

## A method of interpretation of photoelastic data taking into account the deformation of the model thickness

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IN STANDARD photoelasticity, changes in thickness of the model caused by external loading are usually neglected. In most practical cases such an approach is fully justified because the effects of changes in thickness are small. However, in some cases when, for example, the material used for the model has a small Young modulus or when the stress concentrations in the model are very large, these second order effects should be taken into account. A method is proposed allowing to include these second order effects into the process of interpreting photoelastic data.

W klasycznej metodzie elastoptycznej pomija się zwykle zmiany grubości modelu wywołane obciążeniem zewnętrznym. W większości praktycznych przypadków jest to w pełni uzasadnione, ponieważ wpływ zmian grubości na wyniki badań jest niewielki. Jednak w pewnych przypadkach, gdy na przykład materiał użyty do wykonania modelu ma mały moduł sprężystości podłużnej, albo gdy w modelu występują silne koncentracje naprężeń, te efekty drugiego rzędu powinny być uwzględnione przy opracowywaniu wyników badań. Zaproponowano metodę pozwalającą uwzględnić te efekty przy opracowywaniu wyników.

В классическом эластооптическом методе обычно пренебрегается изменениями толщины модели, вызванными внешним нагружением. В большинстве практических случаев это вполне обосновано, т.к. влияние изменений толщины на результаты исследований небольшое. Однако в некоторых случаях, когда, например, материал использованный для изготовления модели имеет малый модуль продольной упругости, или, когда в модели выступают сильные концентрации напряжений, эти эффекты второго порядка должны быть учтены при обработке результатов исследований. Предложен метод позволяющий учитывать эти эффекты при обработке результатов.

### 1. Introduction

IN STANDARD photoelasticity, when transparent plane models are investigated, it is usually assumed that the thickness of the model changes so little when the model is loaded that this effect can be neglected in the process of interpreting photoelastic data. From the pattern of isochromatic fringes, the difference of principal stresses in the model is assumed to be equal to

$$(1) \quad \sigma_1 - \sigma_2 = Km,$$

where  $m$  is the fringe order and  $K$  is the model stress fringe value assumed to be of the same magnitude at every point of the model. This assumption is equivalent to the assumption that the thickness of the model remains unchanged when the model is loaded.

In calculations connected with the interpretation of photoelastic data, the equations of equilibrium in the standard form

$$(2) \quad \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0,$$

or the Laplace equation

$$(3) \quad \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y) = 0$$

for the sum of normal stresses are used. These equations have been derived also by assuming that the thickness of the model remains unchanged and that it is the same everywhere. Owing to the fact that in most practical cases the change in thickness is small (of the order of 0.1 percent), this simplification of the problem seems to be obvious and fully justified. In most books devoted to photoelasticity this simplification is even introduced silently without comments or discussion.

In the early period of the evolution of photoelasticity, changes in thickness of the model were sometimes measured in order to determine the sum of normal stresses because the change in model thickness is proportional to the sum of normal stresses at the point under consideration. Thus

$$(4) \quad \Delta h = -h_0 \frac{\nu}{E} (\sigma_x + \sigma_y),$$

where  $h_0$  is the initial thickness,  $E$  is the modulus of elasticity and  $\nu$  is the Poisson's ratio.

Often, various plastics with a low modulus of elasticity are used to prepare photoelastic models. In such models changes in thickness may be quite large reaching the value of several percent. For example, assume that the model material (e.g., soft epoxy resin) has a modulus of elasticity of 10 MPa, a Poisson's ratio 0.4 and that the sum of normal stresses is about 1 MPa. The above formula would give

$$\Delta h = -h_0 \frac{0.4}{10} = -0.04h_0.$$

Thus the thickness would decrease by four percent. The problem of the effect of changes in thickness on the real distribution of stresses in photoelastic models was analysed by K. H. LAERMANN [1].

## 2. Theory

The second order effects connected with such a change in thickness are significant and should be included into the procedure of interpreting photoelastic data. We will present below a method of calculations in which these effects are included.

Instead of the photoelastic relation (1), its more exact form

$$(5) \quad \sigma_1 - \sigma_2 = m \frac{f}{h}$$

will be used. In this relation  $f$  is the material stress fringe value and  $h$  is the actual thickness at the point under consideration. According to Eq. (4), we can write

$$(6) \quad h(x, y) = h_0 \left[ 1 - \frac{\nu}{E} (\sigma_x + \sigma_y) \right].$$

Instead of the equations of equilibrium (2), we will use the equations of equilibrium

$$(7) \quad \frac{\partial(h\sigma_x)}{\partial x} + \frac{\partial(h\tau_{xy})}{\partial y} = 0, \quad \frac{\partial(h\tau_{xy})}{\partial x} + \frac{\partial(h\sigma_y)}{\partial y} = 0,$$

taking into account changes in model thickness.

To solve the problem, we will introduce two new auxiliary variables, namely, the mean stress  $p = \frac{1}{2}(\sigma_x + \sigma_y)$ , and the angle  $\varphi$  between the larger of the principal stresses  $\sigma_1$

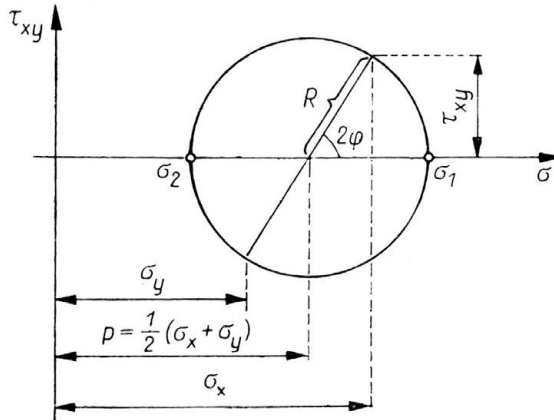


FIG. 1.

and the  $x$ -axis (Fig. 1). The radius of the Mohr circle representing the stress state at a point under consideration is [cf. (5)]

$$(8) \quad R = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{f}{2} \frac{m}{h},$$

where  $h$  is the thickness of the model at that point expressed by the relation (6).

Now the stresses  $\sigma_x, \sigma_y, \tau_{xy}$  can be expressed by means of the two auxiliary functions  $p(x, y)$  and  $\varphi(x, y)$  in the form

$$(9) \quad \begin{aligned} \sigma_x &= p + \frac{fm}{2h} \cos 2\varphi, \\ \sigma_y &= p - \frac{fm}{2h} \cos 2\varphi, \\ \tau_{xy} &= \frac{fm}{2h} \sin 2\varphi. \end{aligned}$$

Substituting Eqs. (9) into the equations of equilibrium and making use of the relation [cf. (6)]

$$(10) \quad h = h_0 \left( 1 - \frac{2\nu}{E} p \right),$$

we obtain a quasi-linear system of two partial differential equations with two unknown functions  $p$  and  $\varphi$ :

$$(11) \quad \begin{aligned} \left(1 - \frac{4\nu}{E} p\right) \frac{\partial p}{\partial x} - K_0 m \sin 2\varphi \frac{\partial \varphi}{\partial x} + K_0 m \cos 2\varphi \frac{\partial \varphi}{\partial y} \\ = -\frac{1}{2} K_0 \left( \cos 2\varphi \frac{\partial m}{\partial x} + \sin 2\varphi \frac{\partial m}{\partial y} \right), \\ \left(1 - \frac{4\nu}{E} p\right) \frac{\partial p}{\partial y} + K_0 m \cos 2\varphi \frac{\partial \varphi}{\partial x} + K_0 m \sin 2\varphi \frac{\partial \varphi}{\partial y} \\ = -\frac{1}{2} K_0 \left( \sin 2\varphi \frac{\partial m}{\partial x} - \cos 2\varphi \frac{\partial m}{\partial y} \right), \end{aligned}$$

where  $K_0 = f/h_0$ .

This system of equations is of the hyperbolic type and has two families of real characteristics determined by the differential equations

$$(12) \quad \begin{aligned} \frac{dy}{dx} = \tan \left( \varphi + \frac{\pi}{4} \right), \\ K_0 m d\varphi + \left(1 - \frac{4\nu}{E} p\right) dp = \frac{1}{2} K_0 \left( \frac{\partial m}{\partial y} dx - \frac{\partial m}{\partial x} dy \right), \end{aligned}$$

for the first family, and

$$(12') \quad \begin{aligned} \frac{dy}{dx} = \tan \left( \varphi - \frac{\pi}{4} \right), \\ K_0 m d\varphi - \left(1 - \frac{4\nu}{E} p\right) dp = \frac{1}{2} K_0 \left( \frac{\partial m}{\partial y} dx - \frac{\partial m}{\partial x} dy \right), \end{aligned}$$

for the second family of characteristics. Thus integration of the basic system of partial differential equations (11) has been reduced to the integration of ordinary differential equations of characteristics (12). The partial derivatives of the isochromatic fringe order with respect to the coordinates  $\partial m/\partial x$  and  $\partial m/\partial y$  appearing in these equations may be determined from the isochromatic fringe pattern by graphical or numerical differentiation of the lines formed by the intersection of the known surface  $m(x, y)$  resulting from the fringe pattern.

If the Poisson's ratio in Eqs. (12) is assumed to be zero ( $\nu = 0$ ), then the equations of characteristics reduce to the form

$$(13) \quad \begin{aligned} \frac{dy}{dx} = \tan \left( \varphi + \frac{\pi}{4} \right), \\ K_0 m d\varphi + dp = \frac{1}{2} K_0 \left( \frac{\partial m}{\partial y} dx - \frac{\partial m}{\partial x} dy \right), \end{aligned}$$

for the first family and

$$(13') \quad \frac{dy}{dx} = \tan\left(\varphi - \frac{\pi}{4}\right),$$

$$K_0 m d\varphi - dp = \frac{1}{2} K_0 \left( \frac{\partial m}{\partial y} dx - \frac{\partial m}{\partial x} dy \right),$$

for the second family.

Equations (13) correspond to the case when second order effects are neglected or in other words the thickness of the model is assumed to remain unchanged during the test. Under this assumption they were derived directly in the previous papers [2] and [3]. The method of calculations based on Eqs. (13) proved to be very effective and was used to solve various practical problems by the author [2], by J. STUPNICKI [4], H. SCHWIEGER [5], R. WOJNAR [6, 7], and by J. STUPNICKI and J. KAPKOWSKI [8].

In Eqs. (13) elastic moduli of the model material do not appear. This means that under the same loading the stress distribution in the photoelastic model will be the same as in a metal plate of the same shape and dimensions. Equations (12) derived on the basis of a more exact theory indicate that this is not exactly true. The stress distribution does depend on the modulus of elasticity and the Poisson's ratio.

The significance of second order effects resulting from the more exact analysis discussed in this paper depends on the magnitude  $4\nu p/E$  appearing in Eqs. (12). For such a standard photoelastic material as the Columbia Resin (CR-39) loaded by stresses close to its proportionality limit, this magnitude equals about 0.01. It will be larger for modern soft photoelastic materials.

### 3. Numerical procedure

Having found experimentally the isochromatic fringe pattern, we can find the stress distribution in the model by numerical integration of the equations of characteristics (12). For explanation of the numerical procedure, a particular problem of the model of a notched plate pulled in tension (Fig. 2a) will be used. The solution of the problem consists in solving successively boundary value problems for the equations of characteristics (12). Owing to double symmetry, we shall consider only one quadrant of the plate (Fig. 2b).

Boundary conditions along the stress free edge  $AB$  are determined by the isochromatic fringe pattern. One of the principal stresses vanishes ( $\sigma_2 = 0$ ). Thus from the photoelastic relation we obtain  $\sigma_1 = mf/h$ . By substituting the relation (10) instead of  $h$  and  $\sigma_1 = 2p$ , we obtain the relation

$$(14) \quad 2p \left( 1 - \frac{2\nu}{E} p \right) = mK_0,$$

which must be satisfied along  $AB$ . For any point  $P$  of the stress free edge, we may find the isochromatic order  $m$  from the fringe pattern and then calculate the value of the auxiliary function  $p$  from the quadratic equation (14). From the two solutions this one should be taken which corresponds to the physical conditions of the problem.

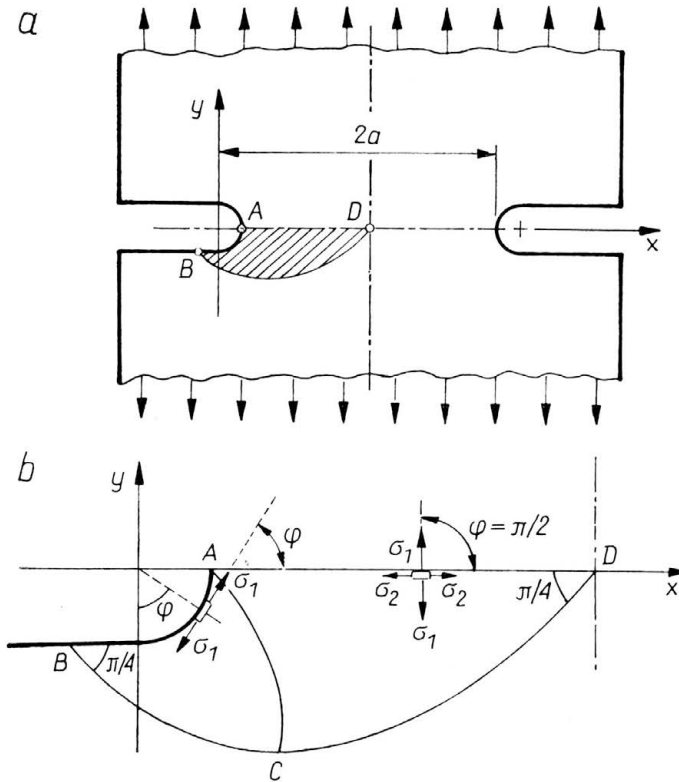


FIG. 2.

The angle  $\varphi$  of inclination of the direction of the larger principal stress  $\sigma_1$  can also be determined along  $AB$  as shown in Fig. 2b.

Along the symmetry line  $AD$  only the angle  $\varphi = \pi/2$  is known. The value of the function  $p$  is to be found along this line.

These boundary conditions are sufficient to calculate the values of the two auxiliary functions  $p$  and  $\varphi$  in the whole region  $ABCD$  bounded by the characteristic  $BCD$  belonging

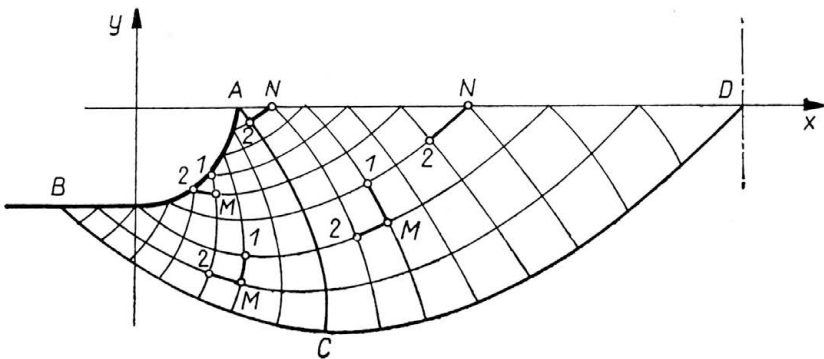


FIG. 3.

to the second family of characteristics. In the curvilinear triangle  $ABC$  we have the so-called Cauchy boundary value problem and in the curvilinear triangle  $ACD$  — a mixed boundary value problem.

Numerical calculations are carried out by means of the method of finite differences (Massau method). Divide now the edge  $AB$  into sufficiently small sectors by a number of points (Fig. 3). At each of these points the values  $p$  and  $\varphi$  are determined by boundary conditions. We will show how to calculate the coordinates  $x_M$  and  $y_M$  of the nodal point  $M$  at which the characteristics of different families through the points 1 and 2 lying on  $AB$  intersect each other, and how to calculate the values  $p_M$  and  $\varphi_M$  at  $M$ .

The characteristic passing through 1 and  $M$  belongs to the first family [cf. Eqs. (12) and Fig. 2b] because its directional coefficient is  $dy/dx = \tan(\varphi + \pi/4)$ , and the characteristic through 2 and  $M$  belongs to the second family (Eqs. (12')). We shall find the coordinates  $x_M$  and  $y_M$  by replacing differentials in Eqs. (12)<sub>1</sub> and (12')<sub>1</sub> by finite differences. In the system of two equations

$$(15) \quad \begin{aligned} y_M - y_1 &= (x_M - x_1) \tan\left(\varphi_1 + \frac{\pi}{4}\right), \\ y_M - y_2 &= (x_M - x_2) \tan\left(\varphi_2 - \frac{\pi}{4}\right), \end{aligned}$$

so obtained, the coordinates  $x_M$  and  $y_M$  only are not known and can be calculated by solving these equations with respect to them. In this numerical procedure the arcs of characteristics have been replaced by chords whose slopes are assumed to have initial values because  $\varphi_1$  and  $\varphi_2$  are the values of the angle  $\varphi$  at points 1 and 2, respectively.

Then the values of  $\varphi_M$  and  $p_M$  can be found from the equations

$$(16) \quad \begin{aligned} K_0 m_1 (\varphi - \varphi_1) + \left(1 - \frac{4\nu}{E} p_1\right) (p_M - p_1) &= \frac{K_0}{2} \left[ (x_M - x_1) \left(\frac{\partial m}{\partial y}\right)_1 - (y_M - y_1) \left(\frac{\partial m}{\partial x}\right)_1 \right], \\ K_0 m_2 (\varphi_M - \varphi_2) - \left(1 - \frac{4\nu}{E} p_2\right) (p_M - p_2) &= \frac{K_0}{2} \left[ (x_M - x_2) \left(\frac{\partial m}{\partial y}\right)_2 - (y_M - y_2) \left(\frac{\partial m}{\partial x}\right)_2 \right], \end{aligned}$$

obtained by replacing in Eqs. (12)<sub>2</sub> and (12')<sub>2</sub> differentials by finite differences. The suffix notations 1 and 2 indicate that the magnitude with the suffix has been taken at the point 1 or 2, respectively. In these equations the values  $\varphi_M$  and  $p_M$  are not known. Thus they can be calculated by solving this system of equations. The accuracy of calculations of  $\varphi_M$  and  $p_M$  may be improved by introducing in Eqs. (16) the mean values

$$\begin{aligned} \frac{1}{2} \left[ \left(\frac{\partial m}{\partial x}\right)_1 + \left(\frac{\partial m}{\partial x}\right)_M \right] &\text{ instead of } \left(\frac{\partial m}{\partial x}\right)_1, \\ \frac{1}{2} \left[ \left(\frac{\partial m}{\partial y}\right)_1 + \left(\frac{\partial m}{\partial y}\right)_M \right] &\text{ instead of } \left(\frac{\partial m}{\partial y}\right)_1, \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \left[ \left( \frac{\partial m}{\partial x} \right)_2 + \left( \frac{\partial m}{\partial x} \right)_M \right] & \text{ instead of } \left( \frac{\partial m}{\partial x} \right)_2, \\ \frac{1}{2} \left[ \left( \frac{\partial m}{\partial y} \right)_2 + \left( \frac{\partial m}{\partial y} \right)_M \right] & \text{ instead of } \left( \frac{\partial m}{\partial y} \right)_2, \\ \frac{1}{2} (m_1 + m_M) & \text{ instead of } m_1, \\ \frac{1}{2} (m_2 + m_M) & \text{ instead of } m_2. \end{aligned}$$

Having found  $\varphi_M$ , we may repeat the calculations of the coordinates  $x_M$  and  $y_M$  by introducing in the system (15) the mean values

$$\begin{aligned} \frac{1}{2} \left[ \tan \left( \varphi_1 + \frac{\pi}{4} \right) + \tan \left( \varphi_M + \frac{\pi}{4} \right) \right] & \text{ instead of } \tan \left( \varphi_1 + \frac{\pi}{4} \right), \\ \frac{1}{2} \left[ \tan \left( \varphi_2 - \frac{\pi}{4} \right) + \tan \left( \varphi_M - \frac{\pi}{4} \right) \right] & \text{ instead of } \tan \left( \varphi_2 - \frac{\pi}{4} \right). \end{aligned}$$

If the second approximation of the coordinates  $x_M$  and  $y_M$  obtained in this way are sufficiently close to their first approximation, the computations can be stopped. If, however, the two approximations give remarkably different values of  $x_M$  and  $y_M$ , calculations should be repeated until the two consecutive approximations become sufficiently close one to the other.

When calculating the second and further approximations of the values of  $\varphi_M$  and  $p_M$ , we may introduce in Eqs. (16) the mean values

$$\begin{aligned} 1 - \frac{4\nu}{E} \frac{p_1 + p'_M}{2} & \text{ instead of } 1 - \frac{4\nu}{E} p_1, \\ 1 - \frac{4\nu}{E} \frac{p_2 + p'_M}{2} & \text{ instead of } 1 - \frac{4\nu}{E} p_2, \end{aligned}$$

where  $p'_M$  is the value of  $p_M$  calculated in the previous iteration.

The numerical procedure described above is analogous to those used in the theory of plastic flow (see for example [9], [10]). By using Eqs. (15) and (16), we may calculate all the data for all points adjacent to the stress free edge  $AB$ . Next, by denoting subsequently by 1 and 2 the determined new nodal points of the set of characteristics, we may in the same manner perform calculations for the next row of nodal points and so on, obtaining finally a solution in the whole curvilinear triangle  $ABC$ . Thus Eqs. (15) and (16) are used as recurrence formulae.

The same numerical procedure with the use of Eqs. (15) and (16) may be used for calculations of nodal points in the region  $ACD$  except for the points located on the symmetry line  $AD$  where  $\varphi = \pi/2$  and  $y = 0$ . Computations should be started from point  $N$  adjacent to point  $A$ . Having calculated previously the data for point 2, we can obtain the coordinate  $x_N$  of the point from the equation

$$(17) \quad y_2 = (x_N - x_2) \frac{1}{2} \left[ \tan \left( \varphi_2 - \frac{\pi}{4} \right) + 1 \right],$$



which has been obtained by replacing in Eq. (12')<sub>1</sub> differentials by finite differences. Since  $\varphi = \pi/2$  we have  $\tan(\varphi - \pi/4) = 1$ . The value of  $p_N$  will be calculated from the equation

$$(18) \quad K_0 m_2 \left( \frac{\pi}{2} - \varphi_2 \right) - \left( 1 - \frac{4\nu}{E} p_2 \right) (p_N - p_2) = \frac{K_0}{2} \left[ (x_N - x_2) \left( \frac{\partial m}{\partial y} \right)_2 + y_2 \left( \frac{\partial m}{\partial x} \right)_2 \right],$$

cf. (12')<sub>2</sub>.

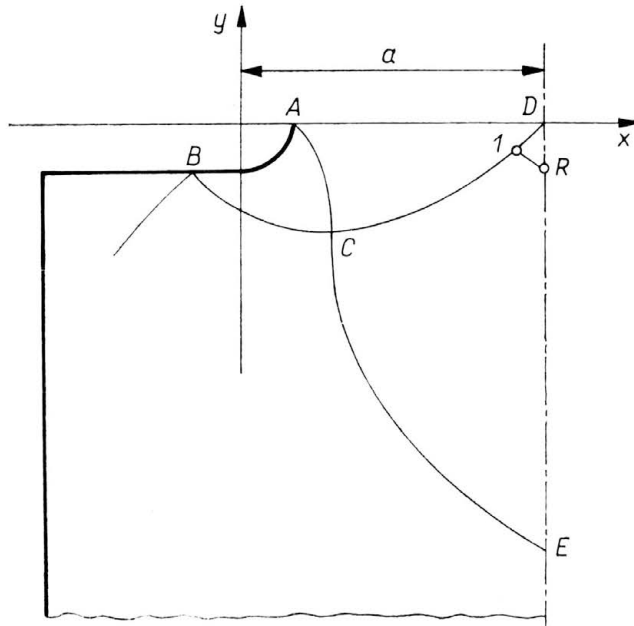


FIG. 4.

The numerical solution for the remaining part of the model (Fig. 4) can be obtained by solving consecutively boundary value problems. For example, in the curvilinear triangle *CDE* we have the mixed problem with the boundary conditions  $\varphi = \pi/2$ ,  $x = a$  along the symmetry line *DE*. The ordinate  $y_R$  of a point *R* on this line may be found from the equation

$$(19) \quad y_R - y_1 = (a - x_1) \frac{1}{2} \left[ \tan \left( \varphi_1 + \frac{\pi}{4} \right) + 1 \right],$$

and the value of  $p_R$  at this point from the equation

$$(20) \quad K_0 m_1 \left( \frac{\pi}{2} - \varphi_1 \right) + \left( 1 - \frac{4\nu}{E} p_1 \right) (p_R - p_1) = \frac{K_0}{2} \left[ (a - x_1) \left( \frac{\partial m}{\partial y} \right)_1 - (y_R - y_1) \left( \frac{\partial m}{\partial x} \right)_1 \right].$$

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