'-\nabla'\cdot\mathbf{T}-\varrho\mathbf{b} = 0, \tag{2.2}$$

where ∇' denotes the gradient operator with respect to $\mathbf{x}'$ and a superposed dot the material time derivative. In a purely mechanical context, $\mathbf{b}$ coincides with the gravity acceleration $\mathbf{g}$ which may reasonably be regarded as a constant vector.

We assume that $\mathscr{R}'(t)$ is very close to a suitable region $\mathscr{R}$ for any time t. This is characterized by saying that the displacement

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{x}'(\mathbf{x}, t)-\mathbf{x}, \qquad \mathbf{x}\in\mathscr{R}$$

and its derivatives can be regarded as "small" quantities at any place $\mathbf{x}\in\mathscr{R}$ and time $t\in\mathsf{R}$. This suggests that we linearize Eqs. (2.1) and (2.2) with respect to $\mathbf{u}$. Though this can be done by paralleling a standard procedure for solids (cf., e. g., [9, 10]), it is convenient to examine the equations in a direct way.

Let $\mathscr{R}$ be an equilibrium configuration and assume that $\mathbf{T} = -p(\varrho)\mathbf{1}+\boldsymbol{\tau}$ with $\boldsymbol{\tau}$ vanishing at equilibrium. Denote by ϱ_0, $p_0 = p(\varrho_0)$, $\mathbf{b}_0$ the values of ϱ, p, $\mathbf{b}$ at $\mathscr{R}$ and let

$$\varrho = \varrho_0+\bar{\varrho}, \qquad \mathbf{b} = \mathbf{b}_0+\bar{\mathbf{b}}.$$

Letting $\nabla = \partial/\partial\mathbf{x}$ and $j = \det(\nabla\mathbf{x}')$, we observe that Eq. (2.1) is equivalent to

$$j\varrho = \varrho_0. \tag{2.1'}$$

Then multiplication of Eq. (2.2) by j gives

$$\varrho_0\ddot{\mathbf{u}}+j[\nabla p(\varrho_0+\bar{\varrho})]\frac{\partial\mathbf{x}}{\partial\mathbf{x}'}-j\nabla'\cdot\boldsymbol{\tau}-\varrho_0(\mathbf{b}_0+\bar{\mathbf{b}}) = 0. \tag{2.3}$$

Hence follows the equilibrium condition

$$\nabla p(\varrho_0)-\varrho_0\mathbf{b}_0 = 0. \tag{2.4}$$

Within linear terms in **u** we have

$$j = 1+\nabla\cdot\mathbf{u}, \qquad \varrho = \varrho_0(1-\nabla\cdot\mathbf{u}), \tag{2.5}$$

where Eq. (2.1′) has been used. Substitution in Eq. (2.3) and accounting for Eq. (2.4) yields

$$\varrho_0\ddot{\mathbf{u}}-\varrho_0\nabla\mathbf{u}\mathbf{b}_0-\frac{\varrho_0^2 p_{\varrho\varrho}}{p_\varrho}\mathbf{b}_0\nabla\cdot\mathbf{u}-\varrho_0 p_\varrho\nabla(\nabla\cdot\mathbf{u})-\nabla\cdot\boldsymbol{\tau}-\varrho_0\bar{\mathbf{b}} = 0, \tag{2.6}$$

where the derivatives p_ϱ, $p_{\varrho\varrho}$ are evaluated at ϱ_0. Incidentally, if b depends on $\mathbf{x}'$, we can write the linear approximation $\bar{\mathbf{b}}(\mathbf{x}') = \mathrm{b}(\mathbf{x})+(\mathbf{u}\cdot\nabla)\bar{\mathbf{b}}(\mathbf{x})$.

Henceforth we are dealing with dissipative fluids and we proceed by considering viscoelastic and viscous fluids at the same time. In both cases p is identified with the pressure. As to the viscoelastic fluid we write $\boldsymbol{\tau}$ in terms of $\mathbf{D} = \mathrm{sym}(\partial\dot{\mathbf{u}}/\partial\mathbf{x})$ as (cf. [11], 4.14.2)

$$\boldsymbol{\tau}(t) = \int_0^\infty 2\mu(s)\mathring{\mathbf{D}}(t-s)\,ds+\int_0^\infty \beta(s)(\mathrm{tr}\mathbf{D})(t-s)\,ds \tag{2.7}$$

a superposed ring denoting the traceless part. Here $\mu(s)$ and $\beta(s)$ are the shear and bulk relaxation functions. The dependence on **X**, or **x**, is understood and not written. The viscous fluid model can be viewed as the limit case when μ and β are Dirac's delta functions, namely $\mu(s) = \mu\delta(s)$, $\beta(s) = \beta\delta(s)$ whence

$$\boldsymbol{\tau} = 2\mu\mathring{\mathbf{D}}+\beta(\mathrm{tr}\mathbf{D})\mathbf{1}. \tag{2.8}$$

Let $\mathbf{u}, \bar{\mathbf{b}}$, and $\bar{\varrho}$, be time-harmonic; e. g.,

$$\mathbf{u}(\mathbf{x}, t) = \tilde{\mathbf{u}}(\mathbf{x})\exp(-i\omega t), \qquad \omega\in\mathbf{R},$$

a twiddle denoting the complex-valued amplitude; of course the physical displacement is the real part of **u**. Then substitution in Eq. (2.6) yields

$$\omega^2\varrho_0\tilde{\mathbf{u}}+\varrho_0(\nabla\tilde{\mathbf{u}})\mathbf{b}_0+\frac{\varrho_0^2 p_{\varrho\varrho}}{p_\varrho}\mathbf{b}_0\nabla\cdot\tilde{\mathbf{u}}+\hat{\mu}\Delta\tilde{\mathbf{u}}+(\hat{\lambda}+\hat{\mu})\nabla(\nabla\cdot\tilde{\mathbf{u}})+\varrho_0\bar{\mathbf{b}}\exp(i\omega t) = 0, \tag{2.9}$$

where

$$\hat{\mu} = -i\omega(\mu_c+i\mu_s), \qquad \hat{\lambda} = \varrho_0 p_\varrho-i\omega\left[\left(\beta_c-\frac{2}{3}\mu_c\right)+i\left(\beta_s-\frac{2}{3}\mu_s\right)\right] \tag{2.10}$$

the subscripts c, s denoting the half-range cosine and sine transforms, namely,

$$\mu_c(\omega) = \int_0^\infty \mu(\xi)\cos(\omega\xi)\,d\xi, \qquad \mu_s(\omega) = \int_0^\infty \mu(\xi)\sin(\omega\xi)\,d\xi.$$

The case of the viscous fluid is obtained by letting

$$\mu_c(\omega)\to\mu, \quad \mu_s(\omega)\to 0, \quad \beta_c(\omega)\to\beta, \quad \beta_s(\omega)\to 0.$$

As a preliminary step toward the analysis of the wave propagation properties, associated with Eq. (2.9), it is worth considering the thermodynamic restrictions on the constitutive quantities $\hat{\mu}$, $\hat{\lambda}$.

3. Thermodynamic restrictions

As to the linearly viscous fluid, the thermodynamic restrictions are well-known (cf. [12], §1.3.2) and consists in the inequalities

$$\mu > 0, \quad \beta > 0 \tag{3.1}$$

for the shear and the bulk viscosity coefficients. By Eqs. (2.10) we can write

$$\mathfrak{F}\left(\frac{\hat{\mu}}{\omega}\right) < 0, \quad \mathfrak{F}\left(\frac{\hat{\lambda}+\gamma\hat{\mu}}{\omega}\right) < 0, \quad \omega \in \mathsf{R} \tag{3.2}$$

for any real γ greater than 2/3.

As to the viscoelastic fluid, the problem is a little more complicated. By paralleling the procedure developed in [13], we start from the statement of the second law whereby, for any rate-of-strain tensor function $\mathbf{D}(t)$ and density $\varrho(t)$ which are periodic, with period $d > 0$, the inequality

$$\int_0^d \mathbf{T}(\varrho(t), \mathbf{D}^t) \cdot \dot{\mathbf{D}}(t)dt > 0 \tag{3.3}$$

holds at any point of the body. This means that, along a cycle, the body really dissipates.

Observe that, by Eq. (2.1), we have

$$\varrho(t) = \varrho(t_0)\exp\Big(-\int_{t_0}^{t} \mathrm{tr}\mathbf{D}(s)ds\Big)$$

and then if $\mathbf{D}$ is periodic, then ϱ as well is periodic. Moreover,

$$-p(\varrho)\mathrm{tr}\mathbf{D} = \frac{d}{dt}h(\varrho),$$

where h is the integral of $p(\varrho)/\varrho$. Hence, if ϱ is periodic with period d, then the integral over $[0, d]$ of $p(\varrho)\mathrm{tr}\mathbf{D}$ vanishes.

Choose the function $\mathbf{D}(t)\cdot \mathsf{R} \to \mathrm{Sym}$ as

$$\mathbf{D}(t) = \mathbf{D}_\omega \sin\omega t, \quad \omega > 0, \quad \mathbf{D}_\omega \in \mathrm{Sym}.$$

Since $\mathbf{D}(t)$ is periodic, it follows that

$$\int_0^d \mathbf{T}(\varrho(t), \mathbf{D}^t) \cdot \dot{\mathbf{D}}(t)dt = \mu_c(\omega)\mathring{\mathbf{D}}_\omega \cdot \mathring{\mathbf{D}}_\omega + \frac{1}{2}\beta_c(\omega)(\mathrm{tr}\mathbf{D}_\omega)^2.$$

By the arbitrariness of $\mathring{\mathbf{D}}_\omega$ and $\mathrm{tr}\mathbf{D}_\omega$ the condition (3.3) yields

$$\mu_c(\omega) > 0, \quad \beta_c(\omega) > 0, \quad \omega > 0. \tag{3.4}$$

These results imply that the inequalities (3.2) hold for the viscoelastic fluid as well. As we expect it to be, when $\mu(s) = \mu\delta(s)$, $\beta(s) = \beta\delta(s)$ then the inequalities (3.4) reduce to the inequalities (3.1).

Thermodynamics does not place any restriction on $\mu_s(\omega)$ and $\beta_s(\omega)$ for viscoelastic fluids. However, if $\mu(s)$ is (positive and) monotone decreasing, then $\mu_s(\omega) > 0$ as $\omega > 0$.

Accordingly it is reasonable to take it that

$$\Re\left(\frac{\hat{\mu}}{\omega}\right) > 0, \quad \Re\left(\frac{\hat{\lambda}+\gamma\hat{\mu}}{\omega}\right) > 0, \quad \omega \in \mathbf{R}, \tag{3.5}$$

where $\gamma > 2/3$.

4. Propagation condition

Return to Eq. (2.9) and search for plane wave solutions in the form

$$\tilde{\mathbf{u}} = \mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{x}).$$

The wave-number $\mathbf{k}$, as well as the polarization vector $\mathbf{p}$, are allowed to be complex vectors. The subscripts 1 and 2 denote the real and the imaginary parts; i.e., $\mathbf{k}_1 = \Re\mathbf{k}$, $\mathbf{k}_2 = \Im\mathbf{k}$. Plane wave solutions are allowed only if $\bar{\mathbf{b}} = 0$, which requires the reasonable approximation that the gravity acceleration be uniform in the domain under consideration. Accordingly, upon substitution Eq. (2.9) yields the propagation condition

$$[\varrho_0\omega^2 - \hat{\mu}\mathbf{k}\cdot\mathbf{k}]\mathbf{p} - [(\hat{\lambda}+\hat{\mu})\mathbf{k}\cdot\mathbf{p} - i\varrho_0\mathbf{b}_0\cdot\mathbf{p}]\mathbf{k} + i\frac{\varrho_0^2 p_{\varrho\varrho}}{p_\varrho}(\mathbf{k}\cdot\mathbf{p})\mathbf{b}_0 = 0 \tag{4.1}$$

which may be viewed as a relation between $\mathbf{p}$ and $\mathbf{k}$. We look for solutions in the form $\mathbf{p} = \mathbf{p}(\mathbf{k})$ which, of course, are affected by the body force $\mathbf{b}_0$. For later convenience it is worth emphasizing that the linearity with respect to $\mathbf{p}$ makes the solution determined up to a complex factor.

Strictly speaking, the polarization $\mathbf{p}$ and the wave-number $\mathbf{k}$ cannot depend on $\mathbf{x}$ and hence Eq. (4.1) rules out plane wave solutions in that ϱ_0 is a function of $\mathbf{x}$. Now, by Eq. (2.4) we have $\varrho_0/|\nabla\varrho_0| = p_\varrho/b_0$. Then, in water, and similarly in solids and other fluids, $\varrho_0/|\nabla\varrho_0| \simeq 10^6$ meters. This makes it highly plausible to regard the material quantities in Eq. (4.1) as constant.

To determine solutions $\mathbf{p} = \mathbf{p}(\mathbf{k})$ to Eq. (4.1), we examine separately the case when $\mathbf{k}\cdot\mathbf{p}$ vanishes and the one when $\mathbf{k}\cdot\mathbf{p}$ does not.

I. $\mathbf{k}\cdot\mathbf{p} = 0$

It follows from Eq. (4.1) that

$$[\varrho_0\omega^2 - \hat{\mu}\mathbf{k}\cdot\mathbf{k}]\mathbf{p} + i\varrho_0(\mathbf{b}_0\cdot\mathbf{p})\mathbf{k} = 0 \tag{4.2}$$

whence

$$\mathbf{k}\cdot\mathbf{k} = \frac{\varrho_0\omega^2}{\hat{\mu}}, \tag{4.3}$$

$$\mathbf{b}_0\cdot\mathbf{p} = 0. \tag{4.4}$$

Quite naturally we can regard this solution as a transverse wave.

The converse also holds, in the sense that any inhomogeneous wave satisfying Eq. (4.3) is necessarily transverse. In fact inner multiplication of Eq. (4.1) by $\mathbf{p}$ and use of Eq. (4.3) yield

$$\mathbf{k}\cdot\mathbf{p}\left[-(\hat{\lambda}+\hat{\mu})\mathbf{k}\cdot\mathbf{p} + i\varrho_0\left(1+\varrho_0\frac{p_{\varrho\varrho}}{p_\varrho}\right)\mathbf{b}_0\cdot\mathbf{p}\right] = 0,$$

whence it follows that either

$$\mathbf{k}\cdot\mathbf{p} = 0$$

or

$$\mathbf{b}_0\cdot\mathbf{p} = \frac{\hat{\lambda}+\hat{\mu}}{i\varrho_0\left(1+\varrho_0\dfrac{p_{\varrho\varrho}}{p_\varrho}\right)}\mathbf{k}\cdot\mathbf{p}.$$

In the former case, substitution into Eq. (4.1) yields Eq. (4.4) again. In the latter case it is found that Eq. (4.1) reduces to

$$-(\hat{\lambda}+\hat{\mu})\mathbf{k}+i\varrho_0\left(1+\varrho_0\frac{p_{\varrho\varrho}}{p_\varrho}\right)\mathbf{b}_0 = 0.$$

Scalar multiplication by $\mathbf{k}$ and $\mathbf{b}_0$, and comparison of the results leads to the requirement

$$\left(i\varrho_0\frac{1+\varrho_0\dfrac{p_{\varrho\varrho}}{p_\varrho}}{\hat{\lambda}+\hat{\mu}}\right)^2 b_0^2 = \frac{\varrho_0\omega^2}{\hat{\mu}},$$

which generally does not hold. Then we conclude that only $\mathbf{k}\cdot\mathbf{p} = 0$ is allowed and then the wave is transverse.

Two possibilities occur according as $\mathbf{k}$ is parallel to $\mathbf{b}_0$, or not. If $\mathbf{k}\times\mathbf{b}_0 = 0$, then any $\mathbf{p}$ orthogonal to $\mathbf{b}_0$ satisfies Eq. (4.4) — and (I) as well. If $\mathbf{k}\times\mathbf{b}_0 \neq 0$, then Eq. (4.4) and (I) yield

$$\mathbf{p} = \zeta\mathbf{k}\times\mathbf{b}_0$$

where ζ is any complex number.

As a comment to Eq. (4.3), we can say that, whenever $\mathbf{k}\cdot\mathbf{p} = 0$, the wave-number $\mathbf{k}$ is unaffected by $\mathbf{b}_0$. Yet, by Eq. (4.4) the wave exists only if $\mathbf{p}$ and $\mathbf{b}_0$ are orthogonal.

II. $\mathbf{k}\cdot\mathbf{p}\neq 0$

Inner multiplication of Eq. (4.1) for $\mathbf{k}$ and $\mathbf{b}_0$ yields two equations which may be viewed as a linear homogeneous system in the unknowns $\mathbf{k}\cdot\mathbf{p}$, $\mathbf{b}_0\cdot\mathbf{p}$. The determinantal equation, which allows nontrivial solutions, is

$$\det\begin{pmatrix} \varrho_0\omega^2-(\hat{\lambda}+2\hat{\mu})\mathbf{k}\cdot\mathbf{k}+i\dfrac{p_{\varrho\varrho}}{p_\varrho}\varrho_0^2\mathbf{b}_0\cdot\mathbf{k} & i\varrho_0\mathbf{k}\cdot\mathbf{k} \\ i\dfrac{p_{\varrho\varrho}}{p_\varrho}\varrho_0^2 b_0^2-(\hat{\lambda}+\hat{\mu})\mathbf{b}_0\cdot\mathbf{k} & \varrho_0\omega^2-\hat{\mu}\mathbf{k}\cdot\mathbf{k}+i\varrho_0\mathbf{b}_0\cdot\mathbf{k}\end{pmatrix} = 0. \tag{4.5}$$

In consistency with the general framework, we regard the unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$ of $\mathbf{k}_1$, $\mathbf{k}_2$ as given and then Eq. (4.5) becomes a system of two equations in the unknowns k_1, k_2. Once k_1 and k_2, and then $\mathbf{k}$, are determined, we can find $\mathbf{b}_0\cdot\mathbf{p}$ in terms of $\mathbf{k}\cdot\mathbf{p}$, namely.

$$\mathbf{b}_0\cdot\mathbf{p} = A\mathbf{k}\cdot\mathbf{p},$$

where

$$A = -\frac{\varrho_0\omega^2-(\hat{\lambda}+2\hat{\mu})\mathbf{k}\cdot\mathbf{k}+i\dfrac{p_{\varrho\varrho}}{p_\varrho}\varrho_0^2\mathbf{b}_0\cdot\mathbf{k}}{i\varrho_0\mathbf{k}\cdot\mathbf{k}}.$$

Substitution in Eq. (4.1) provides

$$\mathbf{p} = \frac{\mathbf{k}\cdot\mathbf{p}}{\varrho_0\omega^2-\hat{\mu}\mathbf{k}\cdot\mathbf{k}}\left[(\hat{\lambda}+\hat{\mu}-i\varrho_0 A)\mathbf{k}-i\frac{\varrho_0^2 p_{\varrho\varrho}}{p_\varrho}\mathbf{b}_0\right]. \tag{4.6}$$

Due to the arbitrariness in $\mathbf{p}$ we can always assume $\mathbf{k}\cdot\mathbf{p} = 1$ and then Eq. (4.6) is the desired relation $\mathbf{p} = \mathbf{p}(\mathbf{k})$.

As motivated in the next section, we regard this solution as a longitudinal wave.

5. Wave modes

In this section we investigate the wave modes and show how they are related to the standard modes in nondissipative media and are affected by the body force.

Transverse waves. Since the subscripts 1 and 2 denote the real and imaginary parts, by Eq. (4.3) we have

$$k_1^2-k_2^2 = \frac{\varrho_0\omega^2\hat{\mu}_1}{\hat{\mu}_1^2+\hat{\mu}_2^2}, \tag{5.1}$$

$$2\mathbf{k}_1\cdot\mathbf{k}_2 = -\frac{\varrho_0\omega^2\hat{\mu}_2}{\hat{\mu}_1^2+\hat{\mu}_2^2}. \tag{5.2}$$

If $\omega > 0$, by the inequality $(3.2)_1$ we conclude that the right-hand side of Eq. (5.2) is positive. Then $\mathbf{k}_1\cdot\mathbf{k}_2 > 0$ whereby the wave amplitude decays while propagating. This shows the direct connection between thermodynamics and wave damping. If, instead, $\omega < 0$, then the same conclusion is reached by observing that the wave propagates along the direction $-\mathbf{k}_1$. Accordingly, we consider the angle θ between $\mathbf{k}_1$ and $\mathbf{k}_2$, $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, and write Eq. (5.2) as

$$k_1k_2 = -\frac{1}{\cos\theta}\,\frac{\varrho_0\omega^2\hat{\mu}_2}{2(\hat{\mu}_1^2+\hat{\mu}_2^2)}. \tag{5.3}$$

As to Eq. (5.1), observe that $\hat{\mu}_1 = 0$ in viscous fluids; in such a case $k_1 = k_2$. By the inequalities (3.5), for viscoelastic fluids $\mu_1 > 0$ and then $k_1 > k_2$. In either case, the moduli k_1, k_2 are expressed in terms of the parameter θ. This is a general feature of inhomogeneous waves [14]. To determine the value of θ ,we need information about how the waves are generated [15].

As to the polarization $\mathbf{p}$, by definition it satisfies

$$\mathbf{k}\cdot\mathbf{p} = 0.$$

Although this can be viewed as the orthogonality condition, it is worth recalling that in general $\mathbf{k}$ and $\mathbf{p}$ are complex-valued and then $\mathbf{k}$ and $\mathbf{p}$ are not orthogonal in the geometrical sense.

Longitudinal waves. We expect that solutions exist as $\cos\theta \geqslant 0$. However, to determine analytically k_1 and k_2 from Eq. (4.5) and to show the compatibility of $\cos\theta \geqslant 0$ with $k_1, k_2 > 0$ is a formidable task. Rather, to find explicit, analytical results we prefer to examine the particular case when $\mathbf{k}$ is orthogonal to the body force $\mathbf{b}_0$ (and then $\mathbf{k}_1$, $\mathbf{k}_2$ horizontal if $\mathbf{b}_0$ is the gravity acceleration). In such a case Eq. (4.5) yields

$$[\varrho_0\omega^2-(\hat{\lambda}+2\hat{\mu})\mathbf{k}\cdot\mathbf{k}][\varrho_0\omega^2-\hat{\mu}\mathbf{k}\cdot\mathbf{k}]+\varepsilon\mathbf{k}\cdot\mathbf{k} = 0, \tag{5.4}$$

where $\varepsilon = \varrho_0^3 b_0^2 p_{\varrho\varrho}/p_\varrho$. Incidentally, by the previous proof, we know that in general $\varrho_0\omega^2 - \hat{\mu}\mathbf{k}\cdot\mathbf{k} \neq 0$ because $\mathbf{k}\cdot\mathbf{p} \neq 0$. Then we look for solutions to Eq. (5.4) with two conditions. First, we require that the solution $\mathbf{k}\cdot\mathbf{k}$ to Eq. (5.4) reduce to $\varrho_0\omega^2/(2\hat{\mu}+\hat{\lambda})$ as $\varepsilon = 0$, namely the known solution for longitudinal waves when the body force is disregarded. Second, we let ε be small enough so that the sign chosen for the root when $\varepsilon = 0$ is maintained for any ε. Accordingly, by Eq. (5.4) we obtain

$$\mathbf{k}\cdot\mathbf{k} = \frac{\varrho_0\omega^2}{2\hat{\mu}+\hat{\lambda}}\left[1+\frac{\hat{\mu}+\hat{\lambda}}{2\hat{\mu}}\left(1-\sqrt{1-\frac{\dfrac{2(3\hat{\mu}+\hat{\lambda})}{\varrho_0\omega^2}\varepsilon-\dfrac{1}{\varrho_0^2\omega^4}\varepsilon^2}{(\hat{\mu}+\hat{\lambda})^2}}\right)\right]. \tag{5.5}$$

At the linear approximation in ε we have

$$\mathbf{k}\cdot\mathbf{k} = \frac{\varrho_0\omega^2}{2\hat{\mu}+\hat{\lambda}}\left[1+\frac{3\hat{\mu}+\hat{\lambda}}{2\hat{\mu}(\hat{\mu}+\hat{\lambda})\varrho_0\omega^2}\varepsilon\right].$$

Let $\omega > 0$. By the inequality $(3.2)_2$ it follows that, as the contribution of ε is small enough, $\cos\theta > 0$ whereby the wave amplitude decreases while the wave propagates. The same conclusion follows when $\omega < 0$.

The effect of the body force on the wave-number shows up only in connection with longitudinal waves. To have an estimate of such an effect we consider Eq. (5.5) and let $\hat{\lambda}$ and $\hat{\mu}$ be of the same order of magnitude. We conclude that the effect is negligible if $|\varepsilon/\varrho_0\omega^2\hat{\mu}| \ll 1$, namely

$$\left|\frac{\varrho_0^2 b_0^2 p_{\varrho\varrho}}{p_\varrho\hat{\mu}\omega^2}\right| \ll 1. \tag{5.6}$$

To fix ideas look at water, viewed as a viscous fluid, at 1 atmosphere and 20°C. We have

$$\mu = i\frac{\hat{\mu}}{\omega} = 10^{-2} \text{ g/cm s}.$$

As to the constitutive equation $p = p(\varrho)$ we consider the modified form of the Tait equation along the adiabatic passing through 1 atmosphere and 20°C [16]. It follows that

$$\frac{\varrho_0 p_{\varrho\varrho}}{p_\varrho} = 6.$$

Let

$$\omega_c = \left(\frac{\varrho_0^2 b_0^2 p_{\varrho\varrho}}{\mu p_\varrho}\right)^{1/3}.$$

Then the inequality (5.6) holds for

$$\omega \gg \omega_c \simeq 8.33\ 10^2 \text{ s}^{-1}.$$

This means that, quantitatively, in water the effect of the body force is negligible for any reasonable value of the angular frequency ω in wave propagation experiments. The same conclusion holds for other common fluids. Of course the higher is the viscosity ocefficient, the lower is the critical angular frequency ω_c.

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References

1. L. M. BREKHOVSKIKH, *Waves in layered media*, Academic Press, New York 1980.
2. J. A. STRATTON, *Electromagnetic theory*, McGraw-Hill, New York 1941.
3. M. A. HAYES and R. S. RIVLIN, *Longitudinal waves in a linear viscoelastic material*, ZAMP, **23**, 153–157, 1972.
4. J. D. ACHENBACH, *Wave propagation in elastic solids*, North Holland, Amsterdam 1975.
5. M. A. HAYES and R. S. RIVLIN, *Propagation of sinusoidal small-amplitude waves in a deformed visco-elastic solid, I*, J. Acoust. Soc. Am., **46**, 610–616, 1969.
6. M. F. MCCARTHY and M. A. HAYES [Eds.], *Elastic wave propagation*, North-Holland, Amsterdam 1989.
7. Y. H. PAO and V. VARATHARAJULU, *Huygens' principle, radiation conditions, and integral formulas for the scattering of elastic waves*, J. Acoust. Soc. Am., **59**, 1361–1371, 1976.
8. M. E. GURTIN and E. K. WALSH, *Extrinsically induced acceleration waves in elastic bodies*, J. Acoust. Soc. Am., **41**, 1320–1324, 1967.
9. A. C. ERINGEN and E. S. ŞUHUBI, *Elastodynamics*, vol. 1, Academic Press, New York 1974.
10. D. IESAN, *Prestressed bodies*, Longman, Burnt Mill 1989.
11. E. H. DILL, in: Continuum Physics, A. C. ERINGEN [ed.], Academic Press, New York 1975.
12. I. MÜLLER, *Thermodynamics*, Pitman, Boston 1985.
13. M. FABRIZIO and A. MORRO, *Viscoelastic relaxation functions compatible with thermodynamics*, J. Elasticity, **19**, 63–75, 1988.
14. G. G. CAVIGLIA, A. MORRO and E. PAGANI, *Inhomogeneous waves in viscoelastic media*, Wave Motion, 12, 143-159, 1990.
15. G. CAVIGLIA and A. MORRO, *Inhomogeneous waves and sound absorption in viscous fluids*, Phys. Lett., A **134**, 127–130, 1988.
16. J. G. KIRKWOOD and H. A. BETHE, *Progress report on "The pressure wave produced by an underwater explosion I"*, in: Shock and Detonation Waves, W. W. WOOD [ed.], Gordon and Breach, New York 1967.

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