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The stored energy in metals and the concept of residual microstresses in plasticity

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Different mezomechanical models were analysed in an attempt to estimate how large is the portion of elastic energy stored in plastically deformed polcrystalline metals which may be attributed to the residual microstresses, its remaining part being connected with the creation of defects inside the particular crystallites. It has been indicated that durring complex plastic deformation of a metal, the portion of elastic energy stored due to the residual microstresses may be estimated with the use of the kinematical strain hardening law, in which such stresses play an important role. Such an approach is supported by results of a simple experiment.

1. Introduction

When metals udergo plastic deformations, a part of expended energy reappears in the form of heat, and the remaining part is retained inside the metal. The latter part is called "latent energy" or alternatively, the "stored energy". This phenonomenon was experimentally studied in the classical work by W. S. Farren and G. I. Taylor in 1925 [1], and in the following papers by G. I. Taylor and H. Quinney [2, 3]. In these and in other early experiments, the latent energy in metals after plastic deformation was found to lie between 5 and 15 percent of the work done by external forces during the deformation process (cf. [4]). However, recently (see e.g. [5, 6, 7, 8]) it has been demonstrated that in some cases it can reach the amount of up to 50-60 percent in the initial stage of plastic deformation as shown for example in Fig. 1 (see [6]). The relation E_s/E_w, where E_s is the stored energy and E_w stands for the work of plastic deformation, exhibits a distinct maximum at the relatively small permanent deformation.

Usually it assumed that the stored energy is connected with the creation of crystal defects, mainly dislocations, in the crystallites of the metal structure (cf.e.g. [6, 7, 8].) However, metals subjected to complex plastic deformations display the so-called generalized Bauschinger effect connected with the residual microstresses remaining in their internal structure after the load is removed. Associated with these microstresses is the elastic energy remaining in a metal after plastic deformation. In order to assess this part of the whole stored energy, simple mezomechanical models will be used below.

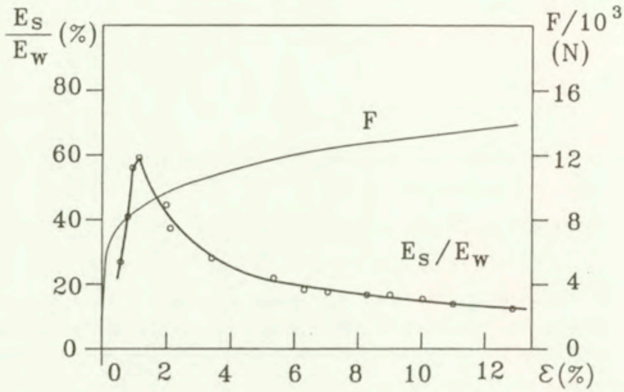

FIG. 1. The ratio E_s/E_w of the stored-to-expended energy and the pulling force F versus tensile deformation of a stainless steel specimens pulled in uniaxial tension – after [6].

2. Stored energy connected with residual microstresses

The concept of residual microstresses in plasticity has been proposed in 1958 by YU. I. KADASHEVICH and V. V. NOVOZHILOV [9] as the basic factor of the so-called kinematic strain–hardening hypothesis. If the initial non-deformed material is assumed to obey the Huber-Mises yield condition

$$s_{ij}s_{ij} = 2k^2, \tag{2.1}$$

then after plastic deformation, the yield condition may be written as

$$(s_{ij} - \alpha_{ij})(s_{ij} - \alpha_{ij}) = 2k^2, \tag{2.2}$$

where s_{ij} is the stress deviator, k is the initial yield locus in simple shear, and α_{ij} stands for an internal parameter, which is interpreted as the tensor of residual microstresses. According to such a strain–hardening law, the initial yield surface (2.1) in the six-dimensional stress space is shifted as a rigid whole. Parameter α_{ij} is represented in this space as the translation vector of the central point of the initial yield surface. The concept of residual microstresses in plasticity was later discussed in [10].

All experimental studies of the energy stored in plastically deformed metals have been done under conditions of uniaxial tension. A simple mezomechanical model shown in Fig. 2 will be used for the theoretical analysis of the process of energy storing due to residual microstresses remaining after uniaxial tension. Note, however, that the model may also be used for a more general analysis of any two-dimensional plastic deformation process. The model represents a regular array of cuboidal elements A and B, each of a different yield locus (cf.[11]). The elastic properties of elements A and B are assumed to be identical. They are defined by the elastic modulus E and by Poisson's ratio ν. Both elements may deform plastically without strain-hardening. In a more advanced variant of the model the strain-hardening effect may be accounted for.

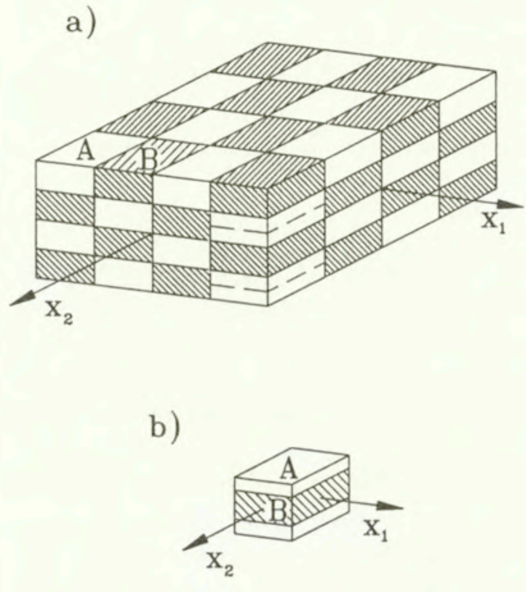

FIG. 2. Simple model of structure of polycrystalline metals with grains A and B having various yield loci: a) general layout of the model, b) its element analysed in calculations.

To simplify the analysis, plane stress states only will be taken under consideration. They will be defined by the principal stresses σ_1 and σ_2 directed along the axes x_1 and x_2 respectively. The yield loci of elements A and B are different as shown in Fig. 3. Elements A begin to deform plastically when the stresses in them satisfy the Huber-Mises yield condition

$$(\sigma_1^A)^2 - \sigma_1^A \sigma_2^A + (\sigma_2^A)^2 = (\sigma_{pl}^A)^2. \tag{2.3}$$

Similarly, the yield condition for elements B may be written as

$$(\sigma_1^B)^2 - \sigma_1^B \sigma_2^B + (\sigma_2^B)^2 = (\sigma_{pl}^B)^2. \tag{2.4}$$

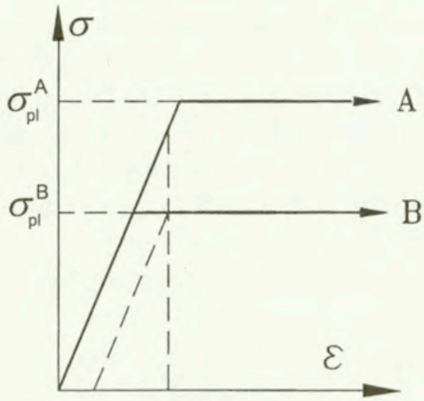

FIG. 3. Assumed theoretical stress-strain relations for elements A and B forming the model shown in Fig. 2.

The exact analysis of behavior of the model undergoing plastic deformation would be very complex, requiring application of numerical techniques with dividing each element into a number of small subelements. However, qualitative conclusions may be obtained with the use of a simplified procedure in which plastic deformation of a segment of the model shown in Fig. 2b will be separately analysed.

As an example, let us analyse the evolution of residual microstresses in the segment of the model shown in Fig. 2b undergoing complex plastic deformation. At first it is uniaxially loaded by tensile stresses σ_2 until point C (Fig. 4) beyond the initial yield stresses $\sigma^0 = \sigma_{pl}^B$, at which weaker element B begins to deform plastically. Point C has been so chosen that element A is still in the elastic state. Then after unloading until point 0, the model is reloaded by uniaxial tensile stresses σ_1 until point D. After the following total unloading until point 0 we can find the residual stresses in the model by analysing the succesive positions of the shifted yield ellipse. The initial yield ellipse shown in Fig. 4 by dashed line has been shifted after this loading programme to the position I with central point 0_D.

The coordinates of this central point 0_D represent the components of the tensor of residual microstresses σ_1^r and σ_2^r remaining in the model. Thus we can calculate the stored energy E_s

$$E_s = \frac{1}{2E}\left[\left(\sigma_1^r\right)^2 + \left(\sigma_2^r\right)^2 - 2\nu\sigma_1^r\,\sigma_2^r\right]. \tag{2.5}$$

If the model is loaded by uniaxial tension by stresses σ_2 only, then the initial yield ellipse is shifted to the position II (see Fig. 4) with centre 0_C. The stored energy due to residual microstresses is

$$E_s = \frac{1}{2E}\left(\sigma_2^r\right)^2. \tag{2.6}$$

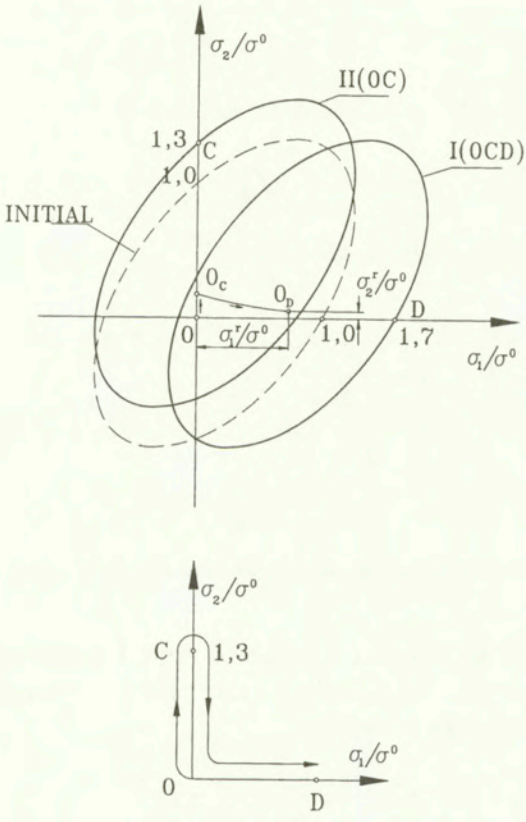

FIG. 4. An example illustrating how residual microstresses σ_1^r and σ_2^r my be calculated when a metal is biaxially deformed plastically-inset shows the loading programme.

The stress-strain relation for the initial stage of uniaxial tensile loading of the model is shown in Fig. 5. Along the segment 0-1 of the loading path, the deformation of the two elements A and B is fully elastic. When stresses in element B reach the yield locus σ_{pl}^B it begins to deform plastically, while element A remains in the elastic state and deforms along the segment 1-b of the diagram. The deformation of the entire model is then represented by the segment 1-2 until point 2 at which also element A begins to deform plastically. From this point the deformation of the model is represented by the horizontal line 2-3. Thus starting from the limiting value of strain ε^*, the model yields under constant limiting value of stress σ^*.

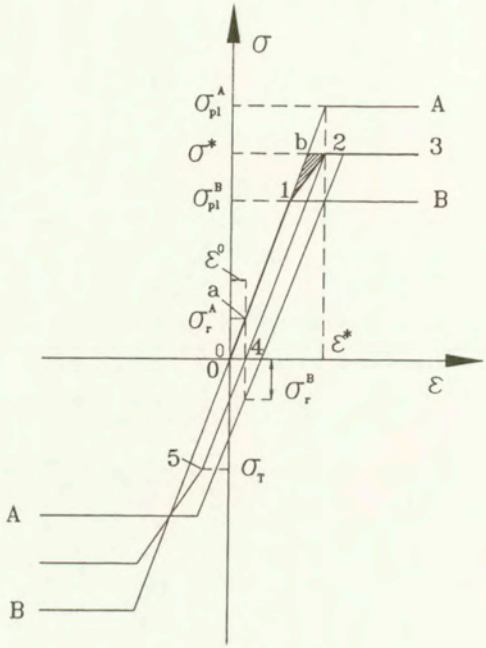

FIG. 5. Stress-strain relation for the model shown in Fig. 2b subject to uniaxial tension.

After unloading from, for example, point 2 until point 4 along straight segment 2-4, there remains in the model a certain plastic deformation ε^0. In element A remain the tensile residual stresses

$$\sigma_r^A = \frac{1}{2}\left(\sigma_{pl}^A - \sigma_{pl}^B\right). \tag{2.7}$$

The residual compressive stresses in element B are of the same absolute value.

Thus in the unit volume of the model is stored the elastic energy

$$E_s = \frac{1}{2E}\sigma_r^2, \tag{2.8}$$

where $\sigma_r = \sigma_r^A = -\sigma_r^B$. Note that this energy is represented in Fig. 5 by the area of the triangle 0-4-a or, in other words, by the dashed area of triangle 1-2-b.

Thus, having found the stress-strain diagram from the tension test we may assess the internal energy stored in the metal due to residual microstresses remaining in it, by measuring the area dashed in Fig. 6 (compare e. g. [12]). This method will be used here for the estimation of the partition of the stored energy into a part connected with the microstresses playing the key role in the strain-hardening hypotheses of the theory of plasticity, and the other part due to the creation of defects inside the crystallites of the metal structure. In Fig. 7 is presented the quotient E_s/E_w calculated for the model shown in Fig. 2b on the basis

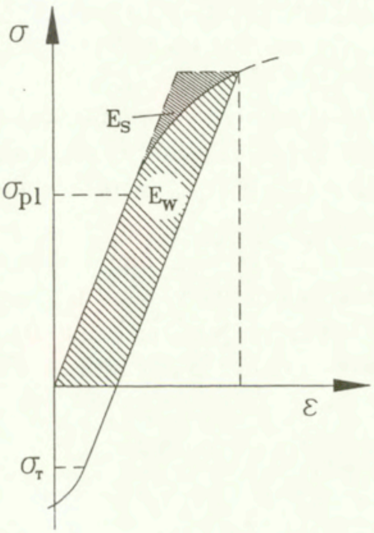

FIG. 6. Schematic diagram showing how to assess the elastic energy E_s stored due to the formation of residual microstresses in plastically deformed metals.

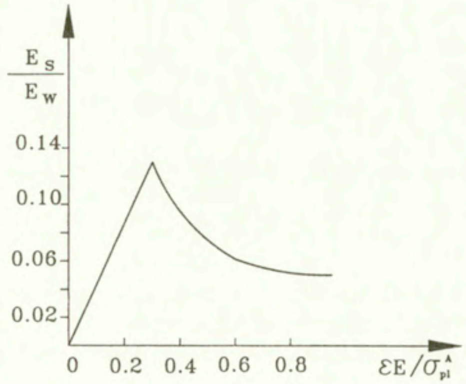

FIG. 7. Relation E_s/E_w versus plastic strain calculated for the model shown in Fig. 2b undergoing uniaxial tension as shown in Fig. 5.

of the diagram for uniaxial tension shown in Fig. 5. The general layout of the diagram is similar to the analogous diagrams found experimentally – e.g. Fig. 1 and diagrams given in [7] and [8]. However, in real metals the quotient E_s/E_w is remarkably larger that obtained for our model. It is possible to increase the maximum value of this quotient in the model up to the value say 0.2 by assuming for example that $\sigma_{pl}^A = 2\sigma_{pl}^B$. Note that such a large difference between the yield loci of crystallites may be expected in real polycrystalline metals – (see [15]) and Sec. 4 of the present paper.

Applying the method of assessing the amount of stored energy shown in Fig. 6 to the analysis of the stress-strain diagram of the stainless steel given in Fig. 1, we find the maximum value $(E_s/E_w)_{max} = 0.1$, while the real maximum value measured experimentally was found to be about 0.6. Thus that part of the stored energy which is connected with residual microstresses constitutes at the initial stage of plastic deformation a relatively small fraction (about one fifth) of the total stored energy, the remaining portion being attributed to the creation of defects (mainly dislocations) inside the crystallites of the metal structure.

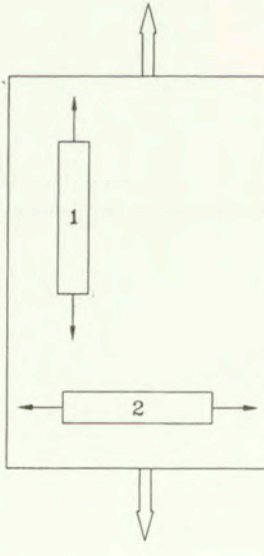

FIG. 8. Scheme of the performed experiment. A large specimen was plastically deformed by uniaxial tension. Then after unloading two small specimens 1 and 2 have been cut out from it, each of them being later plastically deformed under uniaxial tensile loading.

Let us now use this method to the analysis of results of a simple experiment performed according to the loading programme shown previously in Fig. 4. A sheet 6.5 mm thick of an AlMg aluminium alloy (4.7 percent of Mg) was used.

At first, a large specimen cut out from the sheet was loaded by uniaxial tensile stresses until the permanent deformation reached 1.92 percent. Then after unloading, two small specimens, 15 mm wide and 180 mm long, were cut out from it, one in the direction of the previous prestressing of large specimen, and the other in the perpendicular direction (Fig. 8).

Each of these small speciment was then loded by uniaxial tension. In Fig. 9 is shown the stress-strain diagram for the specimen cut out in the direction of previous deformation of the large specimen. For comparison, in the figure is also shown the stress-strain diagram for another specimen cut out in the same direction from the non-deformed sheet. It is seen that using the procedure shown in Fig. 6, we do not find any considerable increase of the stored energy during the initial stage of plastic deformation of our specimen.

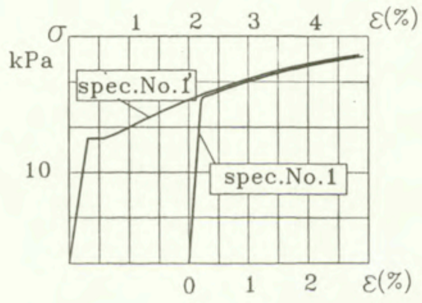

FIG. 9. Stress-strain diagram for specimen No. 1 cut out from the deformed large specimen and then loaded by uniaxial tension, compared with the diagram for the analogous specimen No. 1′ cut out from the non-deformed sheet.

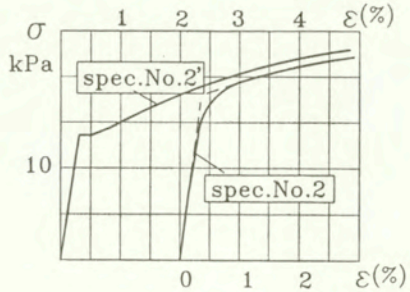

FIG. 10. Stress-strain diagram for specimen No. 2 cut out from the deformed large specimen and then loaded by uniaxial tension, compared with the diagram for the analogous specimen No. 2′ cut out from the non-deformed sheet.

Figure 10 presents an analogous stress-strain diagram for the other specimen cut out in the direction perpendicular to the previous uniaxial deformation of large specimen. In the figure is also shown the diagram for a specimen cut out in the same direction from the non-deformed sheet. It is seen that now the material retains the ability to store the elastic energy due to redistribution of residual microstresses during the process of plastic deformation. Figure 11 shows how the quotient E_s/E_w changes the value during the initial stage of deformation. This diagram was obtained by analysing the diagram shown in Fig. 10 with the use of the procedure shown schematically in Fig. 6.

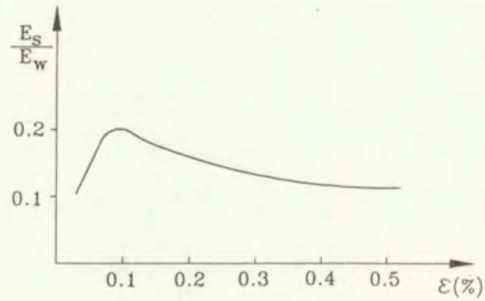

FIG. 11. Relation E_s/E_w versus plastic strain experimentally determined for specimen No. 2 on the basis of the stress-strain diagram shown in Fig. 10.

3. Stored energy and Bauschinger effect

Let use note, coming back to Fig. 5, that the amount of that part of the stored energy, which is connected with the residual microstresses, may also be estimated by measuring the Bauschinger effect. When the model after previous deformation by uniaxial tensile stresses is then loaded by uniaxial compressive stresses along the segment 4-5 of the diagram, its plastic deformation begin at stresses σ_T. In the particular case shown in Fig. 5 we can write

$$\sigma_r^A = \frac{1}{2}\Big(\sigma^* - |\sigma_T|\Big). \tag{3.1}$$

In practice the magnitude of residual microstresses may be estimated on the basis of the stress-strain diagram by measuring the ordinate of the central point of sector 2-5. The elastic energy stored due to these microstresses may be than calculated from formuła (2.8). This procedure holds valid for any value of stresses along the segment 1-2 followed by successive compression.

4. Another model for the study of residual microstresses

In the previous model (Fig. 2) of plycrystalline structure of metals it was assumed that its elements have different yield loci. Plastic deformations in metals are caused by shearing along certain allowable slip-planes identified with the appropriate planes of their crystalline structure. In the previous model it was implicitly assumed that any arbitrary plane inside its elements may be activated as a slip-plane.

However, in real polycrystalline metals there exist in each crystallite only certain admissible directions of slip-planes connected with its atomic structure. A model accounting for this fact was proposed by T. H. LIN and M. ITO [13] and used for the analysis of global plastic properties of polycrystalline aggregates. Analogous model was later used by the same authors for the study of latent energy in plastically deformed such aggregates [14]. In the latter work the model is formed by 64 cubic-shaped "crystals" having different orientations. Each "crystal" was assumed to have one slip-planes orientation only, on which there are three slip directions.

Despite the simplifications introduced in the model, the calculations were complicated and required large computation time. The latent energy remaining in the plastically deformed model after unloading was calculated when at first the residual microstresses were found. Particular calculations have been done for an aggregate of zinc crystals at the temperature $-196°C$. The calculated latent energy was found to reach the astonishing value of about 90 percent of the work done during plastic deformation of the model.

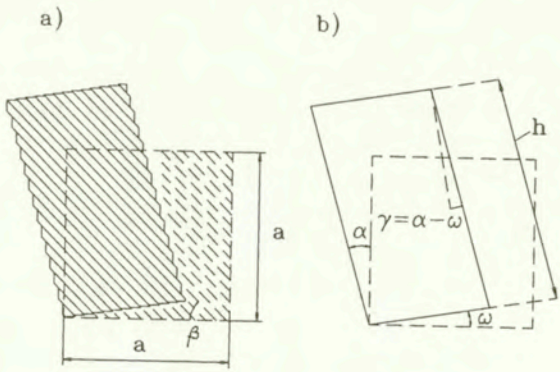

FIG. 12. Sketch of deformation of a crystal by slip.

In what follows, much simpler models of aggregates of elements with different orientations of allowable slip-planes will be analysed in order to estimate how much elastic energy may be stored in them after plastic deformations. In Fig. 12 is shown the basic mechanism of plastic deformation of a single crystal with one allowable orientation of slip-planes. The initially quadratic crystal is deformed into a rhomboid. Its angle of rotation is

$$\omega = \arctan \frac{\tan\beta \sin\beta \sin\alpha}{\cos(\alpha+\beta) - \sin\beta\sin\alpha}. \tag{4.1}$$

Dimension h characterizing the elongation of the crystal is

$$h = a\Big[\cos\alpha + \sin\alpha\tan(\alpha+\beta)\Big]\cos\gamma, \tag{4.2}$$

where $\gamma = \alpha - \omega$ is the angle of shearing distortion.

Let us now consider a simple model composed of two elements with the same but oppositely oriented inclination angles β of allowable slip-planes (Fig. 13). Let the model be loaded by tensile uniaxial stresses as shown in the figure.

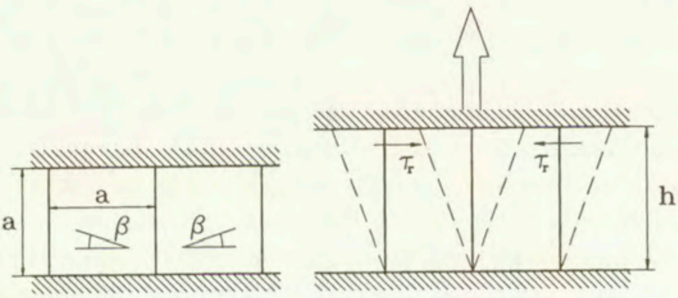

FIG. 13. Simple two-component model with oppositely oriented slip-planes.

Assume that to begin the sliding process, the shear stresses on the slip-planes must reach a certain constant limiting value k. No strain-hardening on these planes is taken into account. Thus the model begins to deform plastically when the tensile stresses in its components reach the value

$$\sigma^* = 2k/\sin 2\beta. \tag{4.3}$$

The components of the model are assumed to be attached to rigid external plates. Thus they cannot rotate and suffer shearing distortion. To remove such a distortion shown in the figure by dashed lines, the elastic residual stresses τ_r must be introduced.

Theoretically the limiting value τ_r^* of these residual shear stresses is that at which the shear stresses on the slip-plane are equal to the yield stress k assumed in the model. Thus we can write

$$\tau_r^* = k/\cos 2\beta. \tag{4.4}$$

Note, however, that for $\beta = 45°$ we get infinite value of the allowable residual shear stress remaining in the model. Such singular properties of models with elements in which only a single orientation of slip-planes is assumed may lead to unrealistic theoretical estimations of the stored elastic energy, such as that calculated in [14].

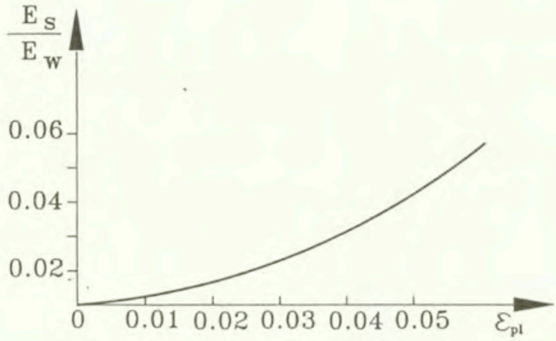

FIG. 14. Theoretical relation E_s/E_w versus plastic deformation calculated for the model shown in Fig. 13.

Nevertheless, our model may be used for rough estimation of the energy stored due to the residual shear microstresses at the initial stage of plastic deformation. In Fig. 14 is shown the diagram E_s/E_w versus plastic longitudinal deformation calculated for $\beta = 45°$ and $k = \mathrm{E}/2000$. Thus the energy stored due to residual shear microstresses is rather small. Let us notice that for $\varepsilon_{pl} = 0.023$ these microstresses reach the value $\tau_r = k$. For this value of residual microstresses we get, assuming $\beta = 30°$, the value $E_s/E_w = 0.031$.

In the model presented in Fig. 13, elongations of two its elements were of the same magnitude – only their distortions had opposie signs.

Now let us analyse a slightly more advanced model consisting of elements with different inclinations of allowable slip-planes (Fig. 15a). In the initial stage of deformation at first begin to deform plastically elements B according to the scheme shown in Fig. 12, while elements A having smaller inclination angle β_A of allowable slip-planes remain in the elastic state. The shear stresses on these planes are smaller then the yield stresses k. In Fig. 15b dashed lines represent the natural shapes of elements A and B after plastic deformation of the model followed then by total unloading. In order to assure the compatibility of the aggregate, the residual shear microstresses τ_r^B must be applied to the elements and moreover, the residual normal microstresses σ_r^A and σ_r^B must appear in the two elements. Thus, ignoring the shear microstresses τ_r^B we have arrived to the situation analogous to that in the previous model shown in Fig. 2. Thus the two models (Fig. 2 and Fig. 15) initially deform plastically almost identically. However, different are the physical mechanisms of their similar behaviour. In the first model any arbitrary plane in its elements may be taken as the slip-plane, while in the second model one orientation of slip-planes only was permissible.

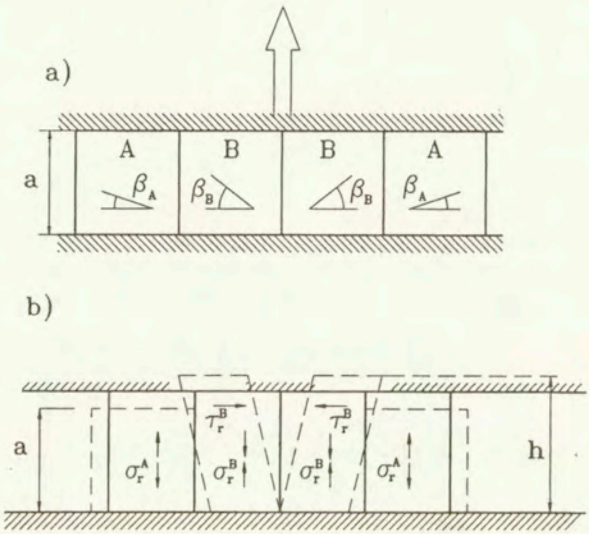

FIG. 15. Model with variously oriented slip-planes in elements.

5. Final remarks

Different simple mezomechanical models of polycrystalline aggregates were analysed in an attempt to estimate how large is the portion of elastic energy stored in plastically deformed metals which may be attributed to the residual microstresses, the remaining part of the energy being connected with the creation of defects in particular grains (mainly dislocations). It was found that the contribution of residual microstresses is rather small but not negligible. Theoretical analysis of these models confirmed by a simple experiment shows that the diagram of the quotient E_s/E_w (E_s is the stored energy and E_w is the work of plastic deformation) versus the permanent deformation has a distinc maximum at the relatively small plastic strains. Such a maximum observed in numerous experimental studies concerning the total stored energy seems to be caused, according to the present study, by the residual microstresses rather than by the creation of defects in particular grains of a polycrystal. It was shown that during complex plastic deformation, the portion of stored energy due to residual microstresses may be theoretically estimated on the basis of the kinematical strain-herdening hypothesis of the theory of plasticity. The residual microstresses are the basic parameter in this hypothesis.

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Variational principles of bending problems of thin plates

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IN THIS PAPER, via the semi-inverse method proposed previously by the present author, a family of variational principles of bending problems of plates is derived directly from their governing equations and boundary conditions, without using the Lagrange multiplier method. In this method, an energy-like trial functional is constructed with a certain unknown function, which can be identified step by step. A new generalized variational principle is obtained.

Key words: Thin Plate, Variational Theory, Semi-Inverse Method, Trial-Functional

1. Introduction

CHIEN [1] STUDIED THE GENERALIZED variational principles with multi-variables of bending problems of thin plates by means of the method of high-order Lagrange multipliers [2] and the involutory transformations, for the purpose of reducing the order of differentiations for the variables in the minimum potential energy principle and minimum complementary energy principle. Reissner gave several generalizations for elasticity [3] and the plate theory [4], WASHIZU [5] suggested a functional to deal with the "corner forces" which appear on the edge at the points of discontinuity of the torsional moments. Classification of various variational theorems was given by CHIEN in [6].

In using the Lagrange multiplier method to eliminate the constraints, however, one might always come across variational crisis [1, 2, 7–12] (some of Lagrange multipliers become zero during the process of variation, or some constraints can be eliminated even in the case the multiplier can be identified; in some special cases, wrong field equations might be obtained after substituting the identified multiplier into the augmented functional). We explained this phenomenon in our previous publications [7, 8, 9, 11] and pointed out in [12] that the so-called variational crisis is the inherent attribution of the Lagrange multiplier method. Several ways have been proposed to eliminate the crisis: a modified Lagrange multiplier method is suggested in [10], and a semi-inverse method is proposed in [13, 14]. Some applications of the semi-inverse method can be found in [15–17].

It has been shown in [1] that we cannot obtain a generalized variational principle by the Lagrange multiplier method due to the variational crisis (see Eq. (7.5) in [1]); in this paper, we apply the semi-inverse method [13, 14] to derive a family of generalized variational principles directly from the differential equations and boundary conditions, and no variational crisis occurs due to absence of Lagrange multipliers in our procedure.

2. Generalized variational principle of thin plate bending problems

The differential equation of deflection due to bending of a thin plate reads [1,18]

$$\nabla^2\nabla^2 w = \frac{\bar{f}}{D}, \tag{2.1}$$

where D is the flexural rigidity, $\bar{f}$ is the given lateral load, w is the lateral deflection of the plate. It can be seen clearly that the differential equation (2.1) requires strong local differentiability (smoothness). While the variable in its corresponding variational partner(see Eq. (2. 2)) is in second order of differentiations, it can be written in the form (the boundary conditions will be taken into consideration at the end of this section)

$$J(w) = \iint \left\{ \frac{1}{2}\left(\nabla^2 w\right)^2 - \frac{\bar{f}}{D} w \right\} dxdy. \tag{2.2}$$

The field associated with w must be continuous and must possess continuous second-order derivatives. In the context of finite elements, it is well known that satisfaction of the continuity of the second-order derivatives across the element boundaries is difficult to achieve [19]. So the high order of differentiation in the variational functional (2.2) leads to complications in the finite element calculation, and consequently, inconveniences appear in numerical computations. For the purpose of simplification in the finite element computation, we often introduce some additional canonical variables by means of involutory transformations [1] to reduce the order of differentiations. According to CHIEN [1], we have the following transformations:

$$\varphi_\alpha = w_{,\alpha}, \tag{2.3}$$

$$k_{\alpha\beta} = -w_{,\alpha\beta}, \tag{2.4}$$

$$w_{,\alpha\beta} = \frac{1}{2}(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}), \tag{2.5}$$

(2.6) $$Q_\alpha = M_{\alpha\beta,\beta},$$

(2.7) $$M_{\alpha\beta} = D_{\alpha\beta\nu\delta} k_{\gamma\delta},$$

where the Greek indices are the dummy indices taking the values 1 or 2, $w_{,\alpha\beta} = \partial^2 w / \partial x_\alpha \partial x_\beta$, $w_{,\alpha}$ is the slope of the deflection surface, Q_α and $M_{\alpha\beta}$ are the shearing force and the bending moment, respectively.

The equilibrium equation (2.1) can be rewritten in the form

(2.8) $$M_{\alpha\beta,\alpha\beta} + \bar{f} = 0.$$

As illustrated in [1], it is difficult to search for a generalized variational principle with five sets of independent variations $(w, \varphi_\alpha, k_{\alpha\beta}, M_{\alpha\beta}, Q_\alpha)$ by Lagrange multipliers. However, it is a straightforward approach to deduce various generalized variational principles by the semi-inverse method [13, 14].

In order to establish a generalized variational principle with five sets of independent variations $(w, \varphi_\alpha, k_{\alpha\beta}, M_{\alpha\beta}, Q_\alpha)$, whose stationary conditions satisfy the field equations (2.3), (2.5), (2.6), (2.7) and (2.8), we can construct a trial functional in the form

(2.9) $$J(w, \varphi_\alpha, k_{\alpha\beta}, M_{\alpha\beta}, Q_\alpha) = \iint \left(\frac{1}{2} D_{\alpha\beta\nu\delta} k_{\alpha\beta} k_{\nu\delta} + F\right) dx dy.$$

Note: Eqs. (2.4) are still considered as constraints.

There exist other ways to construct the trial functionals, details can be found in the [13, 14]. Let us illustrate the procedure of identification of the unknown function F step by step.

STEP 1

Taking variation with respect with $k_{\alpha\beta}$, i.e.

$$\delta_k J = \iint (D_{\alpha\beta\nu\delta} k_{\nu\delta} + \delta F / \delta k_{\alpha\beta}) \delta k_{\alpha\beta} dx dy = 0,$$

we have the following trial Euler equation:

$(2.10)_1$ $$D_{\alpha\beta\nu\delta} k_{\nu\delta} + \frac{\delta F}{\delta k_{\alpha\beta}} = 0,$$

where $\delta F / \delta \varphi = \partial F / \partial \varphi - (\partial F / \partial \varphi_{,\alpha})_{,\alpha}$ is called the variational derivative.

The above trial Euler equation $(2.10)_1$ should satisfy Eq. (2.7), accordingly we can set

$(2.10)_2$ $$\frac{\delta F}{\delta k_{\alpha\beta}} = -M_{\alpha\beta}.$$

The unknown F can be preliminarily identified as follows:

$$F = -k_{\alpha\beta} M_{\alpha\beta} + f. \tag{2.11}$$

The trial functional (2.9), therefore, can be rewritten as follows:

$$J = \iint (\frac{1}{2} D_{\alpha\beta\nu\delta} k_{\alpha\beta} k_{\nu\delta} - k_{\alpha\beta} M_{\alpha\beta} + f) dxdy, \tag{2.12}$$

where f is a newly introduced unknown, and should be, in general, free of the variations $k_{\alpha\beta}$ and their derivatives.

STEP 2

By the same manipulation as that used in STEP 1, we can obtain the following trial Euler equations for $\delta M_{\alpha\beta}$.

$$(2.13)_1 \qquad \delta M_{\alpha\beta} : \qquad - k_{\alpha\beta} + \frac{\delta f}{\delta M_{\alpha\beta}} = 0.$$

We assume that the above Euler equations $(2.13)_1$ satisfy the field equations (2.5), and in view of constraints (2.4), the above trial Euler equations reduce to

$$(2.13)_2 \qquad \frac{\delta f}{\delta M_{\alpha\beta}} = -w_{,\alpha\beta} = -\frac{1}{2}(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}).$$

Thus we can preliminarily identify the unknown f as follows:

$$(2.14)_1 \qquad f = -\varphi_{\alpha,\beta} M_{\alpha\beta} + g_1,$$

or

$$(2.14)_2 \qquad f = \varphi_{\alpha} M_{\alpha\beta,\beta} + g_2,$$

or in a more general form

$$(2.14)_3 \qquad f = -m\varphi_{\alpha,\beta} M_{\alpha\beta} + n\varphi_{\alpha} M_{\alpha\beta,\beta} + g, \quad \text{with} \quad m + n = 1,$$

where g, g_1 and g_2 are unknowns to be determined later, and they should be expressed by the functions $w, \varphi_\alpha, Q_\alpha$ and/or their derivatives.

Substituting $(2.14)_3$ into (2.12) we obtain a modified trial functional

$$J = \iint (\frac{1}{2} D_{\alpha\beta\nu\delta} k_{\alpha\beta} k_{\nu\delta} - k_{\alpha\beta} M_{\alpha\beta} - m\varphi_{\alpha,\beta} M_{\alpha\beta} + n\varphi_{\alpha} M_{\alpha\beta,\beta} + g) dxdy. \tag{2.15}$$

STEP 3

Continually we have the following trial Euler equations for $\delta\varphi_\alpha$:

$$(2.16)_1 \qquad \delta\varphi_\alpha : \qquad (m+n)M_{\alpha\beta,\beta} + \tfrac{\delta g}{\delta\varphi_\alpha} = 0,$$

which should satisfy the equations (2.6); therefore we have

$$(2.16)_2 \qquad \frac{\delta g}{\delta\varphi_\alpha} = -Q_\alpha.$$

From above relation, we can express the unknown g as follows:

$$(2.17) \qquad g = -Q_\alpha\varphi_\alpha + h,$$

where h is a newly introduced unknown to be determined later, and should in general be expressed as w, Q_α and/or their derivatives.

STEP 4

Substituting (2.17) in (2.15) to modify the trial functional, making the modified trial functional stationary with respect to Q_α, we have

$$(2.18)_1 \qquad \delta Q_\alpha : \ -\varphi_\alpha + \frac{\delta h}{\delta Q_\alpha} = 0,$$

which should satisfy the equations (2.3); accordingly, we have

$$(2.18)_2 \qquad \frac{\delta h}{\delta Q_\alpha} = w_{,\alpha},$$

$$(2.19) \qquad h = Q_\alpha w_{,\alpha} + h',$$

where h' is an unknown to be determined, and should be only the function of w and/or its derivative.

STEP 5

Substituting (2.19) into the last modified trial functional, finally we can obtain the last trial Euler equations:

$$(2.20)_1 \qquad \delta w : \ -Q_{\alpha,\alpha} + \tfrac{\delta h'}{\delta w} = 0,$$

which should satisfy Eq.(2.8). In view of (2.6), we have

$$(2.20)_2 \qquad \frac{\delta h'}{\delta w} = M_{\alpha\beta,\alpha\beta} = -\bar{f},$$

$$(2.21) \qquad h' = -\bar{f}w.$$

Finally we obtain the following functional:

$$(2.22)_1 \qquad J = \iint L dx dy,$$

in which

$$(2.22)_2 \quad L = \frac{1}{2} D_{\alpha\beta\nu\delta} k_{\alpha\beta} k_{\nu\delta} - \bar{f} w - M_{\alpha\beta}(k_{\alpha\beta} + m\varphi_{\alpha,\beta}) + n\varphi_\alpha M_{\alpha\beta,\beta} - Q_\alpha(\varphi_\alpha - w_{,\alpha}),$$

and where m and n are arbitrary constants with $m + n$=1. It follows that the continuity requirements for the variables in (2.22) are less stringent. The presence of free parameters (m and n) offers an opportunity for a systematic derivation of the energy-balanced finite elements [15].

Next we will illustrate how to use the semi-inverse method to eliminate the constraints of boundary conditions. The boundary conditions usually encountered are given below,

$$(2.23) \qquad H_n = \bar{H}_n, \quad \text{on } \Gamma_\sigma$$

$$(2.24) \qquad w = \bar{w}, \quad \text{on } \Gamma_w$$

where $H_n = Q_n + M_{ns,s}$, and $\Gamma = \Gamma_\sigma + \Gamma_w$ covers the complete boundary.

To eliminate the constrains (2.23) and (2.24) , we construct a trial functional as follows:

$$(2.25)_1 \qquad J_{GVP} = \iint \tilde{L} dx dy + \int_{\Gamma_\sigma} G dS + \int_{\Gamma_w} P dS,$$

where G and P are unknowns to be determined, and $\tilde{L}$ is defined as (by setting m =1, n=0 in $(2.22)_2$)

$$(2.25)_2 \qquad \tilde{L} = \frac{1}{2} D_{\alpha\beta\nu\delta} k_{\alpha\beta} k_{\nu\delta} - \bar{f} w - M_{\alpha\beta}(k_{\alpha\beta} + \varphi_{\alpha,\beta}) - Q_\alpha(\varphi_\alpha - w_{,\alpha}).$$

With the help of Green's theorem, on the boundary Γ_σ we have following trial Euler equations:

$$(2.26) \qquad \delta w : \quad \frac{\delta G}{\delta w} = -Q_\alpha n_\alpha = -Q_n,$$

$$(2.27) \qquad \delta\varphi_s : \quad \frac{\delta G}{\delta \varphi_s} = M_{ns},$$

$$\delta M_{ns}: \quad \frac{\delta G}{\delta M_{ns}} = 0. \tag{2.28}$$

The above trial Euler equations should satisfy the boundary conditions (2.23) or the identities including the equations (2.3)-(2.8).

In view of (2.23), from (2.26) we have

$$\frac{\delta G}{\delta w} = -(\bar{H} - M_{ns,s}). \tag{2.29}$$

Accordingly, the unknown G can be written as follows:

$$G = -\bar{H}w - M_{ns}w_{,s} + G_1, \tag{2.30}$$

in which G_1 should be free of w or its derivatives.

In combination with (2.28), and in view of (2.3), we obtain

$$\frac{\delta G_1}{\delta M_{ns}} = w_{,s} = \varphi_s. \tag{2.31}$$

From (2.27) and (2.31) we can determine the unknown G_1 as follows:

$$G_1 = M_{ns}\varphi_s. \tag{2.32}$$

Therefore we have finally identified the unknown G which is written in the form

$$G = -\bar{H}w - M_{ns}(w_{,s} - \varphi_s). \tag{2.33}$$

Using the same procedure we have the following trial Euler equations on the boundary Γ_w:

$$\delta w: \quad \frac{\delta P}{\delta w} = -Q_n, \tag{2.34}$$

$$\delta \varphi_s: \quad \frac{\delta P}{\delta \varphi_s} = M_{ns}, \tag{2.35}$$

$$\delta Q_n: \quad \frac{\delta P}{\delta Q_n} = 0, \tag{2.36}$$

$$\delta M_{ns}: \quad \frac{\delta P}{\delta M_{ns}} = 0. \tag{2.37}$$

From (2.34) and (2.35), we determine the unknown P as follows:

$$P = -Q_n w + M_{ns}\varphi_s + P_1, \tag{2.38}$$

where P_1 should be free of w and φ_s or their derivatives; if not, the unknown P should be determined again.

In combination with (2.36), and in view of the boundary conditions (2.24), we have

$$\frac{\delta P_1}{\delta Q_n} = w = \bar{w}, \tag{2.39}$$

$$P_1 = Q_n\bar{w} + P_2. \tag{2.40}$$

Substituting (2.38) and (2.40) into (2.36), we obtain

$$\frac{\partial P_2}{\partial M_{ns}} = -\varphi_s = -w_{,s}. \tag{2.41}$$

We temporarily express the unknown P_2 as follows:

$$P_2 = -M_{ns}w_{,s} + P_3. \tag{2.42}$$

It should be stressed that in P_2 there exist the terms involving $w_{,s}$, therefore the unknown P should be determined again. The unknown P can be rewritten as follows:

$$P = -Q_n(w - \bar{w}) + M_{ns}(\varphi_s - w_{,s}) + P_3, \tag{2.43}$$

where P_3 should be expressed as w or its derivative.

Substituting (2.43) into (2.33), and in view of (2.34), we obtain:

$$\frac{\delta P_3}{\delta w} = -M_{ns,s}. \tag{2.44}$$

Accordingly, the unknown P_3 can be written, in general, as follows:

$$P_3 = -M_{ns,s}(w - \bar{w}). \tag{2.45}$$

It can be proved that such unknown P_3 satisfies, in view of equations (2.3) and (2.24), the equations (2.37). Finally we obtain the following generalized variational principle

$$J_{GVP} = \iint \tilde{L} dx dy + \int_{\Gamma\sigma} \{-\bar{H}w - M_{ns}(w_{,s} - \varphi_s)\} dS + \int_{\Gamma w} \{-(Q_n + M_{ns,s})(w - \bar{w}) + M_{ns}(\varphi_s - w_{,s})\} dS. \tag{2.46}$$

where $\tilde{L}$ is defined by Eq.$(2.25)_2$.

P r o o f. Making the above functional stationary, we have the following Euler's equations:

$$\delta k_{\alpha\beta} : \text{equations } (2.7);$$

$$\delta M_{\alpha\beta} : \kappa_{\alpha\beta} + \varphi_{\alpha\beta} = 0;$$

$$\delta Q_{\alpha} : \text{equations } (2.3);$$

$$\delta \varphi_{\alpha} : \text{equations } (2.6);$$

$$\delta w : \text{equations } (2.8);$$

on the Γ_σ:

$$\delta w : \text{equation } (2.23);$$

$$\delta \varphi_s : -M_{ns} + M_{ns} = 0;$$

$$\delta M_{ns} : w_{,s} - \varphi_s = 0;$$

on the Γ_w:

$$\delta w : Q_n - Q_n = 0;$$

$$\delta \varphi_s : -M_{ns} + M_{ns} = 0;$$

$$\delta Q_n : \text{equation } (2.24);$$

$$\delta M_{ns} : -(w - \bar{w})_{,s} + (\varphi_s - w_{,s}) = 0.$$

It can be clearly seen that on the boundary, some of Euler equations are identities or they satisfy the boundary conditions or they have been already derived as Euler equations in the process of variation.

Let us consider the singular points where M_{ns} is discontinuous. The singular corner might exist on the surface Γ_w or Γ_σ or at $\Gamma_w \cap \Gamma_\sigma$. In this paper, we only discuss the later case. We assume that on the boundary, the corner exists at transition points from Γ_σ to Γ_w and *vice versa,* while all the remaining parts of the boundary are assumed to be smooth.

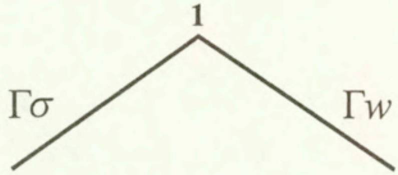

FIG. 1. Singular point

By a similar analysis, we can transform Eq. (2.45) into another functional

$$(2.47)\quad J_{GVP} = \iint \tilde{L}dxdy + \int\limits_{\Gamma\sigma} \{-\bar{H}w - M_{ns}(w_{,s} - \varphi_s)\}dS + \int\limits_{\Gamma w} \{-(Q_n + M_{ns,s})(w - \bar{w}) + M_{ns}(\varphi_s - w_{,s})\}dS - \sum (w - \bar{w})(Q_n + M_{ns,s} - \bar{H}_n)\Big\|_{\Gamma u},$$

where $H_n = Q_n + M_{ns,s}$, and the notation $\sum(\cdot)(\cdot)\|_{\Gamma_u}$ has the same meaning as that in [6,18], which means summation over all the Γ_u. If Eq. (2.23) is replaced by

$$(2.48)\qquad M_{ns} = \bar{M}_{ns}, \quad \text{on } \Gamma_\sigma$$

then the last term at the right-hand side of Eq. (2.46) should be replaced by

$$-\sum (w - \bar{w})(M_{ns} - \bar{M}_{ns})\Big\|_{\Gamma u}.$$

From the generalized variational principle (2.46), various variational principles with smaller number of independent variables can be readily obtained by constraining the functional (2.46) by some field equations or boundary conditions. For example, enforcing the functional (2.46) by the field equation (2.3) results in a new functional, which reads

$$(2.49)_1\qquad J_1 = \iint \left\{\frac{1}{2}D_{\alpha\beta\nu\delta}k_{\alpha\beta}k_{\nu\delta} - \bar{f}w - M_{\alpha\beta}(k_{\alpha\beta} + \varphi_{\alpha,\beta})\right\}dxdy + IB,$$

where

$$(2.49)_2 \quad IB = \int_{\Gamma\sigma} \{-\bar{H}w - M_{ns}(w_{,s} - \varphi_s)\}dS$$

$$+ \int_{\Gamma w} \{-(Q_n + M_{ns,s})(w - \bar{w}) + M_{ns}(\varphi_s - w_{,s})\}dS.$$

The functional $(2.49)_1$ is under constraints of the Eqs. (2.3) and (2.4).

Constraining the functional (2.47) by the Eq. (2.4), we obtain

$$(2.50) \qquad J_2 = \iint \left\{\frac{1}{2}D_{\alpha\beta\nu\delta}k_{\alpha\beta}k_{\nu\delta} - \bar{f}w\right\}dxdy + IB,$$

which is in agreement with the Eq. (2.2).

3. Conclusion

In this paper, the author has systematically discussed the semi-inverse method of establishing generalized variational principles in thin plate bending problems. The variational model can be readily obtained without any variational crisis. A family of generalized variational principles can be readily obtained by specifying the parameters m and n. The Lagrange function $(2.22)_2$, as far as the author knows, has never appeared in any literature.

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Heat transfer over an exponentially stretching continuous surface with suction

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SIMILARY SOLUTIONS of the laminar boundary layer equations describing heat and flow in a quiescent fluid driven by an exponentially stretching surface subject to suction are examined numerically. The direction and amount of heat flow were found to be dependent on the magnitude of "γ " (parameter of temperature) for the same Prandtl number. Nusselt number increases with increasing "γ " and the Prandtl number. The effect of decreasing suction parameter is found to be significant particularly for the Prandtl number.

1. Introduction

THE CONTINUOUS SURFACE heat transfer problem has many practical applications in industrial manufacturing processes. Such processes are hot rolling, wire drawing, glass fiber production, and paper production. Since the pioneering work of SAKIADIS [1], various aspects of the problem have been investigated by many authors. Most studies have been concerned with constant surface velocity and temperature (see, for example TSOU *et al.* [2]) but for many practical applications the surface undergoes stretching and cooling or heating that cause surface velocity and temperature variations. CRANE [3] and VLEGGAAR [4] have analysed the stretching problem with constant surface temperature while SOUNDALGEKAR and RAMANA MURTY [5] investigated the constant surface velocity case with power law temperature. ALI [6] has examined flow and heat transfer characteristics on a stretched surface subject to a power velocity and temperature. Recently MGYARI and KELLER [7] have analysed the exponential stretching problem by discussing a further type of similarity solution of the governing equations. These solutions involve an exponential dependence of the similarity variable as well as of the stretching velocity and temperature distribution on the coordinate in the direction parallel to that of the stretching.

Suction or injection of a stretched surface was introduced by ERICKSON *et al.*, [8] and FOX *et al.*, [9] for uniform surface velocity and temperature. GAUPTA and GAUPTA [10] extended Erickson's work in which the surface was moving with a linear speed for various values of parameters. Furthermore, stretching surface subject to suction or injection was studied by ALI [11] for uniform and variable surface temperature while ELBASHBESHY [12] investigated the uniform and variable surface heat flux.

The present work analyses the heat transfer over an exponentially stretching continuous surface with suction.

2. Formulation of the problem

The laminar velocity and thermal boundary layers on a continuous stretching surface with velocity $U_w \equiv U_w(x)$ and temperature $T_w \equiv T_w(x)$ moving axially through a stationary incompressible fluid with constant physical properties and temperature T_∞ may be described using the normal boundary approximations by the following continuity, momentum and energy equations [7, 11]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$u\,\frac{\partial u}{\partial x} + \nu\,\frac{\partial u}{\partial y} = \nu\,\frac{\partial^2 u}{\partial y^2}, \tag{2.2}$$

$$u\,\frac{\partial T}{\partial x} + \nu\,\frac{\partial T}{\partial y} = \alpha\,\frac{\partial^2 T}{\partial y^2}, \tag{2.3}$$

with the boundary conditions

$$\begin{aligned} &u = U_w(x) = U_0 \exp(x/L), \quad \nu = -V_w, \qquad T = T_w \quad \text{at } y = 0\,, \\ &u = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\; T = T_\infty \quad \text{at } y \to \infty. \end{aligned} \tag{2.4}$$

The x-axis runs along the continuous surface in the direction of motion and the y-axis is perpendicular to it, u and v are the velocity components in the directions of x and y respectively, ν is the kinematic viscosity, T is the temperature, α is the thermal diffusivity, T_∞ is the free stream temperature, U_0 is a constant, L is the reference length, w is the condition at the surface and $y \to \infty$ concerns the condition at the ambient medium.

The solution of Eq. (2.1) may be written in terms of the function $\psi(x, y)$ defined by the relations

$$u = \frac{\partial \psi}{\partial y}, \qquad \nu = -\frac{\partial \psi}{\partial x}.$$

Introducing the usual similarity transformation and dimensionless temperature

$$\eta = y\sqrt{\frac{U_0}{2\nu L}}\,\exp(x/2L)\,, \tag{2.5}$$

(2.6) $$\psi(x,\, y) = \sqrt{2\nu\, LU_0}\, f(\eta)\, \exp\,(x/2L)\;,$$

(2.7) $$\theta(\eta) = \frac{T - T_\infty}{T_0 \exp(\gamma x/2L)}\;.$$

Where γ and T_0 are parameters of the temperature distribution in the stretching surface.

The momentum Eq. (2.2) and energy Eq. (2.3) can be written as

(2.8) $$f''' + f\,f'' - 2f'^2 = 0\;,$$

(2.9) $$\theta'' + \mathrm{Pr}\,(f\,\theta' - \gamma\, f'\,\theta) = 0\;,$$

with the boundary conditions

(2.10) $$\begin{aligned} &f(0) = I\;,\;\; f'(0) = 1\;,\;\; \theta(0) = 1\;,\\ &f'(\infty) = 0\;,\;\; \theta(\infty) = 0\;, \end{aligned}$$

where the prime denotes differentiation with respect to η, and $I = V_w\sqrt{\dfrac{2L}{\nu\, U_0}}$.

3. Numerical solution

Equations (2.8) and (2.9) can be written in the integral form

(3.1) $$f'' + f\,f' = S + I + 3\int_0^\eta f'^2(\eta_1)\, d\eta_1\;,$$

(3.2) $$\theta' + \mathrm{Pr}\; f\theta = H + (\gamma\, +\, 1)\;\mathrm{Pr}\;\theta(\eta)\int_0^\eta f'^2(\eta_1)\, d\eta_1\;,$$

where $S = f''(0)$ and $H = \theta'(0)$.
For $\eta \to\infty$

(3.3) $$I + S = -3\int_0^\eta f'^2(\eta)\, d\eta\;,$$

(3.4) $$H = -(\gamma + 1)\;\mathrm{Pr}\int_0^\infty \theta(\eta)\; f'(\eta)\, d\eta\;.$$

For $\eta = 0$

$$f''(0) = 2 - SI, \tag{3.5}$$

$$\theta''(0) = \gamma \Pr . \tag{3.6}$$

We return to the integral Eq. (3.1). By integrating this equation once more, we get

$$f' + \frac{1}{2} f^2 = \frac{1}{2} I^2 + 1 + I\eta + S\eta + 3 \int_0^{\eta} \left(\int_0^{\eta_2} f'^2(\eta_1) d\eta_1 \right) d\eta_2 . \tag{3.7}$$

The iteration algorithm has to be started by substituting on the right hand side (RHS) as adequate zero-order approximation $f_0'(\eta)$ for $f'(\eta)$. By so doing, the procedure is reduced to the sequential solution of the Riccati-type equations:

$$f_n' + \frac{1}{2} f_n^2 = \mathrm{RHS}\,(f_{n-1}'), \qquad n = 1,\ 2,\ \ldots\ . \tag{3.8}$$

We suggest for the initiating function of the iteration scheme the expression

$$f_0'(\eta) = \exp(-S\eta)\,, \tag{3.9}$$

yielding

$$f_0(\eta) = I + \left[\frac{1 - \exp(-S\eta)}{S} \right] . \tag{3.10}$$

By substituting this into the right hand side of Eq. (3.10) and by requiring that the first iteration f_1 on the left-hand side satisfies the boundary conditions (2.10), one obtains in the zero-order approximation

$$S = S_0 = \frac{I \pm \sqrt{I^2 + 6}}{2}\,, \quad f_0''(0) = S_0\,.$$

The equation for the first-order iteration f_1 becomes

$$f_1' + \frac{1}{2} f_1^2 = 1 + \frac{1}{2} I^2 + \left(\frac{3}{2} I + \frac{\sqrt{I^2+6}}{2} - \frac{3}{I + \sqrt{I^2+6}} \right) \eta$$
$$- \left(\frac{3}{6 + 2I^2 + 2I\sqrt{I^2+6}} \right) \left(1 - \exp\left[- \left(I + \sqrt{I^2+6} \right) \eta \right] \right).$$

We now turn to solving the energy Eq. (2.9) by using f and f' the zero-order approximation. Introducing a new variable ξ as

$$\xi = -\Pr \exp(-S\eta),$$

and substituting the solution for f into Eq. (2.9) gives

$$\xi \frac{\partial^2\theta}{\partial\xi^2} + \left[1 - \Pr^* - \frac{\xi}{S^2}\right]\frac{\partial\theta}{\partial\xi} + \gamma^*\theta = 0\,, \tag{3.11}$$

where $\Pr^* = \left(\dfrac{1+I}{S^2}\right)\Pr$, $\gamma^* = \dfrac{\gamma}{S^2}$ with the approximation boundary conditions

$$\theta(-\Pr) = 1, \qquad \theta(0) = 0\,.$$

It can be readily demonstrated that the solution (3.11) in terms of Kummer's function [14] is

$$\theta = \left(\frac{-\xi}{\Pr^*}\right)^{\Pr^*} \frac{M(\Pr^* - \gamma,\ \Pr^* + 1,\ \xi)}{M(\Pr^* - \gamma,\ \Pr^* - 1,\ \Pr^*)} \tag{3.12}$$

where

$$M(a,\ b,\ z) = 1 + \sum_{n=1}^{\infty} \frac{a_n}{b_n}\frac{z^n}{a!}\,,$$

$$a_n = a(a+1)(a+2)\ \ldots\ (a+n-1)\,,$$

$$b_n = b(b+1)(b+2)\ \ldots\ (b+n-1)\,.$$

The local dimensionless surface temperature gradient corresponding to Eq. (3.2) is

$$\theta'(0) = -\Pr^* + \frac{\Pr^*(\Pr^* - \gamma)}{(\Pr^* + 1)} \frac{M(\Pr^* - \gamma + 1,\ \Pr^* + 2,\ -\Pr^*)}{M(\Pr^* - \gamma,\ \Pr^* + 1,\ -\Pr^*)}\,. \tag{3.13}$$

The accuracy of the numerical solutions has been verified – (for the case $I = 0$) – by comparing them with published results [7], (see Table 1).

Table 1. Results for $-f''(0)$ for different values of I

I	0	0.2	0.4	0.6
$f''(0)$	1.28181 1.28180*	1.37889	1.4839	1.59824

* results by ref. [7].

4. Results and discussion

The shear stress on the surface is defined by

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0}, \qquad (\mu \text{ is the viscosity})$$

$$\tau_w = \mu \frac{U_o}{L}\sqrt{\frac{\mathrm{Re}}{2}}\,\exp\left(\frac{3x}{2L}\right) f''(0), \tag{4.1}$$

where $f''(0)$ is the friction coefficient and $\mathrm{Re} = L\,U_o/\,\nu$ is the Reynolds number. The total Nusselt number for heat transfer in the present case is defined by

$$\mathrm{Nu} = x\,\frac{\partial T}{\partial y}\bigg|_{y=0}\bigg/ (T_w - T_\infty)\,,$$

$$\frac{\mathrm{Nu}}{\sqrt{\mathrm{Re}}} = \sqrt{\frac{x}{2L}}\,\theta'(0), \qquad \mathrm{Re} = \frac{U_w\,x}{\nu}. \tag{4.2}$$

Results for the dimensionless temperature profiles and Nusselt numbers are obtained for various values of Prandtl numbers 0.72, 1, 3 and 10 for different values of γ and I. The nondimensional shear stress at the stretched surface presented by Eq. (4.1) is shown in Table 1 for various values of I. It is clear from the table that the friction coefficient increases as suction decreases. Also from Fig. 1, it is seen that the velocity decreases with suction.

Table 2. Results for $-\theta'$ *(0)* for different values of I, Pr and γ

		$-\theta'$ (0)					
I	Pr	$\gamma=-1.5$	$\gamma=-1.0$	$\gamma=-0.5$	$\gamma=0.0$	$\gamma=1.0$	$\gamma=3.0$
0.0	0.72	-0.304049	0.0	0.234344	0.434717	0.767778	1.274760
	1.0	-0.377410	0.0	0.299874	0.549641	0.954779	1.560290
	3.0	- 0.923855	0.0	0.634114	1.122090	1.869070	2.938530
	10.0	-2.200988	0.0	1.308610	2.257430	3.660370	5.628200
0.6	0.72		-0.235096	0.595220	0.7453955	1.014517	1.463863
	1.0		-0.265921	0.802771	0.9872843	1.313957	1.851375
	3.0		-290646	2.166029	2.4933760	3.063438	3.986393
	10.0		-0.111818	6.560464	7.0727307	7.987197	9.518049

From Table 2, it is observed that for $\gamma = -1$ and no suction ($I = 0$), there is no heat transfer between the continuous surface and the medium, and for

$\gamma > -1$ and $I > 0$ it was found that the heat is transferred to the moving surface. For $\gamma > 0$, $I > 0$ the heat is transferred from the surface to the medium.

From Fig. 2, it is clear that the suction decreases the thermal boundary layer. In other words, the suction can be used as a means for cooling.

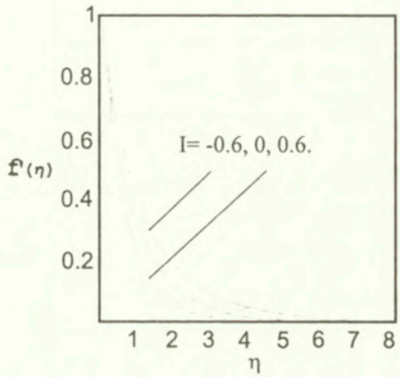

FIG. 1. Velocity profiles for various values of suction

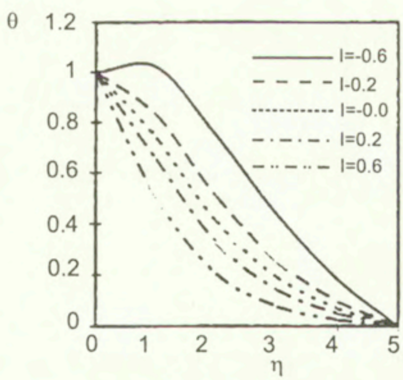

FIG. 2. Temperature profiles aganis η for selected values of I at Pr = 0.72

Sample of the boundary layer temperature for $\gamma = 1$ are presented in Fig. 3. The effect of Prandtl number is such that the thermal boundary layer decreases sharply with increasing Prandtl number. Figure 4 is constructed to present the effect of increasing γ (parameter of temperature) on temperature profile for Pr = 0.72 and the heat is transferred from the continuously stretching surface to the fluid medium.

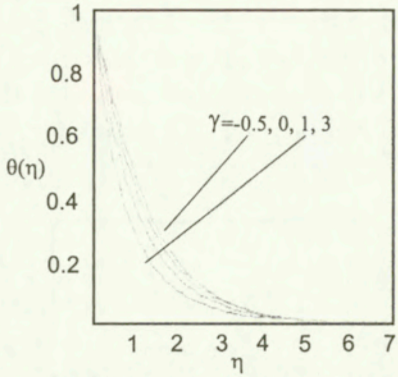

FIG. 3. Teperature profiles for I=0.6,
Pr = 0.72 at γ = -0.5, 0, 1, 3.

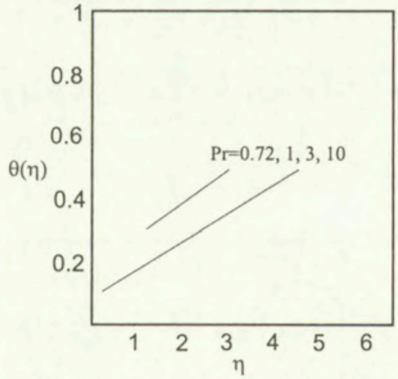

FIG. 4. Temperature profiles for I=0.6,
γ =1, at Pr = 0.72, 1. 3, 10.

5. Conclusions

The heat transfer over an exponentially stretching continuous surface with suction have been examined and compared with the well known results. The heat transfer characteristics for the suction parameter I, temperature parameter γ and the Prandtl number are analyses. The magnitude of γ in the presence of suction affects the direction and quantity of heat flow. For $\gamma = -1$ and no suction ($I = 0$), there is no heat transfer occurring between the moving surface and medium. In other words, the suction enhance heat transfer coefficient and friction coefficient.

The suction can be used as means for cooling the moving continuous surface.

The thickness of the thermal boundary layer decreases with increasing parameter of temperature and suction for all the Prandtl number.

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On material objectivity and reduced constitutive equations

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THE PRINCIPLE OF MATERIAL frame indifference, as it is usually stated, actually consists of two distinct assumptions. Firstly, that the stresses transform like objective tensors under change of the observer, and secondly, that the constitutive equations do not depend on the observer (form-invariance). As a consequence, superimposed rigid body motions also do not effect the material response. In the present work these three statements are formulated independently. The mutual relations between them can be clearly and generally worked out by group-theoretical concepts. If only two of these principles hold for a certain class of materials, then *reduced forms* exist, i. e. forms of constitutive equations that identically fulfil these principles. A general definition of reduced forms is given, its existence is proven, and a method for their construction is formulated and applied to the case of simple elastic materials.

1. Introduction

IN HIS DISSERTATION published in 1955, NOLL stated that under a change of frame, the stresses transform in an objective manner. By this "principle of isotropy of space", or of "objectivity of material properties" as he called it later, he obtained *reduced forms* for simple materials that identically satisfy this principle. This notion of objectivity became quite popular. It was exploited in all branches of continuum physics where material equations are formulated, in order to reduce them. More recently, TRUESDELL/NOLL (1965 Sec. 19 p. 44 ff.) discussed the history of the concept of *material frame-indifference*. Here, one finds the statement. "In fact, *two* principles have been stated and studied. According to the first, which may be called the "Hooke-Poisson-Cauchy form", constitutive equations must be invariant under a superimposed rigid rotation of the *body*. According to the second, which may be called the "Zaremba-Jaumann form", an arbitrary change of *observer* is allowed." This fact could not be expressed clearer. In contrast to this, however, in the rest of the article, the only distinction between the two versions is attributed to orientation, always positive for the Hooke-Poisson-Cauchy form, and either positive or negative for the Zaremba-Jaumann form.

To our knowledge, the consequences of the distinction between these two approaches or interpretations has apparently never really been considered, or worked out, in detail. One purpose of the current work is to attempt just to do this, building on an earlier work (SVENDSEN and BERTRAM [16]). As it turns out, with the help of precise group-theoretic definitions for change of observers or frames, sometimes called *Euclidean observer transformations*, on the one hand, and for superimposed rigid body motions on the other, one can identify three distinct concepts contained in standard formulations of "objectivity" or "frame-indifference". In a sense, the weakest of the three, *Euclidean frame-indifference* (EFI) requires physical quantities associated with Euclidean observers such as stress and heat flux to transform corrotationally between them. Secondly, *form-invariance* (FI) requires (the *form* of) the constitutive relation to be independent of observer. Lastly, *indifference with respect to superimposed rigid-body motions* (IRBM) requires the material response to be independent of arbitrary rotations of the material body with respect to a single observer. As shown in this work, EFI and FI together represent the concept of *material frame indifference* as stated by Truesdell/Noll. In addition, we show that *material frame indifference* is equivalent to IRBM. More precisely, one can show that any two of the concepts EFI, FI and IRBM imply the third one. Or in other words, if one of these principles holds, the remaining two become equivalent.

As discussed in detail in earlier work (e.g., SVENDSEN and BERTRAM [16]), the concepts of FI and IRBM or *material frame indifference* are constitutive in nature, while EFI represents a general principle, apparently holding for all materials[1)] As such assumptions, IRBM or *material frame indifference* appear to be quit reasonable for most material classes subject to non-extreme (i.e., in terms of acceleration and spin) conditions. As is clear from the work of Noll and others, the exploitation of *material frame indifference* leads to a drastic simplification of or *reduction in* the form of constitutive equations. Indeed, invariance with respect to superimposed rigid body motions or, equivalently, material–frame indifference, lead to *reduced forms*. Neither more, nor less can be obtained[2)].

[1)]The subtlety of the concepts and issues inherent in the notion of "objectivity" has led to a number of disagreements between various authors in the literature. One such disagreement had to do with the material behaviour of rarified gases. Using the results from IKENBERRY/TRUESDELL [6], MÜLLER [8] showed that such gases violate IRBM, or equivalently *material frame indifference*. More recently, MURDOCH [9] attempted to refute Müller's conclusion by showing that such materials actually satisfy *material frame indifference*. Because, like Truesell/Noll before him, he tacitly assumed FI from the start in his treatment, however, he could not have shown this, despite his conclusion to the contrary. Indeed, in effect, what he showed was that such gases satisfy EFI. Note that, in contrast to his successors, NOLL [11, 12] clearly stated "In any system of reference, Galilean or not, the constitutive equations must be the same".

[2)]APPLEBY and KADIANAKIS [1] clearly demonstrate that Euclidean frame-indifference is

A second purpose of the present work is the development of a general representaion for such reduced forms and the formulation of a procedure for their construction. More precisely, with the help of an abstract group-theoretic representation for constitutive equations, we are able to (i) define the concept of a *reduced form* in a rather abstract way, (ii) show their existence, and (iii) give a general procedure to construct them.[3)]

2. Kinematics in the euclidean space

Let $\mathscr{V}$ be the three-dimensional vector-space associated with the Euclidean point space. For brevity, we introduce the following tensor sets:

$\mathscr{Lin}$:= the set of all tensors or linear mappings on $\mathscr{V}$

$\mathscr{Inv}$:= the set of all invertible tensors

$\mathscr{Skw}$:= the set of all skew–symmetric tensors

$\mathscr{Sym}$:= the set of all symmetric tensors

$\mathscr{Psym}$:= the set of all symmetric positive-definite tensors

$\mathscr{Orth}$:= the set of all orthogonal tensors.

A subscript + indicates the subset of tensors with positive determinants. Thus, e. g., $\mathscr{Orth}^+$ is the set of *proper* orthogonal tensors. As usual, R stands for the reals. For two sets $\mathscr{A}$ and $\mathscr{B}$, let

- $\mathscr{Map}(\mathscr{A}, \mathscr{B})$ denote the set of all mappings from $\mathscr{A}$ into $\mathscr{B}$
- $\mathscr{Bij}(\mathscr{A}, \mathscr{B})$ denote the set of all bijections from $\mathscr{A}$ onto $\mathscr{B}$

We now come to the basic Euclidean kinematics. As usual, an **observer** or a *frame of reference* can be represented via a reference point for the position

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not enough to obtain reduced forms, by expressing the whole matter in a frame-independent or "intrinsic" way. "It is interesting to note that this approach does not eliminate the need for an invariance principle for equations of state equivalent in effect to the principle of frame-indifference". They use invariance with respect to superimposed rigid-body motions in the form of "invariance under rotations of space time", i. e. essentially the same as we do. This is, however, by no means *equivalent in effect* to Euclidean frame-indifference

[3)] In WANG [20, 21] and in WILLIAMS [22], the theories of invariant forms have been suggested, which fulfil the restrictions imposed by these three principles and by an assumed material symmetry at the same time. However, these presentations become rather complicated and not practical. The present suggestion is not based on either of those and takes only into account the universal restrictions and not the individual ones.

vectors and a vector triad. Let $\mathscr{B}$ represent a material body manifold, $\mathscr{T}$ an open time interval, and ξ an observer, and

$$\kappa_\xi \; : \mathscr{B} \times \mathscr{T} \to \mathscr{V}$$

$$(P, t) \mapsto \mathbf{r}_\xi$$

the *motion of the body* $\mathscr{B}$ during $\mathscr{T}$ with respect to ξ, assigning to each material point P and each instant t the position vector in the Euclidean space.

This mapping is subject to certain regularity requirements which depend on the specific context and thus, shall not be specified in general. The same holds for all time-dependent mappings in the rest of this work, without further mention.

As usual, we have the *velocity*

$$\mathbf{v}_\xi(P,t) = \frac{\partial}{\partial t}\kappa_\xi(P,t) = \kappa_\xi(P,t)^\bullet$$

and the *acceleration*

$$\mathbf{a}_\xi(P,t) = \frac{\partial^2}{\partial t^2}\kappa_\xi(P,t) = \kappa_\xi(P,t)^{\bullet\bullet}.$$

Further, the spatial differential of κ_ξ is the linear mapping from the tangent space $\mathscr{T}_P\mathscr{B}$ to the body manifold at P onto the space of the Euclidean shifters

$$\mathbf{K}_\xi(P,t) = d\kappa_\xi(P,t) : \mathscr{T}_P\mathscr{B} \to \mathscr{V}$$

at $(P,\ t)$, called the *local placement* at P and time t in the motion κ_ξ. It is customary, but not necessary (as we know from NOLL [11, 12]), to use a *reference-placement*

$$\kappa_0 \; : \mathscr{B} \to \mathscr{V}$$

$$P \mapsto \mathbf{X}$$

of the body, and to define the *motion of the body* relative to it by

$$\chi_\xi(X,t) := \kappa_\xi(\kappa_0^{-1}(\mathbf{X}), t).$$

which induces the mapping

$$\chi_\xi \; : \kappa_0(\mathscr{B}) \times \mathscr{T} \to \mathscr{V}$$

$$(\mathbf{X}, t) \mapsto \mathbf{r}_\xi.$$

Note that the time derivatives of χ_ξ and κ_ξ coincide

$$\mathbf{v}_\xi(\kappa_0^{-1}(\mathbf{X}), t) = \frac{\partial}{\partial t}\chi_\xi(\mathbf{X}, t),$$

$$\mathbf{a}_\xi(\kappa_0^{-1}(\mathbf{X}),t) = \frac{\partial^2}{\partial t^2}\chi_\xi(\mathbf{X},t),$$

whereas the corresponding differential, the *deformation gradient*, is according to the chain rule

$$\mathbf{F}_\xi(\mathbf{X},t) = d\chi_\xi(\mathbf{X},t) = d\kappa_\xi(d\kappa_0^{-1}(\mathbf{X}),t) = \mathbf{K}_\xi\,\mathbf{K}_0^{-1} \in \mathscr{Inv}.$$

The function

$$\chi_\xi^X(\mathbf{Y},t) := \chi_\xi(\mathbf{X},t) + \mathbf{F}_\xi(\mathbf{X},t)(\mathbf{Y}-\mathbf{X}) \tag{2.1}$$

stands for the motion of an infinitesimal neighborhood of $\mathbf{X}$; in what follows, this will be referred to as an *infinitesimal motion around* $\mathbf{X}$. The determinant

$$\det\ \mathbf{F}_\xi(\mathbf{X},t) = \frac{\rho_{\xi 0}(\mathbf{X})}{\rho_\xi(\mathbf{X},t)}$$

relates the mass density in the reference placement to that of the current placement. For invertibility we have det $\mathbf{F}_\xi(\mathbf{X},t) \neq 0$ at all times and points. So, its sign is either strictly positive *or* negative, but never changes sign. As the choice of the reference placement is arbitrary, one can do it in such a way, that det $\mathbf{F}_\xi > 0$ without loss of generality. The advantage is twofold. Firstly, we avoid the ambiguity in the above expression. Secondly, it is then *possible* (but not necessary) that the body occupies the reference placement at some time during its motion.

Other important kinematic quantities are the spatial *velocity gradient*

$$\mathbf{L}_\xi := \mathbf{F}_\xi^\circ\,\mathbf{F}_\xi^{-1} \in \mathscr{Lin},$$

$$\mathbf{D}_\xi := \frac{1}{2}\left(\mathbf{L}_\xi + \mathbf{L}_\xi^T\right) \in \mathscr{Sym},$$

and its skew-symmetric part, the *spin tensor*

$$\mathbf{W}_\xi := \frac{1}{2}(\mathbf{L}_\xi - \mathbf{L}_\xi^T) \in \mathscr{Skw},$$

such that

$$\mathbf{L}_\xi = \mathbf{W}_\xi + \mathbf{D}_\xi,$$

T denoting the transpose of a tensor.

The deformation gradient can be subjected to the polar decomposition

$$\mathbf{F}_\xi = \mathbf{R}_\xi\mathbf{U}_\xi \in \mathscr{Inv}^+, \quad \mathbf{U}_\xi \in \mathscr{Psym}, \quad \mathbf{R}_\xi \in \mathscr{Orth}^+$$

with the *right stretch tensor* $\mathbf{U}_\xi$ and the *local rotation tensor* $\mathbf{R}_\xi$.

3. Euclidean observers

Again, all these kinematical concepts generally depend on the *Euclidean observer* denoted by the suffix ξ. As usual, any such observer can be represented via a reference point for the position vectors and a vector triad, both appearing time-independent to the observer by definition.

Although each observer has his own reference instant $t = 0$, this plays no role in continuum mechanics, as most concepts are based on time *differences*. Consequently, this effect of change of observer will not be taken into account here. On the other hand, the spatial part of the transformation is of central importance. If we change the observer from ξ to η, the position vectors are subjected to the **Euclidean transformation**

$$\mathbf{r}_\eta = \mathbf{Q}_\eta^\xi \mathbf{r}_\xi + \mathbf{c}_\eta^\xi \tag{3.1}$$

with $\mathbf{Q}_\eta^\xi \in \mathscr{O}\!rth^+$ and $\mathbf{c}_\eta^\xi \in \mathscr{V}$, both time-dependent. Although it is possible that different observers use different orientations, we assume for simplicity and without loss of generality, that $\det \mathbf{F} > 0$ for all observers. Consequently, all Euclidean transformations are orientation preserving, and $\mathbf{Q}$ is proper orthogonal at all times. We emphasize that $\mathbf{Q}_\eta^\xi$ and $\mathbf{c}_\eta^\xi$ are uniquely determined (as functions of time) by the two involved observers ξ and η, being the same for all bodies and all motions. As such, the pair $E := \{\mathbf{Q}, \mathbf{c}\}$ of time-dependent $\mathbf{Q}$ in $\mathscr{O}\!rth^+$ and $\mathbf{c}$ in $\mathscr{V}$ completely determine such change of observer or Euclidean transformation.

Now, let η, ξ, ζ, be three observers. Analogously to (3.1), we have the transformations

$$\mathbf{r}_\zeta = \mathbf{Q}_\zeta^\eta \mathbf{r}_\eta + \mathbf{c}_\zeta^\eta$$

and

$$\mathbf{r}_\zeta = \mathbf{Q}_\zeta^\xi \mathbf{r}_\xi + \mathbf{c}_\zeta^\xi$$

between them, which yield

$$\mathbf{Q}_\zeta^\xi = \mathbf{Q}_\zeta^\eta \mathbf{Q}_\eta^\xi$$

and

$$\mathbf{c}_\zeta^\eta = \mathbf{Q}_\zeta^\xi \mathbf{c}_\xi^\eta + \mathbf{c}_\zeta^\xi.$$

For the inverse Euclidean transformation we find

$$\mathbf{Q}_\xi^\eta = (\mathbf{Q}_\eta^\xi)^{-1} = (\mathbf{Q}_\eta^\xi)^T$$

and

$$\mathbf{c}_\xi^\eta = -\mathbf{Q}_\xi^\eta \mathbf{c}_\eta^\xi,$$

or, equivalently,

$$E = \{\mathbf{Q}, \mathbf{c}\} \;\Leftrightarrow\; E^{-1} = \{\mathbf{Q}^T, -\mathbf{Q}^T\mathbf{c}\}.$$

Trivially, there is a neutral Euclidean transformation $I = \{\mathbf{I}, \mathbf{o}\}$ such that $\mathbf{Q}$ is the identity tensor $\mathbf{I}$ and $\mathbf{c}$ is the null-vector $\mathbf{o}$ at all times. Clearly,

$$E \circ E^{-1} = I = E^{-1} \circ E$$

holds for all Euclidean transformations E. Here, $\circ$ denotes the composition of mappings. In fact, by these properties the Euclidean transformations form a group under composition, the **Euclidean group** $\mathscr{G}$.

Euclidean transformations with $\mathbf{c}^{\bullet\bullet} \equiv \mathbf{o}$ and $\mathbf{Q}^{\bullet} = \mathbf{0}$ are called *Galilean transformations* which become important as the invariance groups of balance equations, what is beyond the scope of the present considerations.

Clearly, the set of all observers is equipotent to the set of all time-functions with values in $\mathscr{O}rth^+ \times \mathscr{V}$. Therefore, it is equivalent whether a certain property is (i) valid for *one* observer and remains valid under *all* transformations in $\mathscr{G}$, or is (ii) valid for *all* observers.

As almost all physical quantities depend on observers, we have to specify the actions of Euclidean transformations on these quantities. For kinematical quantities, these actions can be uniquely derived from (3.1). It is a common practice to introduce the reference placement to be the same for all observers. This is by no means necessary, but it simplifies some transformations without real loss of generality.

The actions of an $E_\xi^\eta \in \mathscr{G}$ on the following quantities are

- on the velocity

$$\mathbf{v}_\eta = \mathbf{Q}_\eta^\xi \mathbf{v}_\xi + \mathbf{Q}_\eta^{\xi\bullet}\ \mathbf{Q}_\eta^\xi(\mathbf{r}_\xi - \mathbf{c}_\eta^\xi) + \mathbf{c}_\eta^{\xi\bullet},$$

- on the acceleration

$$\mathbf{a}_\eta = \mathbf{Q}_\eta^\xi \mathbf{a}_\xi + \mathbf{c}_\eta^{\xi\bullet\bullet} + 2\mathbf{Q}_\eta^{\xi\bullet}\mathbf{v}_\xi + \mathbf{Q}_\eta^{\xi\bullet\bullet}\mathbf{r}_\xi,$$

- on the infinitesimal motion around $\mathbf{X}$

$$\chi_\eta^{\mathbf{X}}(\mathbf{Y}, t) = \mathbf{Q}_\eta^\xi[\chi_\xi(\mathbf{X}, t) + \mathbf{F}_\xi(\mathbf{X}, t)(\mathbf{Y} - \mathbf{X})] + \mathbf{c}_\eta^\xi = \mathbf{Q}_\eta^\xi(\chi_\xi^{\mathbf{X}}(\mathbf{Y}, t)) + \mathbf{c}_\eta^\xi,$$

- on the deformation gradient

$$\mathbf{F}_\eta = \mathbf{Q}_\eta^\xi \mathbf{F}_\xi, \tag{3.2}$$

- on the velocity gradient

$$\mathbf{L}_\eta = \mathbf{Q}_\eta^\xi \mathbf{L}_\xi \mathbf{Q}_\xi^\eta + \mathbf{Q}_\eta^{\xi\bullet} \mathbf{Q}_\xi^\eta,$$

- on the deformation rate

$$\mathbf{D}_\eta = \mathbf{Q}_\eta^\xi \mathbf{D}_\xi \mathbf{Q}_\xi^\eta,$$

- on the spin tensor

$$\mathbf{W}_\eta = \mathbf{Q}_\eta^\xi \mathbf{W}_\xi \mathbf{Q}_\xi^\eta + \mathbf{Q}_\eta^{\xi\bullet} \mathbf{Q}_\xi^\eta,$$

- on the right stretch tensor

$$\mathbf{U}_\eta = \mathbf{U}_\xi,$$

- and on the local rotation tensor

$$\mathbf{R}_\eta = \mathbf{Q}_\eta^\xi \mathbf{R}_\xi.$$

If we also include the mass density, the *temperature* $\theta_\xi \in \mathscr{R}$, and the (spatial) *temperature gradient* $\mathbf{g}_\xi \in \mathscr{V}$, we assume in addition the actions of $\mathscr{G}$

- on the densities

$$\rho_\eta = \rho_\xi; \qquad \rho_{0\eta} = \rho_{0\xi},$$

- on the temperature

$$\theta_\eta = \theta_\xi,$$

- on the temperature gradient

$$\mathbf{g}_\eta = \mathbf{Q}_\eta^\xi \, \mathbf{g}_\xi.$$

Generally speaking, the action a of a group $\mathscr{G}$ on some set $\mathscr{A}$ is a group-(homo) morphism

$$a: \mathscr{G} \to \mathscr{B}ij(\mathscr{A}, \mathscr{A})$$

from $\mathscr{G}$ to the group of all automorphisms (= bijections) of $\mathscr{A}$. This means that it is compatible with the four group axioms

- $a(E_2 \circ E_1) = a(E_2) \circ a(E_1)$
- $a((E_3 \circ E_2) \circ E_1) = a(E_3 \circ (E_2 \circ E_1))$
- $a(I) = I_{\mathscr{A}}$
- $a(E_1^{-1}) = a(E_1)^{-1}$

$\forall \; E_i \in \mathscr{G}.$

By the above examples, we see that the actions of $\mathscr{G}$ on different quantities are in general different. While some of them depend on both $\mathbf{Q}$ and $\mathbf{c}$ (like velocity, acceleration), others do not depend on $\mathbf{c}$ (like all gradients), or depend neither on $\mathbf{Q}$ nor on $\mathbf{c}$ (mass density, temperature). Most, but not all of the above actions are **instantaneous**, i. e., only the momentary values of $\mathbf{Q}$ and $\mathbf{c}$ enter the transformation. To make this clearer, we write the arguments of an example for such an action, namely that of the deformation gradient

$$\mathbf{F}_\eta(\mathbf{X},t) = \mathbf{Q}^\xi_\eta(t)\mathbf{F}_\xi(\mathbf{X},t).$$

As a counter example, the action for the spin tensor is not instantaneous in this sense, as also $\mathbf{Q}^{\xi\bullet}_\eta$ enters.

In many cases, there are more actions than just that induced by the unique identity $I = \{\mathbf{I}, \mathbf{o}\}$ in $\mathscr{G}$ that leave some physical quantity unchanged. And for some $E \in \mathscr{G}$, there are often more inverse actions than just the one belonging to $E^{-1} \in \mathscr{G}$. It also happens that certain actions commute, whereas $\mathscr{G}$ is clearly a non-commutative group.

Two kinds of physical quantities are very important for what follows, namely the corrotational ones and the invariant ones. We call a quantity φ **corrotational** (sometimes also called *objective* or *tensorial*) if

$$\begin{aligned} \varphi_\eta &= \varphi_\xi && \text{for scalars,}\\ \varphi_\eta &= \mathbf{Q}^\xi_\eta\,\varphi_\xi && \text{for vectors,}\\ \varphi_\eta &= \mathbf{Q}^\xi_\eta\,\varphi_\xi\,\mathbf{Q}^\eta_\xi && \text{for tensors,}\end{aligned}$$

and **invariant** if

$$\varphi_\eta = \varphi_\xi \qquad \text{in all cases.}$$

Clearly, the actions on corrotational and invariant quantities are instantaneous.

Note that only $\mathbf{Q}^\xi_\eta$ acts on corrotational and invariant quantities. In particular, the translational acceleration $\mathbf{c}^{\xi\bullet\bullet}_\eta$ and the angular velocity $\mathbf{Q}^{\xi\bullet}_\eta$ do not influence corrotational quantities. Examples include:

- for invariant quantities: ρ, θ, $\mathbf{U}$,
- for corrotational quantities: ρ, $\mathbf{g}$, $\mathbf{D}$,

whereas

- $\mathbf{F}$, $\mathbf{R}$ are acted on instantaneously, and
- the others $\mathbf{v}$, $\mathbf{a}$, $\mathbf{L}$, $\mathbf{W}$ are neither corrotational nor instantaneous.

4. Superimposed rigid body motions

Apart from (and independent of) observer changes, another transformation class is very useful in continuum mechanics, namely that of *superimposed rigid body motions*. Here, all quantities of this section are taken with respect to a fixed, but arbitrary observer, if not otherwise stated.

DEFINITION 1. *Let χ be the motion of a body $\mathscr{B}$. Let $\{\boldsymbol{Q}(t), \boldsymbol{c}(t)\}$ be functions of time with values in $\mathscr{O}rth^+ \times \mathscr{V}$. Then*

$$\chi^*(\boldsymbol{X},t) = \boldsymbol{Q}(t)\chi(\boldsymbol{X},t) + \boldsymbol{c}(t) \tag{4.1}$$

is called ***superimposed rigid body motion*** *(RBM) of χ.*

Clearly, χ^* is a motion of $\mathscr{B}$ iff χ is. Conversely, then, χ is also a superimposed rigid body motion of χ^*. Mathematically, the superimposed rigid body motions are identical to Euclidean transformations, and thus form the same group $\mathscr{G}$. Physically, however, two observers watching the same motion is something quite different from one observer watching two different motions. This distinction must be kept in mind, even if formally the same notations appear in the formulas.

As a consequence, the actions of RBMs on all kinematical quantities are the same as those of the Euclidean group in the preceding section, if we drop the observer indices ξ and η. If the temperature is considered as a material state property, then the same holds for the temperature and its gradient. For other quantities, however, the actions of Euclidean transformations can be different from those of RBM's, as we will see later.

5. Constitutive equations

In continuum mechanics, it is customary to consider the kinematical quantities as independent variables, and all dynamical ones such as stresses, couples, forces, etc. as dependent ones. If generalized to thermodynamics, motion and temperature are considered as independent, whereas heat flux, energy, entropy, stresses, etc. are taken as dependent.

For elastic materials, only the current values of the variables appear in the constitutive equations. For materials with memory, however, past values can also influence the present values of the dependent variables. In such cases, higher time derivatives, finite kinematic process segments or even the (semi-infinite) history of the motion may appear as arguments in the constitutive equations.

Let $\mathscr{X}$ be a set or space of such **independent variables** of a certain class of materials, and $\mathscr{Y}$ a set of corresponding **dependent variables**. In most cases, the identification of $\mathscr{Y}$ is clear and the same for a broad class of materials.

The set $\mathscr{X}$, however, depends on the specific framework and/or materials under consideration.

For non-polar, purely mechanical behavior, for example, $\mathscr{Y}$ is just the set of all symmetric tensors $\mathscr{Sym}$, each of them being a candidate for the Cauchy stress tensor. On the other hand, for simple materials, $\mathscr{X}$ could be chosen as

- the set of all semi-infinite deformation histories $\mathbf{F}(\tau)$, $-\infty < \tau \leqslant t$;
- the set of all finite deformation processes $\mathbf{F}(\tau)$, $0 \leqslant \tau \leqslant t$.

The easiest case is that of a simple non-polar elastic material. Here, the Cauchy stresses are assumed to depend on the current value of the deformation gradient $\mathbf{F}(t)$, and $\mathscr{X}$ equals the set of all invertible tensors with positive determinant.

For a simple viscous gas or fluid, the stresses depend on the mass density ρ and the current velocity gradient $\mathbf{L}$, and $\mathscr{X}$ is the set-product of the positive reals and second order tensors.

The identification of the spaces of variables is, in general, not a trivial task (see Bertram [2, 4]). But it becomes easier to solve, if it is restricted to specific material classes.

With this we can give the notion of a constitutive equation a rather general form.

Principle of Determinism: *For a given class of materials, there exist two sets, $\mathscr{X}$ and $\mathscr{Y}$, and for any observer ξ a* ***constitutive equation*** $f_\xi \in \mathscr{Map}(\mathscr{X}, \mathscr{Y})$.

We assume that a change of observer, represented by an element of $\mathscr{G}$, induces the actions

- on the independent variables

$$a : \mathscr{G} \to \mathscr{Bij}(\mathscr{X}, \mathscr{X}) \mid E \quad \mapsto \quad a_E := a(E);$$

- on the dependent variables

$$b : \mathscr{G} \to \mathscr{Bij}(\mathscr{Y}, \mathscr{Y}) \mid E \mapsto b_E := b(E).$$

If these actions are specified, then the action

- on the constitutive equations

$$c : \mathsf{G} \to \mathscr{Bij}\{\mathscr{Map}(\mathscr{X}, \mathscr{Y}), \mathscr{Map}(\mathscr{X}, \mathscr{Y})\} \mid E \mapsto c_E := c(E)$$

is determined by

$$f_\eta = c_E(f_\xi) := b_E \circ f_\xi \circ a_E^{-1} \qquad \forall\, E \in \mathscr{G}. \tag{5.1}$$

So, if a constitutive equation has once been identified by one observer, then by virtue of EFI it is determined for all other observers.

How can these actions be determined? As the members of $\mathscr{X}$ are kinematical quantities, their transformations can be deduced from (3.1) uniquely. For the dependent variables, however, the action b cannot be deduced from (3.1), but rather is the subject of another principle. In particular, Cauchy stresses, heat flux, internal or free energy, and entropy are assumed to be corrotational. Nobody has ever been able to prove this assumption generally, and it will probably never be possible to do so. Therefore, the following assumption is axiomatic in nature.

EFI: Euclidean frame-indifference (ZAREMBA, 1903; JAUMANN, 1906)[4)] *The dependent variables in $\mathscr{Y}$ are corrotational (or objective) under the action b of the Euclidean group.*

This means for the Cauchy stresses

$$\mathbf{T}_\eta = \mathbf{Q}_\eta^\xi \, \mathbf{T}\xi \, \mathbf{Q}_\eta^{\xi T}, \tag{5.2}$$

for the heat flux

$$\mathbf{q}_\eta = \mathbf{Q}_\eta^\xi \, \mathbf{q}_\xi,$$

and for certain scalar variables φ like internal or free energy, entropy, etc.

$$\varphi_\eta = \varphi_\xi.$$

Clearly, the corrotationality of the dependent variables does not hold for all choices of them. If the Cauchy stresses are corrotational, then the 2. Piola-Kirchhoff stresses are invariant under Euclidean transformations. Therefore, the above form of EFI depends on the specific choice of $\mathscr{Y}$ as spatial ones.

As an example, we consider an *elastic material*, where the independent variables consist of the current deformation gradient $\mathbf{F}$, and the dependent ones of the Cauchy stress tensor $\mathbf{T}$. Hence,

$$\mathscr{X} \equiv Inv^+, \quad Y \equiv \mathscr{S}ym.$$

Then, by (3.2), and the EFI in the form of (5.2), we obtain the transformation

$$f_\eta(\mathbf{F}_\eta) = \mathbf{Q}_\xi^\eta \, f_\xi(\mathbf{Q}_\xi^\eta \mathbf{F}_\eta) \, \mathbf{Q}_\eta^\xi$$

between the constitutive equations via the relative rotation $\mathbf{Q}_\xi^\eta$.

In general, the constitutive function also depends on the observer. Only for certain classes of materials constitutive equations themselves are invariant and the following principle is valid (see BERTRAM [3]).

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4) For historical sources see TRUESDELL/NOLL [18], p. 47, App. 19 A, TRUESDELL [19], Ch. 3.

FI: **Form-invariance**[5)] *The constitutive equation f is* **invariant** *under Euclidean transformations, i. e.*

$$f_\eta \equiv f_\xi \tag{5.3}$$

holds for all observers.

Note that *together*, EFI and FI imply that the induced action c_E (5.1) of the Euclidean group on $\mathcal{Map}(\mathcal{X}, \mathcal{Y})$ is the identity, i. e.

$$c_E(f_\xi) = f_\xi \qquad \forall\ E \in \mathcal{G}. \tag{5.4}$$

The consequences of this principle will be investigated later.

Let us next consider the actions of superimposed rigid body motions on the constitutive equations. The action of $\mathcal{G}$ on $\mathcal{X}$ resulting from (4.1) is formally the same as that of the Euclidean group. In contrast to EFI, and like FI, the principle to follow does not generally hold, as counterexamples show.

IRBM: **Indifference with respect to superimposed rigid body motions** (Hooke 1678, Poisson 1831, Cauchy 1829)

The dependent variables are corrotational under the action of the superimposed rigid body motions $\mathcal{G}$ (which thus coincides with b).[6)]
Since the observer ξ is held fixed, here, this takes the form

$$f_\xi \circ a_E = b_E \circ f_\xi \quad \text{or} \quad c_E(f_\xi) = f_\xi, \tag{5.5}$$

formally analogous to (5.4).

For a mechanical material, this would mean that the Cauchy stresses will simply be rotated together with the body, but not modified otherwise. This condition does not hold for rarified gases under fast rotations (see MÜLLER [8]), what is discussed in another paper SVENDSEN, BERTRAM, [16].

By (5.5), we immediately see the following

PROPOSITION 1. *Let $f_\xi \in \mathcal{Map}(\mathcal{X}, \mathcal{Y})$ be a constitutive equation which fulfills* IRBM *or* FI . *Then the following implication holds for all Euclidean transformations E :*

$$a_E \ \text{ is the identity on } \mathcal{X} \Rightarrow\ b_E \ \text{ is the identity on } f_\xi(\mathcal{X}).$$

This will be trivially the case if E is the identity. But this is, by no means, the only case.

Now, by comparison of the different invariance principles in the forms (5.1), (5.3), (5.5), we immediately obtain the following

[5)]In relativity EINSTEIN [5] called the analogous *Principle of Relativity.*

[6)]LEIGH([7], as an exception, introduces this carefully seperated from EFI and FI. See also MUSCHIK [10] and SPEZIALE [15].

PROPOSITION 2. *Let $f_\xi \in \mathcal{M}ap(\mathcal{X}, \mathcal{Y})$ be a constitutive equation. Then the following implications hold:*

- EFI $\wedge$ FI $\Rightarrow$ IRBM : $c_E(f_\xi) = f_\eta \quad \wedge \quad f_\eta = f_\xi \quad \Rightarrow \quad c_E(f_\xi) = f_\xi,$

- EFI $\wedge$ IRBM $\Rightarrow$ FI : $c_E(f_\xi) = f_\eta \quad \wedge \quad c_E(f_\xi) = f_\xi \quad \Rightarrow \quad f_\eta = f_\xi,$

- FI $\wedge$ IRBM $\Rightarrow$ EFI : $f_\eta = f_\xi \quad \wedge \quad c_E(f_\xi) = f_\xi \quad \Rightarrow \quad c_E(f_\xi) = f_\eta.$

In many papers and books, influenced by TRUESDELL/NOLL [18], form-invariance is tacitly assumed as a part of material frame indifference, which is here equivalent to EFI and FI together. As the above proposition shows, EFI and IRBM are then indistinguishable. In such a theory, materials, the response of which is affected by superimposed rigid body motions, cannot be described.

6. Reduced constitutive equations

If a material satisfies EFI, it is sufficient to identify the constitutive equation for one single observer, since by (5.1) it can immediately be transformed to any other Euclidean observer. By this procedure EFI is identically fulfilled, i. e., no further restrictions are imposed on the constitutive equation by EFI.

If in addition to EFI, IRBM or, equivalently, FI holds, the number of candidates for constitutive equations in some $\mathcal{M}ap(\mathcal{X}, \mathcal{Y})$ is drastically reduced. In the rest of this section, this reduction will be worked out. For this purpose, we consider exclusively materials for which both EFI and IRBM are satisfied, so that FI also holds. Let us denote all such constitutive equations in $\mathcal{M}ap(\mathcal{X}, \mathcal{Y})$ by $\mathcal{R}ed(\mathcal{X}, \mathcal{Y})$.

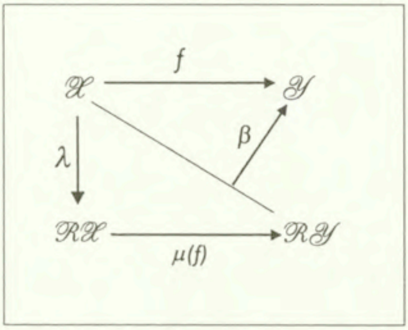
Our aim is to find representations for this set of constitutive equations. For this purpose, we will define two sets $\mathcal{R}\mathcal{X}$ and $\mathcal{R}\mathcal{Y}$, called *reduced sets of independent* and *dependent variables*, respectively, such that any element of $\mathcal{M}ap(\mathcal{R}\mathcal{X}, \mathcal{R}\mathcal{Y})$ uniquely corresponds to an element of $\mathcal{R}ed(\mathcal{X}, \mathcal{Y})$. This means that $\mathcal{R}ed(\mathcal{X}, \mathcal{Y})$ and $\mathcal{M}ap(\mathcal{R}\mathcal{X}, \mathcal{R}\mathcal{Y})$ are isomorphic by means of a natural bijection.

Note that $\mathcal{R}\mathcal{X}$ and $\mathcal{R}\mathcal{Y}$ are not necessarily subsets of $\mathcal{X}$ or $\mathcal{Y}$, respectively, but rather independent. Let us put this in precise terms by the following concept of reduced forms.

DEFINITION 2. *Let $\mathcal{R}ed(\mathcal{X}, \mathcal{Y}) \subset \mathcal{M}ap(\mathcal{X}, \mathcal{Y})$ be as before. Let $\mathcal{R}\mathcal{X}$ and $\mathcal{R}\mathcal{Y}$ be two sets, $\lambda \in \mathcal{M}ap(\mathcal{X}, \mathcal{R}\mathcal{X})$ a surjection, $\beta \in \mathcal{M}ap(\mathcal{X} \times \mathcal{R}\mathcal{X}, \mathcal{Y})$, and $\mu \in \mathcal{B}ij\{\mathcal{R}ed(\mathcal{X}, \mathcal{Y}), \mathcal{M}ap(\mathcal{R}\mathcal{X}, \mathcal{R}\mathcal{Y})\}$. Then $(\mathcal{R}\mathcal{X}, \mathcal{R}\mathcal{Y}, \lambda, \beta, \mu)$ is called a **reduced form**, if*

$$f(x) = \beta[x, \mu(f) \circ \lambda(x)] \qquad \forall\, x \in \mathcal{X}, \ \ \forall\, f \in \mathcal{R}ed(\mathcal{X}, \mathcal{Y}). \tag{6.1}$$

The practical benefit from such a reduced form is the following. We could pick out any function of $\mathscr{M}ap(\mathscr{R}\mathscr{X}, \mathscr{R}\mathscr{Y})$, and this would correspond uniquely (by μ^{-1}) to a constitutive equation in $\mathscr{M}ap(\mathscr{X}, \mathscr{Y})$, which automatically satisfies the three principles, and hence is in $\mathscr{R}ed(\mathscr{X}, \mathscr{Y})$.



The following construction not only assures that the concept of reduced forms is not vacuous, but also gives a general procedure to construct the reduced forms.

7. Construction of a reduced form

The construction consists of three parts. Firstly, we will construct the reduced spaces of the independent variables, secondly, of the dependent variables and finally, the bijection μ is defined so that the condition (6.1) will be fulfilled.

1. *Construction of the reduced state of independent variables*
   We introduce an equivalence relation $\sim$ on $\mathscr{X}$ in the following way. Let x_1, $x_2 \in \mathscr{X}$, then $x_1 \sim x_2$, if there exists an $E \in \mathscr{G}$, such that $a_E(x_1) = x_2$[7]. Let $\underline{\mathscr{X}}$ be the quotient space of $\sim$, and $\varepsilon : \mathscr{X} \to \underline{\mathscr{X}}$ be the natural surjection of $\sim$. Moreover, let η be a section of the fiber bundle $\underline{\mathscr{X}}$ (sometimes called a selection function), i.e., an $\eta : \underline{\mathscr{X}} \to \mathscr{X}$, such that $\varepsilon \circ \eta = \mathbf{I}_{\underline{\mathscr{X}}}$, the identity on $\underline{\mathscr{X}}$. In general, η is not unique.
   We define $\mathscr{R}\mathscr{X} := (\eta \circ \varepsilon)(\mathscr{X})$, which is clearly a subset of $\mathscr{X}$. Then

   $$\lambda := \eta \circ \varepsilon : \mathscr{X} \to \mathscr{R}\mathscr{X}$$

   is surjective by definition. Consequently, λ has the following property: $\lambda(x) \sim x, \forall x \in \mathscr{X}$, therefore an $E(x) \in \mathscr{G}$ exists such that $\lambda(x) = a_{E(x)}(x)$.
2. *Construction of the reduced set of dependent variables*
   Let $\mathscr{R}\mathscr{Y} \equiv \mathscr{Y}$, simply.

[7] This means that x_1 and x_2 lie in the same orbit of a.

3. *Construction of the bijection* μ
Let $f \in \mathcal{Red}(\mathcal{X}, \mathcal{Y})$. We define

$$\mu : \mathcal{Red}(\mathcal{X}, \mathcal{Y}) \to \mathcal{Map}(\mathcal{RX}, \mathcal{RY})$$

by the restriction

$$\mu : f \mapsto f|_{\mathcal{RX}}$$

and

$$\beta : \mathcal{X} \times \mathcal{RY} \to \mathcal{Y}$$

by

$$\beta(x, y) := b_{E(x)}^{-1}(y)$$

with the specific $E(x) \in \mathcal{G}$ that gives $\lambda(x) = a_{E(x)}(x)$, and therefore generally depends on x. Then, by IRBM and this $E(x)$,

$$b_{E(x)} \circ f(x) = f \circ a_{E(x)}(x) = f \circ \lambda(x) = f|_{\mathcal{RX}} \circ \lambda(x) \qquad \forall x \in \mathcal{X}$$

and

$$\beta[x, \mu(f) \circ \lambda(x)] = \beta[x, f_{\mathcal{RX}} \circ \lambda(x)] = \beta[x, f \circ a_{E(x)}(x)]$$
$$= \beta[x, b_{E(x)} \circ f(x)] = b_{E(x)}^{-1} \circ b_{E(x)} \circ f(x) = f(x) \qquad \forall x \in \mathcal{X}.$$

The inverse of μ is given by

$$\mu^{-1} : g(x) \mapsto f(x) = \beta[x, g \circ \lambda(x)] \qquad \forall\, x \in \mathcal{X}.$$

We will next show that such $\mu^{-1}(g)$ fulfils the invariance-requirement (5.5) for all $g \in \mathcal{Map}(\mathcal{RX}, \mathcal{RY})$. We take an arbitrary transformation $Q \in \mathcal{G}$. Clearly, $a_Q(x)$ lies in the same orbit as $\lambda(x)$ and x: $\lambda(x) \sim x \sim a_Q(x)$, for all $x \in \mathcal{X}$. Thus

$$\lambda(x) = \lambda \circ a_Q(x).$$

We now evaluate the above expression for $f := \mu^{-1}(g)$ at $a_Q(x)$ and obtain

$$f(a_Q(x)) = \beta[a_Q(x), g \circ \lambda(a_Q(x))] = f \circ a_Q(x) = \beta[a_Q(x), g \circ \lambda \circ a_Q(x)]$$
$$= b_{\underline{E}^{-1}} \circ g \circ \lambda \circ a_Q(x) = b_{\underline{E}^{-1}} \circ g \circ \lambda(x)$$

with the specific transformation $\underline{E} := E(a_Q(x)) \in \mathcal{G}$ that gives $\lambda(a_Q(x)) = a_{\underline{E}}(a_Q(x))$. It can easily be seen that $\underline{E} = E(x)\, Q^{-1}$ as

$$\lambda(x) = a_{E(x)}(x) = \lambda \circ a_Q(x) = a_{\underline{E}} \circ a_Q(x) = a_{\underline{E}Q}(x) \qquad \forall\, x \in \mathcal{X}.$$

Therefore

$$b_{\underline{E}} = b_{E(x)Q^{-1}} = b_{E(x)} \circ b_{Q^{-1}} \Leftrightarrow b_{\underline{E}^{-1}} = b_Q \circ b^{-1}_{E(x)}.$$

We continue with this

$$f \circ a_Q(x) = b_Q \circ b^{-1}_{E(x)} \circ g \circ \lambda(x) = b_Q \circ f(x).$$

Thus, $\mu^{-1}(g) \in \mathscr{Red}(\mathscr{X}, \mathscr{Y})$.

We show that $\mu \circ \mu^{-1}$ is the identity on $\mathscr{Map}(\mathscr{RX}, \mathscr{RY})$.

$$\mu \circ \mu^{-1}(g)(x) = \beta[x, g \circ \lambda(x)]\,|_{\mathscr{RX}} \qquad \forall\, x \in \mathscr{RX}$$

$$= b^{-1}_{E(x)} \circ g \circ a_{E(x)}(x)$$

with

$$\lambda(x) = a_{E(x)}(x).$$

In this particular case with the restriction to $\mathscr{RX}$, λ is the identity and so is $a_{E(x)}$. By Proposition 1, $b_{E(x)}$ as well as $b^{-1}_{E(x)}$ are also identities. Thus

$$\mu \circ \mu^{-1}(g)(x) = g(x) \qquad \forall\, x \in \mathscr{RX}, \qquad \forall\, g \in \mathscr{Map}(\mathscr{RX}, \mathscr{RY}).$$

On the other hand

$$\begin{aligned}
\mu^{-1} \circ \mu(f)(x) =& \beta[x, \mu(f) \circ \lambda(x)] \\
=& \beta[x, f\,|_{\mathscr{RX}} \circ \lambda(x)] \\
=& \beta[x, f \circ a_{E(x)}(x)] \\
=& \beta[x, b_{E(x)} \circ f(x)] \\
=& b^{-1}_{E(x)} \circ b_{E(x)} \circ f(x) \\
=& f(x) \qquad \forall\, x \in \mathscr{X}, \;\; \forall\, f \in \mathscr{Red}(\mathscr{X}, \mathscr{Y}).
\end{aligned}$$

Therefore $\mu^{-1} \circ \mu$ is the identity on $\mathscr{Red}(\mathscr{X}, \mathscr{Y})$, and μ is shown to be a bijection. Thus we have proven the following

THEOREM. *Let $\mathscr{Red}(\mathscr{X}, \mathscr{Y})$ be non-empty. Then $(\mathscr{RX}, \mathscr{RY}, \lambda, \beta, \mu)$ is a reduced form.*

REMARK 1. In the construction no use has been made of the corrotationality of $\mathscr{Y}$ under the action b. The theorem remains valid for any other action b.

REMARK 2. The construction of this reduced form is based on the choice of η . Apart from trivial cases, this selection function is not unique, and each choice gives rise to a different reduced form.

To illustrate these concepts, a simple and well-known example shall be given next.

EXAMPLE. Let us choose the class of *simple elastic materials*, in which the stresses at a material point are assumed to depend on the current infinitesimal motion around the point. For a fixed $\mathbf{X} \in \mathscr{B}$ and $t \in \mathscr{T}$, χ^X is uniquely determined by the vector $\mathbf{r} = \chi_\xi(\mathbf{X},t) \in \mathscr{V}$ and the tensor $\mathbf{F}_\xi(\mathbf{X}, t) \in \mathscr{Inv}^+$ according to (2.1). The following identifications specify the sets and functions for this class:

- $\mathscr{X} \equiv \mathscr{V} \times \mathscr{Inv}^+$

and

- $\mathscr{Y} \equiv \mathscr{Sym}$.

Each element $\{\mathbf{r}, \mathbf{F}\} \in \mathscr{X}$ stands for an infinitesimal motion, and each element $\mathbf{T} \in \mathscr{Sym}$ for the Cauchy stress tensor. An elastic constitutive equation is then, according to the principle of determinism,

$$f : \mathscr{V} \times \mathscr{Inv}^+ \to \mathscr{Sym}$$

$$\{\mathbf{r}, \mathbf{F}\} \mapsto \mathbf{T}.$$

The action of $\mathscr{G}$ on $\mathscr{X}$ is (see 3.1, 3.2)

$$a_E : \mathscr{Inv}^+ \to \mathscr{Inv}^+$$

$$\{\mathbf{r}, \mathbf{F}\} \mapsto \{\mathbf{Qr} + \mathbf{c}, \mathbf{QF}\} \qquad \text{with } E = \{\mathbf{Q}, \mathbf{c}\},$$

and on $\mathscr{Y}$

$$b_E : \mathscr{Sym} \to \mathscr{Sym}$$

$$\mathbf{T} \mapsto \mathbf{Q}\ \mathbf{T}\ \mathbf{Q}^T$$

with $E = \{\mathbf{Q}, \mathbf{c}\}$, i. e. corrotational. Now $\mathscr{Red}(\mathscr{X}, \mathscr{Y})$ consists of all $f \in \mathscr{Map}(\mathscr{X}, \mathscr{Y})$, such that

$$\mathbf{Q} f(\mathbf{r}, \mathbf{F}) \mathbf{Q}^T = f(\mathbf{Qr} + \mathbf{c}, \mathbf{QF}) \qquad \forall \mathbf{c} \in \mathscr{V}, \quad \forall \mathbf{Q} \in \mathscr{Orth}^+.$$

We now exemplify the three steps of the above Proof for this class of material.

1. Two pairs $\{\mathbf{r}_1, \mathbf{F}_1\}$, $\{\mathbf{r}_2, \mathbf{F}_2\} \in \mathscr{X}$ are considered as equivalent, if there exists a $\mathbf{c} \in \mathscr{V}$ and a $\mathbf{Q} \in \mathscr{Orth}^+$, such that $\mathbf{r}_2 = \mathbf{Qr}_1 + \mathbf{c}$ and $\mathbf{F}_2 = \mathbf{QF}_1$. The first condition can always be fulfilled. By the polar decomposition $\mathbf{F}_i$

$= \mathbf{R}_i\, \mathbf{U}_i$, $i = 1, 2$, $\mathbf{R}_i \in \mathscr{O}\!rth^+$, $\mathbf{U}_i = (\mathbf{F}_i^T\, \mathbf{F}_i)^{1/2} \in \mathscr{P}\!sym$, the second one is fulfilled iff $\mathbf{U}_1 = \mathbf{U}_2$, i. e.

$$\{\mathbf{r}_1, \mathbf{F}_1\} \sim \{\mathbf{r}_2, \mathbf{F}_2\} \Leftrightarrow (\mathbf{F}_1^T \mathbf{F}_1)^{1/2} = (\mathbf{F}_2^T \mathbf{F}_2)^{1/2}.$$

Thus, our choice is $E = \{\mathbf{Q} = \mathbf{R}_2\, \mathbf{R}_1^T,\ \mathbf{c} = \mathbf{r}_2 - \mathbf{Q}\mathbf{r}_1\}$. We identify $\mathscr{R}\mathscr{X} \equiv \{\mathbf{o}\} \times \mathscr{P}\!sym \subset \mathscr{X} \equiv \mathscr{V} \times \mathscr{I}\!nv^+$

$$\lambda : \mathscr{V} \times \mathscr{I}\!nv^+ \to \{\mathbf{o}\} \times \mathscr{P}\!sym$$

$$\{\mathbf{r}, \mathbf{F}\} \mapsto \{\mathbf{o}, \mathbf{U} = (\mathbf{F}^T\mathbf{F})^{1/2}\},$$

which is clearly surjective. Obviously, $\lambda(\mathbf{r}, \mathbf{F}) = \{\mathbf{o}, \mathbf{U}\} \sim \{\mathbf{r}, \mathbf{F}\}$, so that with $\mathbf{c} \equiv -\mathbf{Q}\ \mathbf{r} \in \mathscr{V}$ and $\mathbf{Q} \equiv \mathbf{R}^T \in \mathscr{O}\!rth^+$ we obtain

$$\lambda(\mathbf{r}, \mathbf{F}) = \mathbf{a}_{E(x)}(\mathbf{r}, \mathbf{F}) = \{\mathbf{o}, \mathbf{U}\}$$

with

$$E(x) = \{\mathbf{R}^T = (\mathbf{F}^T\mathbf{F})^{-1/2}\mathbf{F}^T, -\mathbf{R}^T\mathbf{r}\} \in \mathscr{G}.$$

Thus $\{\mathbf{o}\} \cup \mathscr{P}\!sym$ stands for the quotient space $\underline{\mathscr{X}}$, and the selection function η is the inclusion of $\{\mathbf{o}\} \cup \mathscr{P}\!sym$ in $\mathscr{V} \times \mathscr{I}\!nv^+$.

2. We take $\mathscr{R}\mathscr{Y} \equiv \mathscr{S}\!ym$, whose elements stand for the back-rotated or *relative stress tensor*

$$\mathbf{T}_{\text{rel}} := \mathbf{R}^T\mathbf{T}\ \mathbf{R}. \tag{7.1}$$

3. Let

$$\beta(\{\mathbf{r}, \mathbf{F}\}, \mathbf{T}_{\text{rel}}) := b^{-1}_{E(x)}(\mathbf{T}_{\text{rel}}) = \mathbf{Q}^T\mathbf{T}_{\text{rel}}\mathbf{Q} = \mathbf{T} \tag{7.2}$$

and

$$\mu(f) := f\,|_{\{\mathbf{o}\} \times \mathscr{P}\!sym}.$$

Now we have by (7.1)

$$b_{E(x)} \circ f(\mathbf{r}, \mathbf{F}) = \mathbf{R}^T f(\mathbf{r}, \mathbf{F})\mathbf{R} = f(\mathbf{o}, \mathbf{U}) = f \circ \lambda(\mathbf{r}, \mathbf{F}) = f\,|_{\{\mathbf{o}\}\times\mathscr{P}\!sym}(\mathbf{o}, \mathbf{U}),$$

and

$$\begin{aligned}\beta[\{\mathbf{r}, \mathbf{F}\}, \mu(f) \circ \lambda(\mathbf{r}, \mathbf{F})] &= \beta[\{\mathbf{r}, \mathbf{F}\}, f\,|_{\{\mathbf{o}\}\times\mathscr{P}\!sym}(\mathbf{o}, \mathbf{U})]\\ &= \beta[\{\mathbf{r}, \mathbf{F}\}, f(\mathbf{o}, \mathbf{R}^T\mathbf{F})]\\ &= \beta[\{\mathbf{r}, \mathbf{F}\}, \mathbf{R}^T f(\mathbf{r}, \mathbf{F})\mathbf{R}]) = \mathbf{T} = f(\mathbf{r}, \mathbf{F}).\end{aligned}$$

As the first argument $\mathbf{o}$ of $\mu(f)$ is trivial, we can drop it. Hence, IRBM does not allow the stresses to depend on $\mathbf{r}$. Moreover, the dependence of the stresses on $\mathbf{F} = \mathbf{RU}$ is only arbitrary in the stretching part $\mathbf{U}$, but rather specific in the rotational part $\mathbf{R}$.

As mentioned before, the selection function η is not unique, and this is not the only reduced form. One could also have taken the right Cauchy-Green tensor $\mathbf{C} = \mathbf{U}^2$ or the Green tensor $1/2(\mathbf{C} - \mathbf{I})$ instead of $\mathbf{U}$, etc.

In the literature, one often finds the following argument. Let

$$\psi : \mathscr{X} \equiv \mathscr{I}nv^+ \to \mathscr{Y} \equiv \mathscr{R}$$

$$\mathbf{F} \mapsto \psi(\mathbf{F})$$

be the *hyperelastic energy*. Then by IRBM

$$(7.3) \qquad \psi(\mathbf{F}) = \psi(\mathbf{Q}\ \mathbf{F}) \qquad \forall \mathbf{Q} \in \mathscr{O}rth^+, \qquad \forall \mathbf{F} \in \mathscr{I}nv^+$$

and by the polar decomposition one obtains the reduced form

$$\psi(\mathbf{F}) = \psi(\mathbf{U})$$

with $\mathbf{Q} \equiv \mathbf{R}^T$, and one concludes that any function $\psi(\mathbf{U})$ identically fulfils the IRBM. Of course, this reasoning is a shorthand[8)] for saying: iff ψ fulfils (7.3), then it can be represented by

$$\psi(\mathbf{F}) = \psi\,|_{\mathscr{P}sym} \circ \lambda_\psi(\mathbf{F})$$

with

$$\lambda_\psi : \mathscr{I}nv^+ \to \mathscr{P}sym$$

$$\mathbf{F} \mapsto \mathbf{U} = (\mathbf{F}^T\mathbf{F})^{1/2}.$$

8. Conclusions

Euclidean frame-indifference (EFI) determines the action of changes of observer or Euclidean transformations on the dependent variables such as Cauchy stresses. On the other hand, IRBM describes the effect of superimposed rigid body motions on the stresses. It is violated, if the response of the material depends on accelerations, spins, etc.

[8)] This shorthand is essentially correct, but has been misunderstood [see RIVLIN and SMITH, [14]. Unfortunately, also these critical authors overlooked the fact that the condition (7.2) has nothing to do with Euclidean frame-invariance].

Although there exist certain mathematical similarities, these two principles represent completely distinct notions. In SVENDSEN and BERTRAM [16], as well as in the current work, we have attempted to work these and other aspects of these principles out in detail. While EFI appears to be generally valid on the basis of our understanding of stresses, IRBM is clearly violated for certain materials such as kinetic gases. As such, both classes of materials, namely those that do satisfy the IRBM, and those that fail to do so, are described in the current formulation.

When investigating the reasons for the long and controversial debate on this issue, a third assumption comes into play. The dependence of the constitutive equations on the observer can be further specified as form-invariant.

It turns out, that this assumption is equivalent to IRBM, if the validity of EFI is assumed. As such, EFI and IRBM are indistinguishable whenever the observer dependence of the constitutive equations is not taken into account, i. e., whenever FI holds. And this is common practice.

We have stated FI with certain emphasis, although it is rather formal and difficult to interpret physically. This has two reasons. Firstly, it is often assumed without mentioning. And secondly, it has strong consequences for the material theory, as we have seen.

Once having understood the structure of the mutual dependences of these principles, the following approach seems to be natural and physically adequate.

1. State EFI as a fundamental principle which is generally valid in continuum physics.
2. Define a special class of materials by the condition IRBM, but not as a general principle for all materials. Under EFI, FI is necessary and sufficient for IRBM to hold.
3. For this specific class of materials that obey IRBM, obtain the reduced forms via the procedure developed in the last section.

We have shown a way to define and to construct reduced forms in a rather general context. The underlying structure is the same for many different applications in physics and other fields. One has a set of independent and dependent variables, $\mathscr{X}$ and $\mathscr{Y}$, respectively. Then one considers mappings from $\mathscr{X}$ to $\mathscr{Y}$ that are restricted by invariance-properties under certain transformation groups. The problem is to find two sets $\mathscr{RX}$ and $\mathscr{RY}$, such that any mapping from $\mathscr{RX}$ to $\mathscr{RY}$ corresponds to exactly one mapping from $\mathscr{X}$ to $\mathscr{Y}$ that fulfills these invariances. Clearly, the stronger these conditions are, the greater is the reduction.

The suggested procedure to construct reduced forms is a mathematical reformulation and generalization of what NOLL [12] suggested in the context of elastic materials.[9] This procedure has to be specified for the individual class of materials under consideration so that we can really benefit from this reduction. For many classes of materials, this has already been done long ago. For complicated material classes such as higher-order (non-simple) materials, non-local materials, Cosserat materials, mixtures, materials with micro-structure or internal length scales, this analysis of reduced forms can be expected to be advantageous in the future.

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Hall effect on thermosolutal instability of Walters' (model B′) fluid in porous medium

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THE THERMOSOLUTAL INSTABILITY of Walters' (model B′) fluid in porous medium is considered in the presence of uniform vertical magnetic field to include the effect of Hall currents. For the case of stationary convection, the stable solute gradient and magnetic field have stabilizing effects on the system, whereas the Hall currents have destabilizing effect on the system. The medium permeability has both stabilizing and destabilizing effects on the system depending on the Hall parameter M. The kinematic viscoelasticity has no effect for stationary convection. The kinematic viscoelasticity, stable solute gradient and magnetic field (and the corresponding Hall currents) introduce oscillatory modes in the system, which were non-existent in their absence. The sufficient conditions for the non-existence of overstability are also obtained.

Key words: Hall effect, thermosolutal instability, Walters' (model B′) fluid, porous medium.

1. Introduction

THE THERMAL CONVECTION in an electrically conducting, Newtonian fluid layer in the presence of magnetic field has been discussed in detail in the celebrated monograph by CHANDRASEKHAR [1]. BHATIA and STEINER [2] have studied the thermal instability of a Maxwellian viscoelastic fluid in the presence of magnetic field while the thermal convection in Oldroydian viscoelastic fluid in hydromagnetics has been studied by SHARMA [3]. The problem of thermohaline convection in a layer of fluid heated from below and subjected to a stable salinity gradient has been considered by VERONIS [4]. The physics is quite similar in the stellar case in which helium acts like salt in raising the density and in diffusing more slowly than heat. The conditions under which convective motions are important in stellar atmospheres are usually far removed from consideration of single component fluid and rigid boundaries and therefore, it is desirable to consider a fluid acted on by a solute gradient and free boundaries. The problem is of great importance because of its application to atmospheric physics and astrophysics, especially in the case of the ionosphere and the outer layer of the atmosphere. The thermosolutal convection problems also arise in oceanography, limnology and engineering.

With the growing importance of non-Newtonian fluids in modern technology and industries, the investigations on such fluids are desirable. There are many elastico-viscous fluids that cannot be characterized by Maxwell's constitutive relations or Oldroyd's constitutive relations. One such class of elastico-viscous fluids is Walters' fluid (model B′). SHARMA and KUMAR [5] have studied the steady flow and heat transfer of Walters' fluid (model B′) through a porous pipe of uniform circular cross-section with small suction.

In recent years, the investigation of flow of fluids through porous media has become an important topic due to the recovery of crude oil from the pores of reservoir rocks. A great number of applications in geophysics may be found in a book by PHILLIPS [6]. When the fluid permeates a porous material, the gross effect is represented by the Darcy law. As a result of this macroscopic law, the usual viscous and viscoelastic terms in the equation of Walters' fluid (model B′) motion are replaced by the resistance term $\left[-\frac{1}{k_1}\left(\mu-\mu'\frac{\partial}{\partial t}\right)\mathbf{q}\right]$, where μ and μ' are the viscosity and viscoelasticity of the Walters' fluid, k_1 is the medium permeability and $\mathbf{q}$ is the Darcian (filter) velocity of the fluid. The problem of thermosolutal convection in fluids in a porous medium is of great importance in geophysics, soil sciences, ground water hydrology and astrophysics. Generally, it is accepted that comets consist of a dusty 'snowball' of a mixture of frozen gases, which in the process of their journey, changes from solid to gas and vice-versa. The physical properties of comets, meteorites and interplanetary dust strongly suggest the importance of porosity in astrophysical context (MCDONNEL [7]). The Hall effect is likely to be important in many geophysical situations as well as in flow of laboratory plasma. SHERMAN and SUTTON [8] have considered the effect of Hall currents on the efficiency of a magneto-fluid-dynamic generator. There is a growing importance of non-Newtonian fluids in chemical technology, industry and geophysical fluid dynamics. The Hall currents have relevance and importance in geophysics, MHD generators and industry. In a recent study, SHARMA *et al.* [9] studied the instability of streaming Walters' viscoelastic fluid B′ in porous medium. More recently, the effect of rotation on thermosolutal instability of Walters' fluid (model B′) in porous medium has been studied by SHARMA *et al.* [10].

Keeping in mind the importance of non-Newtonian fluids in modern technology, and various applications mentioned above, the thermosolutal instability of a electrically conducting Walters' (model B′) fluid in porous medium in the presence of uniform vertical magnetic field to include the effect of Hall currents, has been considered in the present paper.

2. Formulation of the problem and perturbation equations

Here we consider an infinite, horizontal, incompressible Walters' (model B′) fluid layer of thickness d, heated and soluted from below so that the temperatures, densities and solute concentrations at the bottom surface $z = 0$ are T_0, ρ_0 and C_0, and at the upper surface $z = d$ are T_d, ρ_{d} and C_d, respectively, and that a uniform temperature gradient $\beta\,(= |dT/dz|)$ and a uniform solute gradient $\beta'\,(= |dC/dz|)$ are maintained. The gravity field $\mathbf{g}\,(0,0,-g)$ and a uniform vertical magnetic field $\mathbf{H}\,(0,0,H)$ pervade the system. This fluid layer is assumed to be flowing through an isotropic and homogeneous porous medium of porosity ε and medium permeability k_1.

Let $p, \rho, T, C, \alpha, \alpha', g, \eta, \mu_e$,N,e and $\mathbf{q}\,(u, v, w)$ denote, respectively, the fluid pressure, density, temperature, solute concentration, thermal coefficient of expansion, an analogous solvent coefficient of expansion, gravitational acceleration, resistivity, magnetic permeability, electron number density, charge of an electron and fluid velocity. The equations expressing the conservation of momentum, mass, temperature, solute concentration and equation of state of Walters' (model B′) fluid are

$$(2.1)\quad \frac{1}{\varepsilon}\left[\frac{\partial \mathbf{q}}{\partial t} + \frac{1}{\varepsilon}(\mathbf{q}.\nabla)\,\mathbf{q}\right] = -\left(\frac{1}{\rho_0}\right)\nabla p + \mathbf{g}\left(1 + \frac{\delta\rho}{\rho_0}\right) - \frac{1}{k_1}\left(\nu - \nu'\frac{\partial}{\partial t}\right)\mathbf{q} + \frac{\mu_e}{4\pi\rho_0}(\nabla \times \mathbf{H}) \times \mathbf{H},$$

$$(2.2)\quad \nabla.\mathbf{q} = 0,$$

$$(2.3)\quad E\frac{\partial T}{\partial t} + (\mathbf{q}.\nabla)\,T = \kappa\nabla^2 T,$$

$$(2.4)\quad E'\frac{\partial C}{\partial t} + (\mathbf{q}.\nabla)\,C = \kappa'\nabla^2 C,$$

$$(2.5)\quad \rho = \rho_0\left[1 - \alpha\,(T - T_0) + \alpha'\,(C - C_0)\right],$$

where the suffix zero refers to values at the reference level $z = 0$ and in writing Eq. (2.1), use has been made of the Boussinesq approximation. The magnetic permeability μ_e, the kinematic viscosity ν, the kinematic viscoelasticity ν', the thermal diffusivity κ and the solute diffusivity κ' are all assumed to be constants.

The Maxwell's equations yield

$$\varepsilon\frac{d\mathbf{H}}{\mathrm{dt}} = (\mathbf{H}.\nabla)\,\mathbf{q} + \varepsilon\eta\nabla^2\mathbf{H} - \frac{c\varepsilon}{4\pi Ne}\nabla\times[(\nabla\times\mathbf{H})\times\mathbf{H}], \tag{2.6}$$

$$\nabla.\mathbf{H} = 0, \tag{2.7}$$

where $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \varepsilon^{-1}\mathbf{q}.\nabla$ stands for the convective derivative.

Here $E = \varepsilon + (1-\varepsilon)\left(\dfrac{\rho_s c_s}{\rho_0 c_i}\right)$ is a constant and E$'$ is a constant analogous to E but corresponding to solute rather than heat. ρ_s, c_s and ρ_0, c_i stand for density and heat capacity of solid (porous matrix) material and fluid, respectively. The steady state solution is

$$\mathbf{q} = (0,0,0)\,, \quad T = -\beta z + T_0,$$

$$C = -\beta' z + C_0, \quad \rho = \rho_0\left(1 + \alpha\beta z - \alpha'\beta' z\right). \tag{2.8}$$

Here we use linearized stability theory and normal mode analysis method. Consider a small perturbation on the steady state solution and let δp, $\delta\rho$, θ, γ, $\mathbf{h}\,(h_x, h_y, h_z)$ and $\mathbf{q}\,(u, v, w)$ denote, respectively, the perturbations in pressure p, density ρ, temperature T, solute concentration C, magnetic field $\mathbf{H}\,(0,0,\mathrm{H})$ and velocity $\mathbf{q}\,(0,\ 0,\ 0)$. The change in density $\delta\rho$, caused mainly by the perturbations θ and γ in temperature and concentration, is given by

$$\delta\rho = -\rho_0\left(\alpha\theta - \alpha'\gamma\right). \tag{2.9}$$

Then the linearized perturbation equations become

$$\frac{1}{\varepsilon}\frac{\partial\mathbf{q}}{\partial t} = -\frac{1}{\rho_0}(\nabla\delta p) - \mathbf{g}\left(\alpha\theta - \alpha'\gamma\right) - \frac{1}{k_1}\left(\nu - \nu'\frac{\partial}{\partial t}\right)\mathbf{q} + \frac{\mu_e}{4\pi\rho_0}(\nabla\times\mathbf{h})\times\mathbf{H}, \tag{2.10}$$

$$\nabla.\mathbf{q} = 0, \tag{2.11}$$

$$\mathrm{E}\frac{\partial\theta}{\partial \mathrm{t}} = \beta w + \kappa\nabla^2\theta, \tag{2.12}$$

$$\text{(2.13)} \qquad \mathrm{E}'\frac{\partial\gamma}{\partial t} = \beta' w + \kappa'\nabla^2\gamma,$$

$$\text{(2.14)} \qquad \varepsilon\frac{\partial \mathbf{h}}{\partial \mathrm{t}} = (\mathbf{H}.\nabla)\,\mathbf{q} + \varepsilon\eta\nabla^2\mathbf{h} - \frac{c\varepsilon}{4\pi Ne}\nabla\times[(\nabla\times\mathbf{h})\times\mathbf{H}],$$

$$\text{(2.15)} \qquad \nabla.\mathbf{h} = 0.$$

3. The dispersion relation

Decomposing the disturbances into normal modes, we assume that the perturbation quantities are of the form

$$\text{(3.1)} \quad [w, h_z, \theta, \gamma, \zeta, \xi] = [W(z), K(z), \Theta(z), \Gamma(z), Z(z), X(z)]\exp\left(ik_x x + ik_y y + nt\right),$$

where k_x, k_y are the wave numbers along the x- and y- directions respectively, $k = \sqrt{(k_x^2 + k_y^2)}$ is the resultant wave number and 'n' is the growth rate which is, in general, a complex constant. $\zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$ and $\xi = \dfrac{\partial h_y}{\partial x} - \dfrac{\partial h_x}{\partial y}$ stand for the z-components of vorticity and current density, respectively.

Expressing the coordinates x, y, z in the new unit of length d and letting $a = kd$, $\sigma = \dfrac{nd^2}{\nu}$, $p_1 = \dfrac{\nu}{\kappa}$, $p_2 = \dfrac{\nu}{\eta}$, $q = \dfrac{\nu}{\kappa'}$, $F = \dfrac{\nu'}{d^2}$, $P_\ell = \dfrac{k_1}{d^2}$ and $D = \dfrac{d}{dz}$, Eqs. (2.10)-(2.15), using (3.1), yield

$$\text{(3.2)} \quad \left[\frac{\sigma}{\varepsilon} + \frac{1}{P_\ell}(1-\sigma F)\right](D^2 - a^2)\,W + \frac{ga^2d^2}{\nu}(\alpha\Theta - \alpha'\Gamma) - \frac{\mu_e H d}{4\pi\rho_0\nu}(D^2 - a^2)\,DK = 0,$$

$$\text{(3.3)} \qquad \left[\frac{\sigma}{\varepsilon} + \frac{1}{P_\ell}(1-\sigma F)\right] Z = \frac{\mu_e H d}{4\pi\rho_0\nu} DX,$$

$$\text{(3.4)} \qquad (D^2 - a^2 - p_2\sigma)\,K = -\left(\frac{Hd}{\eta\varepsilon}\right) DW + \frac{cHd}{4\pi Ne\eta} DX,$$

$$\text{(3.5)} \qquad (D^2 - a^2 - p_2\sigma)\,X = -\left(\frac{Hd}{\eta\varepsilon}\right) DZ - \frac{cH}{4\pi Ne\eta d}(D^2 - a^2)\,DK,$$

(3.6) $$\left(D^2 - a^2 - Ep_1\sigma\right)\Theta = -\left(\frac{\beta d^2}{\kappa}\right)W,$$

(3.7) $$\left(D^2 - a^2 - E'q\sigma\right)\Gamma = -\left(\frac{\beta' d^2}{\kappa'}\right)W.$$

Consider now the case when both boundaries are free and perfect conductors of both heat and solute concentration, while the adjoining medium is perfectly conducting. The case of two free boundaries is a little artificial but it enables us to find analytical solutions and to make some qualitative conclusions. The appropriate boundary conditions, with respect to which Eqs. (3.2)-(3.7) must be solved, are (CHANDRASEKHAR [1])

(3.8) $$W = D^2W = 0,\ DZ = 0,\ \Theta = 0,\ \Gamma = 0, \quad \text{at z } = 0 \quad \text{and } 1,$$

$DX = 0$, $K = 0$ on a perfectly conducting boundary and $X = 0$, h_x, h_y, h_z are continuous with an external vacuum field on a non-conducting boundary.

The case of two free boundaries, though a little artificial, is the most appropriate for stellar atmospheres (SPIEGEL [11]). Using the above boundary conditions, it can be shown that all the even-order derivatives of W must vanish for $z = 0$ and 1, and hence the proper solution of W characterizing the lowest mode is

(3.9) $$W = W_0 \sin \pi z,$$

where W_0 is a constant.

Eliminating Θ, X, Z, Γ and K between Eqs. (3.2)-(3.7) and substituting the proper solution $W = W_0 \sin \pi z$, in the resultant equation, we obtain the dispersion relation

(3.10) $$R_1 = \left(\frac{1+x}{x}\right)\left[\frac{i\sigma_1}{\varepsilon} + \frac{1}{P}\left(1 - i\sigma_1\pi^2 F\right)\right][1 + x + iEp_1\sigma_1] + S_1\frac{(1 + x + iEp_1\sigma_1)}{(1 + x + iE'q\sigma_1)} + Q_1\frac{\mathbf{A}}{\mathbf{B}}$$

where

$$R_1 = \frac{g\alpha\beta d^4}{\nu\kappa\pi^4}, \quad S_1 = \frac{g\alpha'\beta' d^4}{\nu\kappa'\pi^4}, \quad Q_1 = \frac{\mu_e H^2 d^2}{4\pi\rho_0\nu\eta\varepsilon\pi^2}, \quad M = \left(\frac{cH}{4\pi Ne\eta}\right)^2,$$

$$x = \frac{a^2}{\pi^2}, \quad i\sigma_1 = \frac{\sigma}{\pi^2} \qquad \text{and} \qquad P = \pi^2 P_\ell.$$

$$\mathbf{A} = (1+x)[1 + x + iEp_1\sigma_1]\left\{\left[\frac{i\sigma_1}{\varepsilon} + \frac{1}{P}\left(1 - i\sigma_1\pi^2 F\right)\right][1 + x + ip_2\sigma_1] + Q_1\right\}$$

$$\mathbf{B} = x\Big\{\Big[\frac{i\sigma_1}{\varepsilon} + \frac{1}{P}(1 - i\sigma_1\pi^2 F)\Big]\Big[1 + x + ip_2\sigma_1\Big]^2 + Q_1\Big[1 + x + ip_2\sigma_1\Big] + M\Big(1 + x\Big)\Big[\frac{i\sigma_1}{\varepsilon} + \frac{1}{P}\Big(1 - i\sigma_1\pi^2 F\Big)\Big]\Big\}.$$

Equation (3.10) is the required dispersion relation including the effects of magnetic field, Hall currents, medium permeability, kinematic viscoelasticity and stable solute gradient on the thermosolutal instability of Walters' (model B$'$) fluid in porous medium in hydromagnetics in the presence of Hall currents.

4. The stationary convection

When the instability sets in as stationary convection, the marginal state will be characterized by $\sigma = 0$. Putting $\sigma = 0$, the dispersion relation (3.10) reduces to

$$R_1 = \left(\frac{1+x}{x}\right).\frac{\left(\dfrac{1+x}{P} + Q_1\right)^2 + \dfrac{M\,(1+x)}{P^2}}{\dfrac{1+x}{P} + Q_1 + \dfrac{M}{P}} + S_1, \tag{4.1}$$

which expresses the modified Rayleigh number R_1 as a function of the dimensionless wave number x and the parameters S_1, Q_1, M and P. The parameter F accounting for the kinematic viscoelasticity effect vanishes for the stationary convection.

To study the effects of stable solute gradient, magnetic field and medium permeability, we examine the natures of $\dfrac{dR_1}{dS_1}$, $\dfrac{dR_1}{dQ_1}$ and $\dfrac{dR_1}{dM}$ analytically. Eq. (4.1) yields

$$\frac{dR_1}{dS_1} = +1, \tag{4.2}$$

$$\frac{dR_1}{dQ_1} = \left(\frac{1+x}{x}\right)\frac{\left(\dfrac{1+x}{P} + Q_1\right)}{\left(\dfrac{1+x+M}{P} + Q_1\right)}, \tag{4.3}$$

and

$$\frac{dR_1}{dM} = -\frac{Q_1\,(1+x)}{x}.\frac{\left(\dfrac{1+x}{P} + Q_1\right)}{\left(\dfrac{1+x}{P} + Q_1 + \dfrac{M}{P}\right)^2}. \tag{4.4}$$

Thus for stationary convection, the stable solute gradient and magnetic field are found to have stabilizing effects, whereas the Hall currents have a destabilizing effect on the thermosolutal instability of Walters' (model B′) fluid in porous medium in hydromagnetics in the presence of Hall currents.

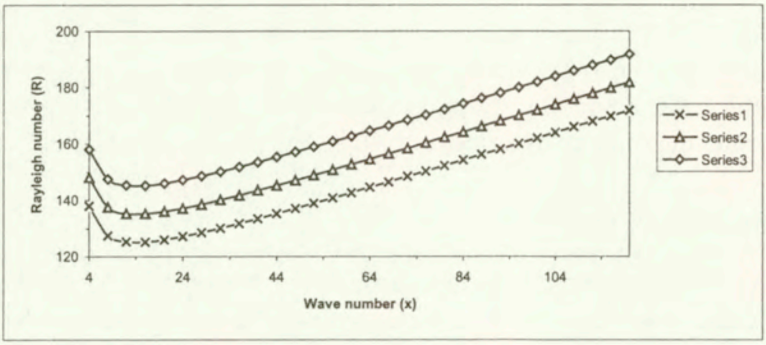

FIG. 1. The variation of Rayleigh number (R_1) with wave number (x) for $M = 0.1$, Q_1=100, $P = 2$; S_1=10 for Series 1, S_1=20 for Series 2 and S_1=30 for Series 3.

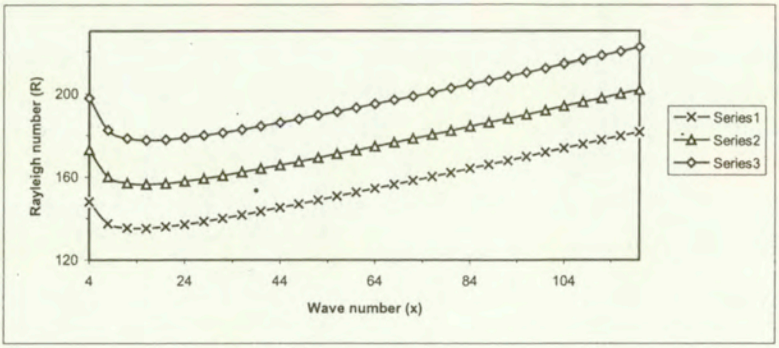

FIG. 2. The variation of Rayleigh number (R_1) with wave number (x) for $M = 0.1$, S_1=20, $P = 2$, Q_1=100 for Series 1, Q_1=120 for Series 2 and Q_1=140 for Series 3.

The dispersion relation (4.1) is analysed numerically. In Fig. 1, R_1 is plotted against x for $P = 2$, Q_1=100, $M = 0.1$ and S_1=10, 20 and 30. The stabilizing role of the stable solute gradient is clear from the increase of the Rayleigh number. with increasing stable solute gradient parameter value. Figure 2 gives R_1 plotted against x for $P = 2$, S_1=20, $M = 0.1$ and Q_1=100, 120 and 140. Here we also find the stabilizing role of the magnetic field as the Rayleigh number increases with the increase in magnetic field parameter Q_1. In Fig. 3, R_1 is plotted against x for Q_1=100, S_1=20, $P = 2$ and $M = 10$, 50 and 100. Here the destabilizing role of the Hall currents is clear from the decrease of the Rayleigh number with increasing Hall currents parameter value.

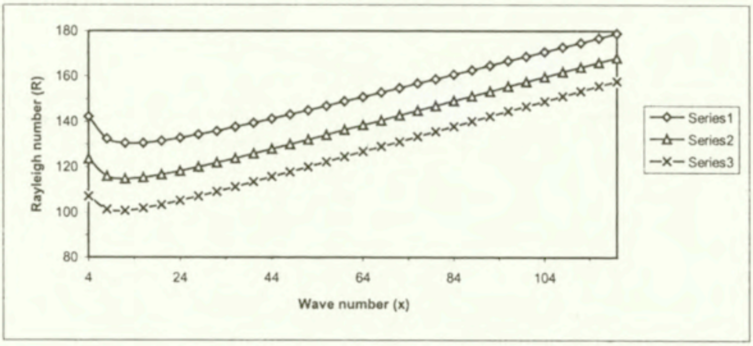

FIG. 3. The variation of Rayleigh number (R_1) with wave number (x) for Q_1=100, S_1=20, $P=2$, $M=10$ for Series 1, $M=50$ for Series 2 and $M=100$ for Series 3.

In order to investigate the effect of medium permeability, we examine the behaviour of $\frac{dR_1}{dP}$ analytically. Equation (4.1) yields

$$\frac{dR_1}{dP} = -\frac{(1+x)}{xP^2} \cdot \frac{\left(\frac{1+x}{P^2}\right)(1+x+M)^2 + \frac{2Q_1(1+x)(1+x+M)}{P} + Q_1^2(1+x-M)}{\left(\frac{1+x+M}{P}+Q_1\right)^2}. \tag{4.5}$$

which is negative. The medium permeability, therefore, has a destabilizing effect (Hall parameter M«1) on thermosolutal instability of Walters' (model B$'$) fluid in porous medium in hydromagnetics in the presence of Hall currents.

It has been shown graphically that for

- $Q_1 = 100, S_1 = 20, M = 0.1$ and $P = 1$, 3; the medium permeability has a destabilizing effect [Fig. 4].
- $Q_1 = 100, S_1 = 20, M = 10$ and $P = 1$, 3; the medium permeability has a stabilizing influence for x<7 and for x>7 they have a destabilizing effect [Fig. 5].
- $Q_1 = 100, S_1 = 20, M = 20$ and $P = 1$, 3; the medium permeability has a stabilizing influence for x<12.5 and for x>12.5 they have a destabilizing effect [Fig. 5].

It has also been shown that as Hall parameter M increases, the stabilizing range of medium permeability also increases.

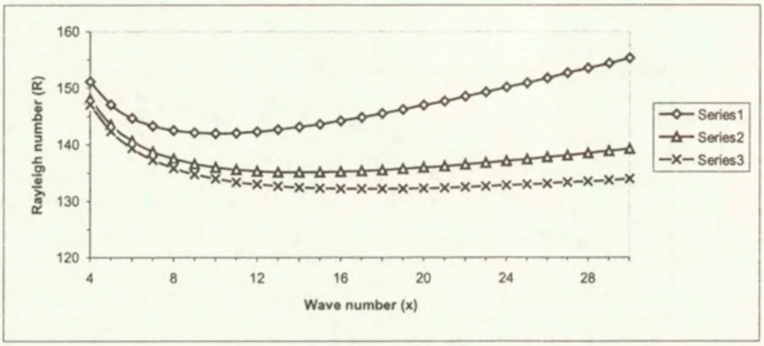

FIG. 4. The variation of Rayleigh number (R_1) with wave number (x) for Q_1=100, S_1=20, $M = 0.1$ and $P = 1$ for Series 1, $P = 2$ for Series 2 and $P = 3$ for Series 3.

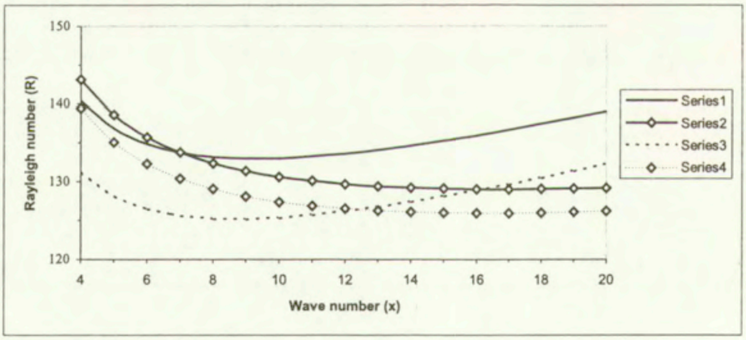

FIG. 5. The variation of Rayleigh number (R_1) with wave number (x) for $Q_1 = 100, S_1 = 20$; and $M = 10, P = 1$ for Series 1, $M = 10$, $P = 3$ for Series 2, $M = 20$, $P = 1$ for Series 3 and $M = 20, P = 3$ for Series 4.

5. Stability of the system and oscillatory modes

Here we examine the possibility of oscillatory modes, if any, on stability problem due to the presence of kinematic viscoelasticity, stable solute gradient and magnetic field. Multiplying (3.2) by $W*$, the complex conjugate of W, and using (3.3)-(3.7) together with the boundary conditions (3.8), we obtain

$$(5.1)\quad \left[\frac{\sigma}{\varepsilon}+\frac{1}{P_\ell}(1-\sigma F)\right]I_1+\left(\frac{g\alpha'\kappa' a^2}{\nu\beta'}\right)[I_4+E'q\sigma^* I_5]$$
$$+\frac{\mu_e\eta\varepsilon}{4\pi\rho_0\nu}[I_6+p_2\sigma^* I_7]+\left(\frac{\mu_e\eta\varepsilon d^2}{4\pi\rho_0\nu}\right)[I_8+p_2\sigma I_9]$$
$$+d^2\left[\frac{\sigma^*}{\varepsilon}+\frac{1}{P_\ell}(1-\sigma^* F)\right]I_{10}-\left(\frac{g\alpha\kappa a^2}{\nu\beta}\right)[I_2+Ep_1\sigma^* I_3]=0,$$

where

$$I_1=\int_0^1\left(|DW|^2+a^2|W|^2\right)dz,\qquad I_2=\int_0^1\left(|D\Theta|^2+a^2|\Theta|^2\right)dz,$$
$$I_3=\int_0^1\left(|\Theta|^2\right)dz,\qquad I_4=\int_0^1\left(|D\Gamma|^2+a^2|\Gamma|^2\right)dz,$$
$$(5.2)\quad I_5=\int_0^1\left(|\Gamma|^2\right)dz,\quad I_6=\int_0^1\left(\left|D^2K\right|^2+2a^2|DK|^2+a^4|K|^2\right)dz,$$
$$I_7=\int_0^1\left(|DK|^2+a^2|K|^2\right)dz,\qquad I_8=\int_0^1\left(|DX|^2+a^2|X|^2\right)dz,$$
$$I_9=\int_0^1\left(|X|^2\right)dz,\qquad I_{10}=\int_0^1\left(|Z|^2\right)dz.$$

The integrals $I_1,...,I_{10}$ are all positive definite. Putting $\sigma=\sigma_r+i\sigma_i$ and equating the real and imaginary parts of Eq. (5.1), we obtain

$$(5.3)\quad \left[\left(\frac{1}{\varepsilon}-\frac{F}{P_\ell}\right)I_1+\frac{g\alpha'\kappa' a^2}{\nu\beta'}E'qI_5+\frac{\mu_e\eta\varepsilon}{4\pi\rho_0\nu p_2}\left(I_7+d^2I_9\right)\right.$$
$$\left.+d^2\left[\frac{1}{\varepsilon}-\frac{F}{P_\ell}\right]I_{10}-\frac{g\alpha\kappa a^2}{\nu\beta}Ep_1I_3\right]\sigma_r$$
$$=-\left[\frac{I_1}{P_\ell}+\frac{g\alpha'\kappa' a^2}{\nu\beta'}I_4+\frac{\mu_e\eta\varepsilon}{4}\pi\rho_0\nu\left(I_6+d^2I_8\right)+d^2\frac{1}{P_\ell}I_{10}-\frac{g\alpha\kappa a^2}{\nu\beta}I_2\right],$$

$$\left[\left(\frac{1}{\varepsilon}-\frac{F}{P_\ell}\right)I_1-\frac{g\alpha'\kappa'a^2}{\nu\beta'}E'qI_5-\frac{\mu_e\eta\varepsilon}{4\pi\rho_0\nu}p_2\left(I_7-d^2I_9\right)\right.$$
$$\left.-d^2\left(\frac{1}{\varepsilon}-\frac{F}{P_\ell}\right)I_{10}+\frac{g\alpha\kappa a^2}{\nu\beta}Ep_1I_3\right]\sigma_i=0. \tag{5.4}$$

It is evident from (5.3) that σ_r is positive or negative. The system is, therefore, stable or unstable. It is clear from (5.4) that σ_i may be zero or non-zero, meaning that the modes may be non-oscillatory or oscillatory. In the absence of kinematic viscoelasticity, stable solute gradient and magnetic field (and hence Hall currents), equation (5.4) reduces to

$$\left[\frac{I_1}{\varepsilon}+\frac{g\alpha\kappa a^2}{\nu\beta}Ep_1I_3\right]\sigma_i=0, \tag{5.5}$$

and the terms in brackets are positive definite. Thus $\sigma_i=0$, which means that oscillatory modes are not allowed and the principle of exchange of stabilities is satisfied for a porous medium, in the absence of kinematic viscoelasticity, stable solute gradient and magnetic field (and hence Hall currents). This result is true for the porous as well as non-porous (CHANDRASEKHAR [1]) medium. The oscillatory modes are introduced due to the presence of kinematic viscoelasticity, stable solute gradient and magnetic field (and hence Hall currents), which were non-existent in their absence.

6. The case of overstability

Here we discuss the possibility of whether instability may occur as overstability. Since we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it suffices to find conditions for which (3.10) will admit the solutions with σ_1 real.

If we equate real and imaginary parts of (3.10) and eliminate R_1 between them, we obtain

$$A_4c_1^4+A_3c_1^3+A_2c_1^2+A_1c_1+A_0=0, \tag{6.1}$$

where we have put $c_1=\sigma_1^2$, $b=1+\mathrm{x}$ and

$$A_4=E'^2q^2p_2^2\left(\frac{1}{\varepsilon}-\frac{\pi^2F}{P}\right)^2\left[\left(\frac{1}{\varepsilon}-\frac{\pi^2F}{P}\right)b+\frac{Ep_1}{P}\right], \tag{6.2}$$

$$(6.3)\quad A_3 = \left[\left(\frac{1}{\varepsilon}-\frac{\pi^2 F}{P}\right)^3\left(2E'^2q^2+p_2^2\right)\right]b^3$$
$$+\left[\frac{Ep_1p_2^2}{P}\left(\frac{1}{\varepsilon}-\frac{\pi^2F}{P}\right)^2+\frac{6E'^2q^2M\pi^2F}{\varepsilon P}\left(\frac{1}{\varepsilon}-\frac{\pi^2F}{P}\right)\right.$$
$$\left.+2E'^2q^2\left\{\frac{Ep_1}{P}\left(\frac{1}{\varepsilon}-\frac{\pi^2F}{P}\right)^2-\frac{M}{\varepsilon^3}+M\left(\frac{\pi^2F}{P}\right)^3\right\}\right]b^2$$
$$+\left[E'^2q^2p_2\left(\frac{1}{\varepsilon}-\frac{\pi^2F}{P}\right)\left\{\frac{p_2}{P^2}-\frac{3Q_1}{\varepsilon}+\frac{3Q_1\pi^2F}{P}\right\}\right.$$
$$\left.+Ep_1E'^2q^2\left(\frac{1}{\varepsilon}-\frac{\pi^2F}{P}\right)^2\left(Q_1-\frac{2M}{P}\right)\right]b$$
$$+p_2\left[\frac{Ep_1E'^2q^2}{P}\left\{\frac{p_2}{P^2}-\frac{2Q_1}{\varepsilon}+\frac{2Q_1\pi^2F}{P}\right\}+S_1(b-1)p_2\left(\frac{1}{\varepsilon}-\frac{\pi^2F}{P}\right)^2\left(Ep_1-E'q\right)\right].$$

Since σ_1 is real for overstability, the four values of c_1 $\left(=\sigma_1^2\right)$ are positive. The sum of roots of (6.1) is $-\dfrac{A_3}{A_4}$, and if this is to be negative, then $A_3>0$, $A_4>0$.

It is clear from (6.2) and (6.3) that A_3 and A_4 are always positive if

$$(6.4)\qquad \frac{\pi^2F}{P}<\frac{1}{\varepsilon},\quad Ep_1>E'q,\quad \frac{Ep_1}{P}\left(\frac{1}{\varepsilon}-\frac{\pi^2F}{P}\right)^2>\frac{M}{\varepsilon^3},\quad p_2>\frac{3Q_1P^2}{\varepsilon}$$
$$\text{and}\quad Q_1>\frac{2M}{P}$$

which imply that

$$(6.5)\qquad \begin{gathered}\nu'<\frac{k_1}{\varepsilon},\ E\frac{\nu}{\kappa}>E'\frac{\nu}{\kappa'},\ E\frac{\nu}{\kappa}>\frac{k_1}{\varepsilon}\left[\frac{cHk_1}{4d\eta Ne\left(k_1-\varepsilon\nu'\right)}\right]^2,\\ \nu>\frac{k_1^2\pi}{\varepsilon d^2}\left(\frac{3\mu_eH^2}{4\nu\rho_0}\right)\ \text{and}\ \nu<\frac{k_1}{\varepsilon}\left(\frac{Ne}{c}\right)^2\left(\frac{2\mu_e\pi\eta}{\rho_0}\right)\end{gathered}$$

i.e.

$$(6.6)\qquad \nu'<\frac{k_1}{\varepsilon},E\frac{\nu}{\kappa}>\max\left[E'\frac{\nu}{\kappa'},\frac{k_1}{\varepsilon}\left\{\frac{cHk_1}{4d\eta Ne\left(k_1-\varepsilon\nu'\right)}\right\}^2\right]$$

and

$$\frac{k_1^2\mu_e}{\varepsilon\rho_o}\left(\frac{3H^2}{4d^2\nu}\right) < \nu < \frac{k_1\mu_e}{\varepsilon\rho_0}\left(\frac{Ne}{c}\right)^2 2\pi\eta.$$

Thus

$$\nu' < \frac{k_1}{\varepsilon}, E\frac{\nu}{\kappa} > \max\left[E'\frac{\nu}{\kappa'}, \frac{k_1}{\varepsilon}\left\{\frac{cHk_1}{4d\eta Ne\,(k_1 - \varepsilon\nu')}\right\}^2\right]$$

and

$$\frac{k_1^2\mu_e}{\varepsilon\rho_o}\left(\frac{3H^2}{4d^2\nu}\right) < \nu < \frac{k_1\mu_e}{\varepsilon\rho_0}\left(\frac{Ne}{c}\right)^2 2\pi\eta$$

are the sufficient conditions for the non-existence of overstability.

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The 7th Conference "Shell Structures, Theory and Applications"
Gdańsk-Jurata, Poland, 9-11 October 2002

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Conferences "Shell Structures, Theory and Applications" (SSTA in short) have been organized in Poland since 1974. The aim of these meetings is always the same: to bring together scientists, designers, engineers and other specialists of shell structures in order to discuss important results and new ideas in this broad field of activity. The goal is to pursue more accurate theoretical models, to develop more powerful and versatile numerical methods for computerized analysis, and to disseminate expertise in design and maintenance of various shell structures and elements commonly used in science, technology and everyday life.

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As a rule, the volume of a paper should not exceed 40 000 typographic signs, that is about 20 type-written pages, format: 210×297 mm, leaded. The papers should be submitted in two copies. They must be set in accordance with the norms established by the Editorial Office. Special importance is attached to the following directions:

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2. The text should be preceded by a brief introduction; it is also desirable that a list of notations used in the paper should be given.
3. The formula number consists of two figures: the first represents the section number and the other the formula number in that section. Thus the division into subsections does not influence the numbering of formulae. Only such formulae should be numbered to which the author refers throughout the paper, and also the resulting formulae. The formula number should be written on the left-hand side of the formula; round brackets are necessary to avoid any misunderstanding. For instance, if the author refers to the third formula of the set (2.1), a subscript should be added to denote the formula, viz. $(2.1)_3$.
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