ERRATA

Growth of voids in a ductile matrix: a review

P. GILORMINI, C. LICHT and P. SUQUET, *Arch. Mech.* **40**, pp. 43–80, 1988

Since all the corrections indicated in the proofs of the paper could not be included by the editor, the authors wish to point out the following errata to the reader. (Some additional minor misprints are not listed, for the sake of brevity.)

Page 45: lines 13 to 30 must be moved to page 44, above *1.2. Damage and micromechanics*, and n_j should be changed into n_i in Eq, (1.3).

Page 47: the last line above *2. Isolated Voids* should be read as follows: *ties, by J. Rice, B. Budiansky, J. W. Hutchinson, A. Needleman, and their coworkers*. A comma should replace the full stop at the end of 6[th] line from the bottom.

Page 48: a *1* is missing between *the* and (Σ_{11}], and a *2* has been omitted between *or* and ($\Sigma_{22} >$ in line 17.

Page 49: remove the *a* before generalized in line 11.

Page 53: change *is* into *was* in line 8, and $\in$ into $\dot{\varepsilon}$ in Eq. (2.5).

Page 57: change *1*); *for* into *1*), *for* in line 6; the beginning of line 21 should be read as *to which a viod will tend has been* etc.; replace *ligh* by *high* in line 23.

Page 61, Eq. (3.4): read $s' = sl(s-l)$ instead of $s' = s(s-l)$.

Page 63: change *0.01,0.4* into *0.01 − 0.4* in line 7.

Page 66: change *in* into *into* (line 17), remove the comma after *i*) *approximate expressions* (line 16), and add one after *uniform strains* in the footnote.

Page 67: read $\dot{\tilde{\mathbf{u}}}^*$ instead of $\dot{\mathbf{u}}^*$ in the 2[nd] equation and instead of $*\dot{\tilde{\mathbf{u}}}*$ in the 5[th].

Page 68: read $s' = s/(s-l)$ instead of $s' = s(s-l)$ in the 4[th] equation, close the parenthesis in the 6[th], and change), into)), in the last line.

Page 69: $c_2 = -\frac{3}{2}^{\frac{s}{s-1}}$ should be replaced by $c_2 = -\frac{3}{2}G^{\frac{1}{s-1}}$.

Page 71: substitute $\dot{\tilde{E}}$ to $\tilde{E}_{\alpha\beta}$ two lines below Eq. (6.19).

Page 77: $\frac{2\mu}{\sigma^2}$ must be replaced by $\frac{2\mu}{\rho^2}$ in line 15, and E_{33} by $\dot{E}_{33}$ at the bottom of the page.

Page 79: substitute *dilatation* to *dilation* in ref. 3, *solids* to *solid* in ref. 5, and *composites* to *composities* in ref. [11].

Page 80: substitute *voids or inclusions* to *voids inclusions* in ref. [17].

Arch. Mech., 41, 1, pp. 61–71, Warszawa 1989

Shakedown of shell-like structures allowing for certain geometrical nonlinearities

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To ANSWER the question whether elastic-plastic structures will adapt to variable loading histories, extensions of the classical shakedown theorems by Melan and Koiter to a certain class of geometrically nonlinear problems are given. Material hardening is taken into account by using the concept of hidden variables. It is shown how the extended theorems fit with existing plate- and shell theories to predict their long-time behaviour. Focussing on the extended Melan's theorem, the method is illustrated by a numerical example.

Aby odpowiedzieć na pytanie, czy konstrukcje sprężysto-plastyczne podlegają przystosowaniu do zmiennych historii obciążenia, wprowadzono pewne rozszerzenie klasycznych twierdzeń Melana i Koitera o przystosowaniu, pozwalające uwzględnić pewną klasę zagadnień geometrycznie nieliniowych. Dzięki koncepcji zmiennych ukrytych uwzględniono efekt wzmocnienia. Pokazano w jaki sposób te rozszerzone twierdzenia pasują do istniejących teorii płyt i powłok, pozwalając zarazem przewidywać ich zachowanie się przy długotrwałych próbach obciążenia. Opierając się na rozszerzonym twierdzeniu Melana zilustrowano metodę postępowania przykładem liczbowym.

Чтобы ответить на вопрос подлежат ли упруго-пластические конструкции приспособлению к переменным историям нагружения, выведено некоторое расширение классических теорем Мелана и Койтера о приспособлении, позволяющих учитывать некоторый класс геометрически нелинейных проблем. Благодаря концепции неявных переменных учтен эффект упрочнения. Показано каким образом эти расширенные теоремы подходят к существующим теориям плит и оболочек, позволяя одновременно предсказывать их поведение при продолжительных испытаниях нагружения. Опираясь на расширенную теорему Мелана, метод иллюстрирован числовым примером.

1. Introduction

THE CLASSICAL shakedown theorems by MELAN [1,2] and KOITER [3] provide methods to predict whether elastic-plastic bodies may or may not fail due to the unlimited accumulation of plastic strains during their loading history. These methods have found broad application and can nowadays be considered as standard tools in the design of structures [4–6]. A drawback, however, is the fact, that they have been derived within the framework of geometrical linearity, so that the influence of progressive changes of the shape of the structure during the deformation process cannot be taken into account. In many cases this may be without any major importance, in other cases, as for example in the assessment of metal plate- and shell-structures, theoretical predictions may be poor compared to experimental results. The first to tackle this problem was MAIER [7] who started from an *a priori* discretized description of structures and gave a criterion for their shakedown, taking so-called "second-order effects" into account and using piecewise linear yield conditions. KÖNIG investigated in several papers [8, 9] the influence of geometrical effects on the stability of the deformation process for particular structures under certain assumptions on the deformation modes; in a similar way, NGUYEN QUOC SON and GARY [10, 11] studied the possibility of destabilization of the shakedown-process due to successive plastic deformations. In several papers [12–14] the authors discussed the shakedown-problem within the framework of geometrically nonlinear continuum mechanics and gave an extension of Melan's theorem applicable to situations where information about the expected deformation pattern is available. Because of the assumed additivity of the purely elastic and purely plastic part of the strain measure, the proposed method is particularly suited for shell-like structures undergoing moderate rotations at small strains [13, 18].

In this paper Melan's and Koiter's shakedown theorems are reviewed in the light of geometrically nonlinear continuum mechanics. Material hardening is taken into account by using the concept of internal ("hidden") variables in the form as developed by HALPHEN and NGUYEN QUOC SON [19] and applied first by MANDEL [20, 21] in the context of the geometrically linear shakedown-theory. Finally it is shown how the extended theorem by Melan can be applied to shell-like structures and a numerical example is presented comparing the shakedown-domains of a cylindrical shell for hardening and ideal-plastic material behaviour.

2. Basic relations

The quasi-static motion of a body $\mathscr{B}$ with the initial volume $\mathscr{V}$ is described by the Cartesian coordinates $[x_1, x_2, x_3]$ of its material points which have at the beginning of the process at time $\tau = 0$ the values $[X_1, X_2, X_3]$ used throughout the paper for reference. The deformation of $\mathscr{B}$ can then be described by the displacement vector $\mathbf{u}$, the deformation gradient $\mathbf{F}$ and the Green–Lagrange strain tensor $\mathbf{E}$, defined by their components

$$u_i = x_i(\mathbf{X}) - X_i, \tag{2.1}$$

$$F_{ij} = \delta_{ij} + u_{i,j}, \tag{2.2}$$

$$2E_{ij} = F_{ki}F_{kj} - \delta_{ij}, \tag{2.3}$$

respectively. Latin indices run from 1 to 3 if not stated otherwise; summation convention over repeated indices is adopted. Here $\boldsymbol{\delta}$ is the Kronecker symbol and the comma denotes the partial derivative of the considered quantity with respect to the coordinate following the comma. We assume that the surface $\mathscr{S}$ of $\mathscr{B}$ consists of disjoint parts $\mathscr{S}_F$ and $\mathscr{S}_K$, where the distributed forces $\mathbf{f}^*$ and displacements $\mathbf{u}^*$ are prescribed respectively. With $\mathbf{t}$ as the first Piola–Kirchhoff stress tensor, $\mathbf{p}^*$ as the given voluminal force and $\mathbf{n}$ as the outer normal vector on $\mathscr{S}$, the equilibrium conditions in $\mathscr{V}$, the statical and kinematical boundary conditions are given by

$$t_{ji,j} = -p_i^* \quad \text{in } \mathscr{V}, \tag{2.4}$$

$$n_j t_{ji} = f_i^* \quad \text{on } \mathscr{S}_F, \tag{2.5}$$

$$u_i = u_i^* \quad \text{on } \mathscr{S}_K, \tag{2.6}$$

respectively. For the constitutive law we assume that the total strains $\mathbf{E}$ can be additively decomposed into a purely elastic part $\mathbf{E}^e$ and a purely plastic part $\mathbf{E}^p$ so that

$$\mathbf{E} = \mathbf{E}^e + \mathbf{E}^p. \tag{2.7}$$

This decomposition, not valid for arbitrary deformation processes, has been shown in [18] to be justified within given error bounds in particular situations, including the case of moderate rotations accompanied by small total strains. This is of special relevance to the investigation of shell-like bodies.

Linear kinematical hardening is taken into account by using internal parameters according to the concept of "generalized standard material" developed in [19] and applied, e.g., in [14, 20–23]. For this, generalized stresses $\mathbf{s} = [\boldsymbol{\sigma}, \boldsymbol{\pi}]$, generalized elastic strains $\mathbf{e}^e = [\mathbf{E}^e, \boldsymbol{\omega}]$ and generalized plastic strains $\mathbf{e}^p = [\mathbf{E}^p, \boldsymbol{\varkappa}]$ are introduced. Here, $\boldsymbol{\sigma}$ is the second Piola–Kirchhoff stress tensor which is related to the first Piola–Kirchhoff stress tensor $\mathbf{t}$ by $F_{ki}\,\sigma_{kj} = t_{ij}$. Then the codimensional vectors $\boldsymbol{\pi}$, $\boldsymbol{\omega}$ and $\boldsymbol{\varkappa}$ of internal ("hidden") parameters describe the actual state of hardening (see also Fig. 1 as the symbolical representation of a material element consisting of sliding-elements (dry friction) and springs to illustrate the role of internal parameters).

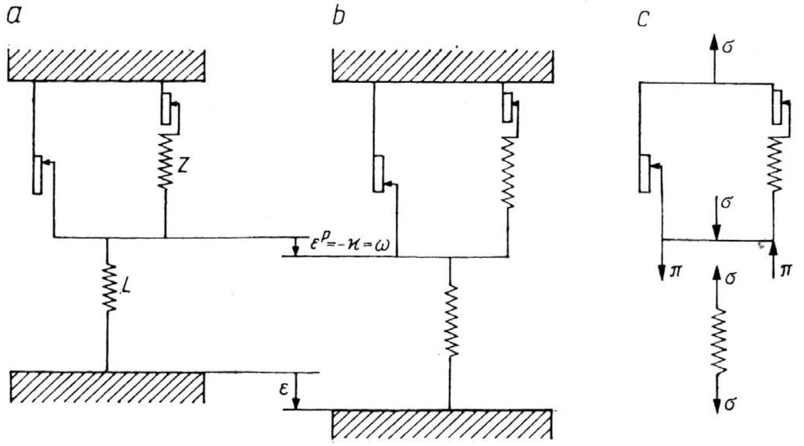

FIG. 1. One-dimensional representation of the material model: (a) undeformed state, (b) kinematical quantities in the deformed state, (c) statical quantities in the deformed state.

In particular, we assume for the elastic part of the material law the linear relationship $\mathbf{e}^e = \mathscr{L} : \mathbf{s}$, or, in index notation,

$$[e^e_{ij}, \omega_m] = [L_{ijkl}\,\sigma_{kl}, Z_{mn}\pi_n], \qquad m, n = 1, 2, \ldots, r. \tag{2.8}$$

The number of internal variables is denoted by r; $\mathbf{L}$ and $\mathbf{Z}$ are constant, positive definite and symmetric material tensors; for ideal plastic material Z_{mn} is equal to zero. By definition we have $\dot{\boldsymbol{\omega}} + \dot{\boldsymbol{\varkappa}} = \mathbf{0}$, so that for processes starting from the virgin state we get $\boldsymbol{\omega} + \boldsymbol{\pi} = \mathbf{0}$ [14, 22, 23].

For the plastic part of the material behaviour we assume the existence of a convex and fixed yield surface $\mathscr{F}$ in the space of generalized stresses $\mathbf{s}$ with

$$\mathscr{F}(\mathbf{s}) \leqslant 0 \tag{2.9}$$

for all physically admissible states of stress. Convexity and the normality rule can then be expressed by the condition $(\mathbf{s} - \hat{\mathbf{s}}) : \dot{\mathbf{e}}^p \geqslant 0$, or, in index notation,

$$(\sigma_{ij} - \hat{\sigma}_{ij})\dot{E}^p_{ij} + (\pi_m - \hat{\pi}_m)\dot{\varkappa}_m \geqslant 0. \tag{2.10}$$

Here $\mathbf{s}$ and $\dot{\mathbf{e}}^p$ denote the true state of generalized stresses and generalized plastic strain rates, respectively, whereas $\hat{\mathbf{s}}$ is any generalized stress field satisfying the inequality (2.9).

We note that in the context of the shakedown theory, kinematical hardening has first been discussed by MELAN [2]. He used Prager's hardening rule and gave a criterion for shakedown under the assumption of unlimited hardening. In recent years this concept has been used by several authors ([24], see also [6]) and the introduction of piecewise linear yieldconditions made problems of this type accessible to methods of linear programming (e.g., [7]). The assumption of unlimited hardening, common to all aforementioned papers, has the advantage that time-independent residual stress fields used in the proof of Melan's theorem and in its numerical application do not have to fulfill any requirement of static admissibility. From the physical point of view, however, it seems to be more realistic to consider an upper limit for hardening as, otherwise, certain loading cases would lead to an unbounded loading capacity and only failure due to alternating plasticity can be detected [6]. This problem can be tackled by checking for limited ductility and imposing limitations on relevant parameters of plastic deformation [25]. Consequently this requires the computation of strains, e.g., by a step-by-step method. The method presented herein includes the limitation of hardening by imposing limits on the internal ("hidden") parameters $\boldsymbol{\pi}$. Practically this can be interpreted as a simple two-surface model for plastic behaviour [15, 16] where the limitation of internal parameters $\boldsymbol{\pi}$ is equivalent to the assumption of a fixed loading surface [15–17] and the calculation of strains can be avoided.

3. The extended shakedown theorems

We assume that at a fixed time τ^R the body $\mathscr{B}$ has already undergone deformations with finite displacements with respect to the initial configuration at time $\tau = 0$ so that $\mathscr{B}$ is at time τ^R in the reference configuration Ω^R in quasi-static and stable equilibrium with the external agencies $\mathbf{a}^R$, consisting of the prescribed loads and surface displacements. In the sequel we restrict our considerations to loading histories characterized by the motion of a fictitious comparison body $\mathscr{B}^0$, having at τ^R the same field quantities as $\mathscr{B}$ but reacting, in contrast to $\mathscr{B}$, purely elastically to the additional external loads $\Delta\mathbf{a}$, superimposed on $\mathbf{a}^R$ for $\tau > \tau^R$. Namely we assume that $\mathscr{B}^0$ would perform under the action of $\Delta\mathbf{a}$ motions $\Delta\mathbf{u}^0$ in the vicinity of Ω^R, small in the sense that $\Delta u^0_{i,j} \ll F^R_{ij}$ where $\mathbf{F}^R$ is the deformation gradient of $\mathscr{B}$ in the configuration $\mathbf{\Omega}^R$ (see Fig. 2). Then the following extension of Melan's theorem holds:

If there exists a time-independent field of generalized residual stresses $\bar{\mathbf{s}}^\varrho = [\bar{\boldsymbol{\sigma}}^\varrho, \bar{\boldsymbol{\pi}}^\varrho]$ satisfying the conditions

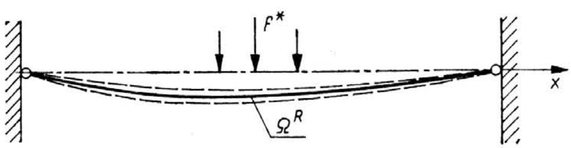

FIG. 2. Motion in the vicinity of a reference configuration.

$$(3.1)\qquad \begin{aligned}(F^R_{kj}\bar{\sigma}^{\varrho}_{ki})_{,j} &= 0 \quad \text{in } \mathscr{V},\\ n_j(F^R_{kj}\bar{\sigma}^{\varrho}_{ki}) &= 0 \quad \text{on } \mathscr{S}_F\end{aligned}$$

and

$$(3.2)\qquad \mathscr{F}(\mathbf{s}^R+\alpha\Delta\mathbf{s}^0+\bar{\mathbf{s}}^{\varrho}) \leqslant 0, \quad \alpha > 1,$$

where $\mathbf{s}^R$ is the state of generalized stresses in the reference-configuration and $\Delta\mathbf{s}^0$ is the time-dependent purely elastic response of $\mathscr{B}^0$ to $\Delta\mathbf{a}$, then the original body $\mathscr{B}$ will shake down.

For the proof given in [12] for an elastic-ideal plastic material the non-negative form

$$(3.3)\qquad W(\mathbf{s}^{\varrho}-\bar{\mathbf{s}}^{\varrho}) = (1/2)\int_V [(\sigma^{\varrho}_{ij}-\bar{\sigma}^{\varrho}_{ij})L_{ijkl}(\sigma^{\varrho}_{kl}-\bar{\sigma}^{\varrho}_{kl})+(\pi^{\varrho}_m-\bar{\pi}^{\varrho}_m)Z_{mn}(\pi^{\varrho}_n-\bar{\pi}^{\varrho}_n)]\,dV$$

is introduced with the following definitions:

$$(3.4)\qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^R+\Delta\boldsymbol{\sigma}^0+\boldsymbol{\sigma}^{\varrho}, \qquad \boldsymbol{\pi} = \boldsymbol{\pi}^R+\boldsymbol{\pi}^{\varrho},$$

$$(3.5)\qquad \boldsymbol{\sigma}^s = \boldsymbol{\sigma}^R+\Delta\boldsymbol{\sigma}^0+\bar{\boldsymbol{\sigma}}^{\varrho}, \qquad \boldsymbol{\pi}^s = \boldsymbol{\pi}^R+\bar{\boldsymbol{\pi}}^{\varrho}.$$

Here $\mathbf{s} = [\boldsymbol{\sigma}, \boldsymbol{\pi}]$ is the actual state of stress and $\mathbf{s}^s = [\boldsymbol{\sigma}^s, \boldsymbol{\pi}^s]$ is a safe state of stress defined by the fact that the inequality (2.9) is satisfied in the strict sense; a superposed bar indicates time-independent fields. Then, with $\Delta\dot{E}^0_{ij} = L_{ijkl}\Delta\dot{\sigma}^0_{kl}$, the derivative of W with respect to time, $\dot{W}$, becomes

$$(3.6)\qquad \dot{W} = \int_V [(\sigma^{\varrho}_{ij}-\bar{\sigma}^{\varrho}_{ij})(\dot{E}_{ij}-\Delta\dot{E}^0_{ij}-\dot{E}^p_{ij})-(\pi^{\varrho}_m-\bar{\pi}^{\varrho}_m)\dot{\varkappa}_m]\,dV.$$

Expanding the Green–Lagrange strain tensors $\mathbf{E}$ and $\mathbf{E}^0$ into a Taylor series in the vicinity of the reference configuration characterized by $\mathbf{F}^R$, and neglecting terms of order higher than one according to the assumption of small motions of $\mathscr{B}$ in the vicinity of Ω^R, we get

$$(3.7)\qquad \dot{E}_{ij} = (1/2)(F^R_{ki}\dot{F}_{kj}+F^R_{kj}\dot{F}_{ki}),$$

$$(3.8)\qquad \Delta\dot{E}^0_{ij} = (1/2)(F^R_{ki}\Delta\dot{F}^0_{kj}+F^R_{kj}\Delta\dot{F}^0_{ki}),$$

where $\Delta\dot{\mathbf{F}}^0$ is the gradient tensor of $\Delta\dot{\mathbf{u}}^0$. With the relations (3.4), (3.5), (3.7), (3.8), Eq. (3.6) transforms to

$$(3.9)\qquad \dot{W} = \int_V [F^R_{ki}(\sigma^{\varrho}_{kj}-\bar{\sigma}^{\varrho}_{kj})(\dot{F}_{ij}-\Delta\dot{F}^0_{ij})\,dV - \int_V [(\sigma_{ij}-\sigma^s_{ij})\dot{E}^p_{ij}+(\pi_m-\pi^s_m)\dot{\varkappa}_m]\,dV.$$

As $\dot{\mathbf{F}}$ and $\Delta\dot{\mathbf{F}}^0$ are the rates of kinematically admissible displacement gradients, the first integral in Eq. (3.9) vanishes as a weak form of Eqs. (3.1). The second integral is non-negative due to Eq. (2.1) so that $\dot{W}$ is always non-positive and negative if plastic deformations occur. Hence Melan's argument holds: As W is non-negative by definition, plastic dissipation is limited and so plastic flow ceases beyond a certain instant if a field $\bar{\mathbf{s}}^{\varrho}$ exists, fulfilling the relations (3.1), (3.2). We say that $\mathscr{B}$ shakes down in this case.

Similarly, an extension of Koiter's theorem [3] can be given: It says that if there exists a kinematically admissible plastic strain rate cycle $\dot{\mathbf{E}}^{p\sim}$, so that the rate of external work exceeds the rate of dissipation due to the associated generalized plastic strain rate cycle $\dot{\mathbf{e}}^{p\sim} = [\dot{\mathbf{E}}^{p\sim}, \dot{\boldsymbol{\varkappa}}^{\sim}]$

$$\int_V p_i \dot{u}_i^{\sim} dV + \int_S f_i \dot{u}_i^{\sim} dS > \int_V (\sigma_{ij}^{\sim} \dot{E}_{ij}^{p\sim} + \pi_m^{\sim} \dot{\varkappa}_m^{\sim}) dV, \tag{3.10}$$

then shakedown will not occur. A kinematically admissible strain rate cycle is defined by

$$\dot{E}_{ij}^{p\sim} = (1/2)(F_{ki}^R \delta \dot{F}_{kj}^{p\sim} + F_{kj}^R \delta \dot{F}_{ki}^{p\sim}) \tag{3.11}$$

with

$$\int_0^T \delta \dot{F}_{ij}^{p\sim} d\tau = \Delta u_{i,j}^{p\sim}, \quad \Delta u_i^{p\sim} = 0 \quad \text{on } \mathscr{S}_K, \tag{3.12}$$

where T is the period of one cycle. The proof, given for an elastic-ideal plastic material in [14], is analogous to Koiter's original proof by contradiction [3]:

HYPOTHESIS. Despite the existence of a kinematically admissible strain rate cycle fulfilling the inequality (3.10), shakedown occurs. Then there exists an admissible field of generalized stresses $\mathbf{s}^a$ with

$$\mathbf{s}^a = \mathbf{s}^R + \Delta\mathbf{s}^0 + \Delta\mathbf{s}^{\sim} + \bar{\mathbf{s}}^{\varrho}, \tag{3.13}$$

so that $\mathscr{F}(\mathbf{s}^a) \leqslant 0$ holds. Here $\Delta\mathbf{s}^{\sim}$ are extra stresses induced by $\dot{\mathbf{e}}^{p\sim}$ in the purely elastically reacting comparison body $\mathscr{B}^0$. As $\mathbf{s}^a$ is statically admissible, the weak form of the equilibrium conditions

$$\int_V p_i \dot{u}_i^{\sim} dV + \int_S f_i \dot{u}_i^{\sim} dS = \int_V \sigma_{ij}^a \dot{E}_{ij}^{\sim} dV \tag{3.14}$$

holds true for any kinematically admissible strain rate $\dot{\mathbf{E}}^{\sim}$, defined by

$$\dot{E}_{ij}^{\sim} = (1/2)(F_{ki}^R \dot{u}_{k,j}^{\sim} + F_{kj}^R \dot{u}_{k,i}^{\sim}), \quad \dot{u}_k^{\sim} = 0 \quad \text{on} \quad S_K. \tag{3.15}$$

(For convenience we restrict our considerations here to homogeneous kinematical boundary conditions.) Transformation of the r.h.s. of Eq. (3.14) with the help of Eq. (3.13) delivers

$$\int_V [(\sigma_{ij}^R + \Delta\sigma_{ij}^0 + \Delta\sigma_{ij}^{\sim} + \bar{\sigma}_{ij}^{\varrho}) \dot{E}_{ij}^{\sim}] dV = \int_V (\sigma_{ij}^a \dot{E}_{ij}^{p\sim} + \pi_m^a \dot{\varkappa}_m^{\sim}) dV$$
$$+ \int_V [(\sigma_{ij}^R + \bar{\sigma}_{ij}^{\varrho}) \dot{E}_{ij}^{e\sim} + (\pi_m^R + \bar{\pi}_m^{\varrho}) \dot{\omega}_m^{\sim}] dV + \int_V [(\Delta\sigma_{ij}^0 + \Delta\sigma_{ij}^{\sim}) \dot{E}_{ij}^{e\sim}] dV. \tag{3.16}$$

All integrals except the first of the r.h.s. of Eq. (3.16) vanish identically under the hypothesis of the proof: If shakedown occurs, the integrals of $\dot{\mathbf{E}}^{e\sim}$ and $\dot{\boldsymbol{\omega}}^{\sim}$ over one cycle must be equal to zero. So, as $\boldsymbol{\sigma}^R$, $\boldsymbol{\pi}^R$, $\bar{\boldsymbol{\sigma}}^{\varrho}$ and $\bar{\boldsymbol{\pi}}^{\varrho}$ are time-independent, the second integral of the r.h.s. of Eq. (3.16) must be equal to zero for any complete period T. To show that the last integral of the r.h.s. of Eq. (3.16) vanishes, Betti's theorem is applied:

$$\int_V [(\Delta\sigma_{ij}^0 + \Delta\sigma_{ij}^{\sim}) \dot{E}_{ij}^{e\sim}) dV = \int_V (\Delta E_{kl}^e L_{ijkl} L_{mnij}^{-1} \dot{\sigma}_{mn}^{\sim}) dV = \int_V \Delta E_{ij}^e \dot{\sigma}_{ij}^{\sim} dV \tag{3.17}$$

with

$$\Delta E_{ij}^e = L_{ijkl}(\Delta\sigma_{kl}^0 + \Delta\sigma_{kl}^{\sim}), \quad \dot{\sigma}_{ij}^{\sim} = L_{ijkl}^{-1} \dot{E}_{kl}^{e\sim}. \tag{3.18}$$

Here $\Delta \mathbf{E}^e$ is kinematically admissible and $\dot{\boldsymbol{\sigma}}^{\sim}$ is a field of residual stress rates so that Eq. (3.17) vanishes as a weak form of the equilibrium conditions for $\dot{\boldsymbol{\sigma}}^{\sim}$. Replacing in the inequality (2.10) $\hat{\mathbf{s}}$ by $\mathbf{s}^a$, we find through the relations (3.10), (3.14), (3.16) that the hypothesis of shakedown violates the inequality (3.10) so that the proof is completed.

Both extended theorems are based upon the linearization of the strain-tensors in the vicinity of the reference configuration Ω^R for which all field quantities are assumed to be given. However, Ω^R is not necessarily physically real; any stable configuration satisfying equilibrium, compatibility, boundary conditions and the constitutive law for the external loads $\mathbf{a}^R$ may serve as reference configuration. This is particularly helpful if only vague information about the expected deformation pattern of $\mathcal{B}$ is available. In this case we may define a domain of possible configurations Ω^R and can find out if the given conditions for shakedown or non-shakedown are fulfilled for all fields from this domain.

4. Applicaton of the extended theorem of Melan to shell-like bodies

The extended theorem of Melan states that shakedown occurs if the relations (3.1), (3.2) are satisfied. For applications to shell-like bodies, we have to find adequate representations of the equilibrium conditions (3.1) and the yield-condition (3.2), accounting for the geometrical particularities and their consequences on the state of stress in the considered bodies. As far as the equilibrium conditions are concerned we note that their work-consistent formulation is material-independent and can thus be obtained directly from the application of the principle of virtual work, the introduction of geometrical constraints and the integration over the shell-thickness, if a 2-D form is required (see, e.g., [26, 27]). Different shell theories, in particular in the case of geometrical nonlinearity, are defined by introducing different geometrical constraints and/or different error-estimates which are the base for omitting higher order terms in the equilibrium conditions, where these terms are considered to be small within a chosen order of accuracy [27–31]. In the case of elastic shells, a commonly used base for consistent aproximations is the elastic strain energy density [28, 29]. This quantity, however, is of reduced meaning in the case of elastic-plastic shells due to the fact that in contrast to elastic shells, no unique relationship between stresses and strains in the shell-body exists for elastic-plastic shells. Here other estimation measures (e.g., displacement-based [30, 31] or work-based [27]) seem to be more meaningful. However, this problem has not yet been answered conclusively.

Considering the material law, in particular the inequality (3.2), we note that in principle their pointwise three-dimensional satisfaction within the shell body is required. Due to the uncoupling of strains and stresses in the case of arbitrary loading histories, a two-dimensional form of the yield-condition (3.2) is approximative even under the assumption of the Kirchhoff–Love hypothesis. Exceptions are sandwich shells, sheets and membrane shells. However, from the practical point of view a complete three-dimensional analysis is too cumbersome, even after appropriate discretization in general. Helpful contributions and propositions for the solution of this problem can be found, e.g., in [5, 30–34]. Only for simplicity, we restrict our attention in this paper to shallow sandwich-shells undergoing moderate rotations about tangents and small rotations about normals to the

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midsurface Γ of the shell (Donnell–Vlasov–Mushtari-type); in this case the governing equilibrium equations derived through energy-based [28, 29] and displacement-based [30, 31] measures of accuracy coincide. They are given by

$$(4.1)\qquad \begin{aligned} &N^{\Delta\Gamma}|_{\Gamma} = -p^{\Delta *}, \\ &N^{\Delta\Gamma}B_{\Delta\Gamma} + (N^{\Delta\Gamma}w|_{\Delta})_{,\Gamma} + M^{\Delta\Gamma}|_{\Delta\Gamma} = -p^{3*} \quad \text{in} \quad \Gamma; \end{aligned}$$

$$(4.2)\qquad \begin{aligned} &N^{\Delta\Gamma}\nu_{\Gamma} = (N^{\Delta\Gamma})^{*}\nu_{\Gamma}, \\ &M^{\Delta\Gamma}\nu_{\Gamma} = (M^{\Delta\Gamma})^{*}\nu_{\Gamma}, \\ &(N^{\Delta\Gamma}w|_{\Delta} + M^{\Delta\Gamma}|_{\Delta})\nu_{\Gamma} = (Q^{\Gamma})^{*}\nu_{\Gamma} \quad \text{on} \quad \partial\Gamma_{F}, \end{aligned}$$

where **N**, **M**, **B** and w are membrane forces, bending moments, the curvature tensor and displacement orthogonal to Γ, respectively; Greek indices run from 1 to 2 and a vertical stroke denotes the covariant derivative with respect to the following coordinate. On the boundary $\partial\Gamma_F$, where statical quantities are prescribed, indicated by ( )*, the outer normal vector is denoted by $\boldsymbol{\nu}$ and $\mathbf{Q}$ is the boundary force orthogonal to Γ.

To apply the theory developed in the foregoing chapter, the external loads $\mathbf{a} = [\mathbf{p}, \mathbf{N}^*, \mathbf{M}^*, \mathbf{Q}^*]$ must be decomposed into a constant part $\mathbf{a}^R$, related to a reference configuration Ω^R which is defined by $\overset{R}{w}$, and a part $\Delta\mathbf{a}$, related to a motion with small amplitude in the vicinity of Ω^R of a purely elastically reacting reference shell $\mathbf{S}^0$. Then particular solutions $[\mathring{\mathbf{N}}, \mathring{\mathbf{M}}]$ for $\mathcal{S}^0$ under the load $\Delta\mathbf{a}$ have to be determined for the system of equations

$$(4.3)\qquad \begin{aligned} &\mathring{N}^{\Delta\Gamma}|_{\Gamma} = -\Delta p^{\Delta *}, \\ &\mathring{N}^{\Delta\Gamma}B_{\Delta\Gamma} + (\mathring{N}^{\Delta\Gamma}\overset{R}{w}|_{\Delta})_{,\Gamma} + \mathring{M}^{\Delta\Gamma}|_{\Delta\Gamma} = -\Delta p^{3*} \quad \text{in} \quad \Gamma; \end{aligned}$$

$$(4.4)\qquad \begin{aligned} &\mathring{N}^{\Delta\Gamma}\nu_{\Gamma} = (\Delta N^{\Delta\Gamma})^{*}\nu_{\Gamma}, \\ &\mathring{M}^{\Delta\Gamma}\nu_{\Gamma} = (\Delta M^{\Delta\Gamma})^{*}\nu_{\Gamma}, \\ &(\mathring{N}^{\Delta\Gamma}\overset{R}{w}|_{\Delta} + \mathring{M}^{\Delta\Gamma}|_{\Delta})\nu_{\Gamma} = (\Delta Q^{\Gamma})^{*}\nu_{\Gamma} \quad \text{on} \quad \partial\Gamma_{F}, \end{aligned}$$

for all fields $\overset{R}{w}$ related to possible configurations Ω^R. If, then, the time-independent fields $(\bar{N}^{\varrho}, \bar{M}^{\varrho})$ can be found, such that Eqs. (4.3), (4.4) with vanishing r.h.s. and the yield-condition (3.2) in an appropriately chosen two-dimensional form

$$\mathcal{F}_2([\mathbf{N}^R + \alpha\mathring{\mathbf{N}} + \bar{\mathbf{N}}^{\varrho}], [\mathbf{M}^R + \alpha\mathring{\mathbf{M}} + \bar{\mathbf{M}}^{\varrho}], [\boldsymbol{\pi}^R + \bar{\boldsymbol{\pi}}^{\varrho}]) \leqslant 0$$

are satisfied, then the considered shell will shake down. The application of the sandwich-model in our case guarantees the exact satisfaction of the yield-condition (3.2) if, alone, the upper and lower layer of the shell are checked.

As a numerical example, we consider a short cylindrical sandwich shell of length L, radius R, wall thickness $2H$, clamped at both ends, under variable internal pressure $p = p^R + \Delta p$, $\Delta p = \varkappa p^R$ (Fig. 3).

Two different kinds of material behaviour are investigated (Fig. 4):

(i) elastic-ideal plastic behaviour with uniaxial yield limit σ_s,

(ii) elastic-plastic linear kinematical hardening behaviour with the yield limit σ_F and ultimate loading limit σ_s.

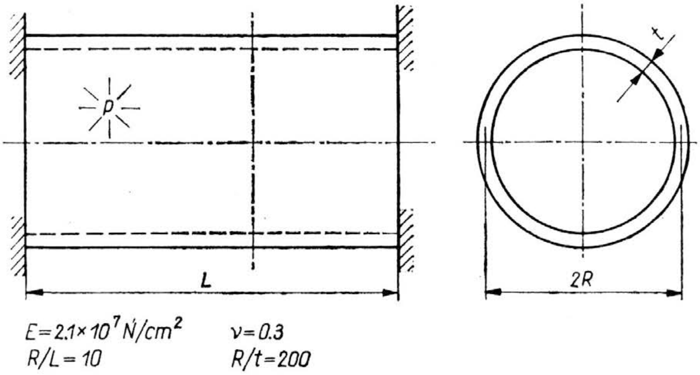

FIG. 3. Shell under internal pressure.

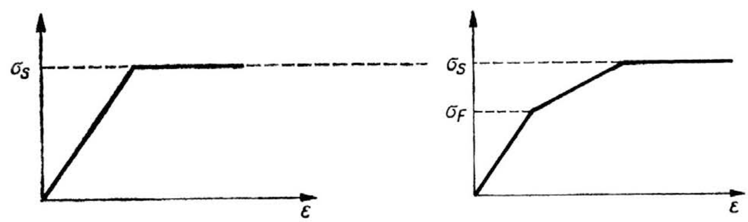

FIG. 4. Uniaxial stress-strain curves of the material model: (a) elastic-ideal plastic, (b) elastic-plastic hardening.

Both initial yield surface and ultimate loading surface are assumed to be of the von Mises-type; we assume the elastic modulus E, Poissons's ratio $\nu = 0.3$, the uniaxial yield-limit σ_s and the ratio σ_s/σ_F to be given. Here we use the reference configurations Ω^R determined by a geometrically non-linear incremental method. We see in Fig. 5 that the shakedown domains are bounded by two families I and II of curves a, b, c and 1, 2, 3, characterized by different values σ_s and σ_s/σ_F, respectively. Family I defines the shakedown limits due to alternating plasticity and we observe no significant difference between geo-

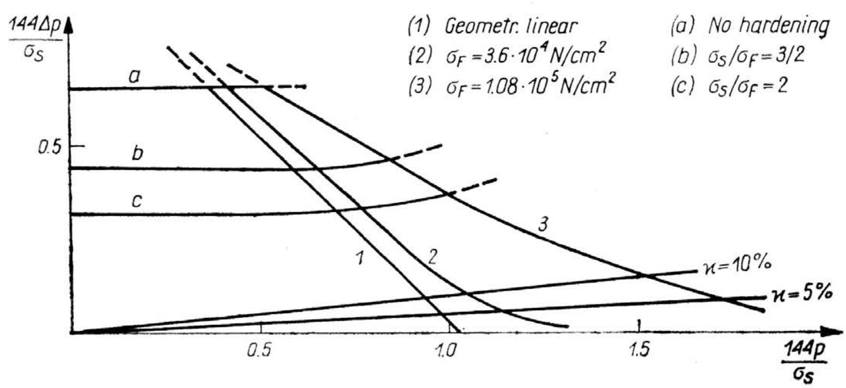

FIG. 5. Shakedown domains.

metrical linear and geometrical nonlinear behaviour (this agrees with observations in [25]). Family II describes the shakedown limit due to incremental collapse and here no significant difference between kinematic hardening and ideal plastic behaviour can be observed.

In our example, the geometrically refined analysis enlarges the domain of admissible loads for the mathematical shell model but we note that in other cases a contraction of the shakedown domain may be observed.

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