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On the disintegration of an arbitrary discontinuity generated by a centrally-cumulated simple wave of finite deformation in an isotropic elastic medium

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The paper presents the solution of the problem of disintegration of an arbitrary discontinuity in an isotropic elastic medium, the discontinuity being generated by a centrally-cumulated simple wave of finite deformation. Two approximations of the motion occurring behind the wave front are considered, isentropic and adiabatic. It is shown that for each approximation there exists only one stable configuration of wave fronts which is created after the disintegration of an arbitrary discontinuity (cf. Figs. 3, 4—isentropic approximation—and Figs. 6, 7—adiabatic approximation). In the isentropic approximation the contact discontinuity is not formed; it appears, however, in the adiabatic approximation. The solution is presented in a closed form; it is one of a very few closed form solutions of dynamic problems in nonlinear elasticity.

W przedstawionej pracy rozwiązano problem rozpadu dowolnej nieciągłości w izotropowym ośrodku sprężystym, wygenerowanej przez centrycznie kumulowaną prostą falę skończonych deformacji. Rozpatrzono dwa przybliżenia ruchu za frontem fali uderzeniowej — izentropowe i adiabatyczne. Wykazano, że dla każdego przybliżenia istnieje tylko jedna stateczna konfiguracja frontów fal jaka powstaje po rozpadzie dowolnej nieciągłości (patrz rys. 3 i 4 — przybliżenie izentropowe oraz rys. 6 i 7 — przybliżenie adiabatyczne). W przybliżeniu izentropowym nie powstaje nieciągłość kontaktowa, natomiast pojawia się ona w przybliżeniu adiabatycznym. Rozwiązanie problemu udało się przedstawić w postaci zamkniętych wzorów. Jest to jedno z nielicznych zamkniętych rozwiązań dynamicznych zagadnień nieliniowej teorii sprężystości.

В представленной работе решена проблема распада произвольного разрыва в изотропной упругой среде, генерированного центрически кумулированной простой волной конечных деформаций. Рассмотрены два приближения движения за фронтом ударной волны — изэнтропическое и адиабатическое. Показано, что для каждого приближения существует только одна устойчивая конфигурация фронтов волн, какая возникает после распада произвольного разрыва (смотри рисунки 3 и 4 — изэнтропическое приближение и рисунки 6 и 7 — адиабатическое приближение). В изэнтропическом приближении не возникает контактный разрыв, появляется же он в адиабатическом приближении. Решение проблемы удалось представить в виде замкнутых формул. Это одно из немногочисленных замкнутых решений динамических задач нелинейной теории упругости.

1. Introduction

THE PROBLEM of disintegration of arbitrary discontinuities was investigated in detail in gas dynamics [1–3]. Papers [4–7] represent a series of contributions to the problem and enlarge the previous state of knowledge on the subject. In these papers considered were, among others, the methods of generation of unstable thermodynamic states in polytropic gases called arbitrary discontinuities. Their disintegration was found to produce stable configurations of weak (expansion waves) and strong (shock) wave fronts.

The problem of particular interest is to prove the possibility of isentropic cumulation of plane compression waves in a single plane by suitable selection of time configuration of the boundary conditions [5], followed by the disintegration of an arbitrary discontinuity generated in this manner and the creation of a stable system of weak and strong waves [7].

Plane one-dimensional motion of an isotropic elastic medium subject to finite deformations is described by an equation similar to that governing the analogous motion of polytropic gases [3, 8, 9]. Here the question may be posed as to the possibility of centric cumulation of a plane acceleration wave and of decay of an arbitrary discontinuity in an isotropic elastic medium under finite deformations. The answer to the first question was discussed by this author in a previous paper [10]. Mereover, in [11] the problem analysed was the generation of a plane shock wave of finite deformations in the isentropic approach.

The present paper deals with the problem of disintegration of an arbitrary discontinuity generated by a centrally cumulated simple wave of finite deformations. The problem will be solved in the isentropic and adiabatic approximations. It will be proved that in both approximations mentioned there exists only one stable configuration of wave fronts which may be generated as a result of centric cumulation of a simple wave of finite deformations. This is a considerable difference as compared to the analogous problem studied in a polytropic gas where three variants of wave systems appear [7]. The solution will be presented in a closed form.

The presented problems are of great significance for the analysis of extremal compression of matter.

2. Formulation of the problem

Let us analyse a plane one-dimensional motion of a half-space filled with the isotropic elastic medium and loaded at the boundary by a suitably designed load $\sigma(0, t)$ which generates the centrally-cumulated simple wave of finite deformations (Fig. 1). The dynamic deformation of the medium is assumed to proceed according to an isentropic or adiabatic process. The process will be described mathematically in Lagrangean coordinates x, t.

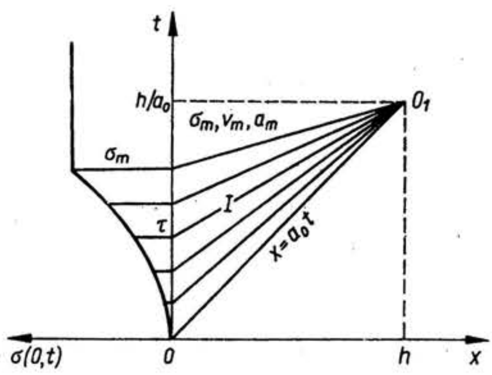

FIG. 1.

A plane one-dimensional motion of a non-conducting, compressible isotropic elastic medium in a uniaxial state of finite deformation is governed by the equations

$$\text{(2.1)}\qquad \begin{aligned} \frac{\partial v}{\partial t} &= a^2(m)\frac{\partial m}{\partial x}, \qquad v = \frac{\partial u}{\partial t},\\ \frac{\partial m}{\partial t} &= \frac{\partial v}{\partial x}, \qquad\quad m = \frac{\partial u}{\partial x}.\end{aligned}$$

The equations may be reduced to a single hyperbolic type equation

$$\text{(2.2)}\qquad \frac{\partial^2 u}{\partial t^2} = a^2(m)\frac{\partial^2 u}{\partial x^2}$$

or replaced by two equivalent ordinary equations to be satisfied on the characteristics

$$\text{(2.3)}\qquad dv = \pm a(m)dm$$

provided

$$\text{(2.4)}\qquad dx = \pm a(m)dt.$$

Here u denotes the displacement of the medium particles, and $a(m)$ is the propagation velocity of disturbances expressed in Lagrangean coordinates. It may also be expressed by the formula

$$\text{(2.5)}\qquad a(m) = \left(\frac{1}{\varrho_0}\frac{d^2 W}{dm^2}\right)^{1/2},$$

where $W(m)$ is the function denoting the dynamic strain energy. In the case of isotropic medium the energy is written in the form

$$\text{(2.6)}\qquad W(m) = \frac{1}{2}E_0 m^2 + \frac{1}{3}E_1 m^3 + 0(m^4)$$

with the notations:

$$E_0 = \lambda + 2\mu, \quad E_1 = 3\left(\frac{\lambda}{2} + \mu + \alpha + \beta + \nu\right),$$

λ and μ are Lamé constants while α, β and ν denote the third-order elastic moduli; a_0 is the direction number of the first characteristic OO_1 of the bunch of Riemann waves, the acceleration (Fig. 1).

The velocity of propagation $a(m)$ of disturbances may be, after applying the reduced form of the expression (2.6) (without the terms $0(m^4)$), expressed by the formula

$$\text{(2.7)}\qquad a(m) = \left(1 + 2\frac{E_1}{E_0}m\right)^{1/2} a_0, \quad a_0 = \sqrt{\frac{E_0}{\varrho_0}}.$$

Equations (2.3) yield, after integration along the characteristics, the relation

$$\text{(2.8)}\qquad v - F(m) = \text{const} \quad \text{if} \quad dx = a(m)dt$$

and

$$\text{(2.9)}\qquad v + F(m) = \text{const} \quad \text{if} \quad dx = -a(m)dt,$$

where

$$(2.10)\qquad F(m) = \int_0^m a(\xi)d\xi = \frac{E_0}{3E_1}\left[\left(1+2\frac{E_1}{E_0}m\right)^{3/2}-1\right]a_0.$$

The equations and relations derived so far hold true for a continuous motion of the medium. However, on the surfaces of strong discontinuities (shock waves) some of the formulae loose their sense. In such cases the conservation laws expressed in finite forms must be used:

$$(2.11)\qquad [\sigma] = -\varrho_0 D[v],$$

$$(2.12)\qquad [v] = -D[m],$$

$$(2.13)\qquad D[W]+\frac{1}{2}\varrho_0 D[v^2] = -[\sigma v],$$

the symbol $[f]$ denoting the jump of the value of f at the strong discontinuity wave front, and D—velocity of propagation of that front.

The constitutive equation for the stress is assumed in the form

$$(2.14)\qquad \sigma = \frac{\partial W}{\partial m} = E_0 m + E_1 m^2 + 0(m^3).$$

Let us now pass to the analysis of disintegration of an arbitrary discontinuity generated by a centrally cumulated simple wave of finite deformations (Fig. 1); to this end formulae and relations derived above will be used. First of all, following the procedure outlined in paper [10], let us derive the solution in the region of wave cumulation (region I in Figs. 1, 3 and 6).

3. Centric cumulation of a simple wave of finite deformation

According to the theory of quasi-linear differential equations, the characteristics of a simple finite deformation waves are represented by straight lines [9]. For a bunch of characteristics shown in Fig. 1 the lines may be expressed by the equation

$$(3.1)\qquad x = a(v)(t-\tau),$$

where τ denotes the time at which the given characteristics originate at the boundary of the half-space (Fig. 1).

According to Eqs. (2.8) and (2.9), along the characteristics either

$$(3.2)\qquad v(m) = -\frac{E_0}{3E_1}\left[\left(1+2\frac{E_1}{E_0}m\right)^{3/2}-1\right]a_0$$

or

$$(3.3)\qquad m = \frac{E_0}{2E_1}\left[\left(1-\frac{3E_1}{E_0}\frac{v}{a_0}\right)^{2/3}-1\right].$$

After substituting the expression (3.3) into Eq. (2.5) we obtain

$$(3.4)\qquad a(v) = \left(1-3\frac{E_1}{E_0}\frac{v}{a_0}\right)^{1/3}a_0.$$

On putting in Eq. (3.1) $x = h$ and $t = h/a_0$ (h is the space coordinate of the cumulation point O_1, Fig. 1) and using Eq. (3.4), we obtain after transformations

$$v(0,\tau) = -\frac{E_0}{3E_1}\left[\left(\frac{h}{h-a_0\tau}\right)^3 - 1\right]a_0. \tag{3.5}$$

In order to set the boundary of the half-space into motion with the velocity expressed by Eq. (3.5), according to the constitutive relation (2.14) and Eqs. (3.3) and (3.5), the boundary must be loaded according to the functional prescription

$$\sigma(0,\tau) = \frac{E_0^2}{4E_1}\left[\left(\frac{h}{h-a_0\tau}\right)^4 - 1\right]. \tag{3.6}$$

The displacement gradient m is then written as

$$m(0,\tau) = \frac{E_0}{2E_1}\left[\left(\frac{h}{h-a_0\tau}\right)^2 - 1\right]. \tag{3.7}$$

Next, according to the Riemann invariants theory [3] and the derived formulae (3.5)–(3.7), the solution of the problem in the zone of the centrally-cumulated finite deformation wave (region I in Figs. 1, 3 and 6) may be written in the form

$$\sigma_1(x,t) = \frac{E_0^2}{4E_1}\left[\left(\frac{h-x}{h-a_0t}\right)^4 - 1\right], \tag{3.8}$$

$$v_1(x,t) = -\frac{E_0}{3E_1}\left[\left(\frac{h-x}{h-a_0t}\right)^3 - 1\right]a_0, \tag{3.9}$$

$$m_1(x,t) = \frac{E_0}{2E_1}\left[\left(\frac{h-x}{h-a_0t}\right)^2 - 1\right]. \tag{3.10}$$

The relations given so far have been derived under the assumption that $E_1 > 0$ and $m > 0$ (waves of expansion, dilatation) or $E_1 < 0$ and $m < 0$ (waves of compression).

Summing up the hitherto presented considerations it may be concluded that if the half-space boundary is loaded by the stress varying in time according to Eq. (3.6) or moves at the velocity (3.5), all the acceleration waves will catch up with each other in the plane $x = h$ at the same instant of time $t_h = h/a_0$. Thus an arbitrary strong discontinuity will be generated in which the conservation laws will not be fulfilled. Hence the discontinuity disintegrates and gives rise to a thermodynamically balanced system of weak and strong discontinuity waves. Configuration of those waves depends, among others, on the properties of the isentrope of the medium $v_s(\sigma)$ and of the shock adiabate $v_H(\sigma)$ [3]. Let us now pass to the analysis of the curves.

4. Properties of isentrope $v_s(\sigma)$ and shock adiabate $v_H(\sigma)$

The relation between the isentropic motion velocity v_s and stress σ follows immediately from Eqs. (3.8) and (3.9)

$$v_s(\sigma) = -\frac{E_0}{3E_1}\left[\left(1+\frac{4E_1}{E_0^2}\sigma\right)^{3/4} - 1\right]a_0. \tag{4.1}$$

The corresponding relation referring to the shock wave front, derived from the Hugoniot conditions (2.11)–(2.13) under the assumption that the medium ahead of the wave is unperturbed, and from the constitutive equations (in the isentropic approximation) (2.14), may be written in the following form:

$$v_H^2(\sigma) = \frac{\sigma m(\sigma)}{\varrho_0}. \tag{4.2}$$

Here

$$m = \begin{cases} \frac{1}{2}\left[\sqrt{\left(\frac{E_0}{E_1}\right)^2 + 4\frac{\sigma}{E_1}} - \frac{E_0}{E_1}\right] & \text{if} \quad \sigma > 0, \quad E_1 > 0, \\ -\frac{1}{2}\left[\sqrt{\left(\frac{E_0}{E_1}\right)^2 + 4\frac{\sigma}{E_1}} + \frac{E_0}{E_1}\right] & \text{if} \quad \sigma < 0, \quad E_1 < 0 \end{cases} \tag{4.3}$$

in the isentropic approximation, and

$$m = \begin{cases} \frac{3}{4}\left[\sqrt{\left(\frac{E_0}{E_1}\right)^2 + \frac{8}{3}\frac{\sigma}{E_1}} - \frac{E_0}{E_1}\right] & \text{if} \quad \sigma > 0, \quad E_1 > 0, \\ -\frac{3}{4}\left[\sqrt{\left(\frac{E_0}{E_1}\right)^2 + \frac{8}{3}\frac{\sigma}{E_1}} + \frac{E_0}{E_1}\right] & \text{if} \quad \sigma < 0, \quad E_1 < 0, \end{cases} \tag{4.4}$$

in the adiabatic approximation.

In order to compare the functions $v_s(\sigma)$ and $v_H(\sigma)$ let us expand them into Maclaurin series; then the isentrope $v_s(\sigma)$ may be written in the form

$$v_s(\sigma) = -a_0\frac{\sigma}{E_0} + \frac{a_0}{2}\frac{E_1}{E_0}\frac{\sigma^2}{E_0^2} - \frac{5a_0}{6}\frac{E_1^2}{E_0^2}\frac{\sigma^3}{E_0^3} + \ldots, \tag{4.5}$$

while the shock adiabate $v_H(\sigma)$ is written either as

$$v_H(\sigma) = -a_0\frac{\sigma}{E_0} + \frac{a_0}{2}\frac{E_1}{E_0}\frac{\sigma^2}{E_0^2} - \frac{5.25a_0}{6}\frac{E_1^2}{E_0^2}\frac{\sigma^3}{E_0^3} + \ldots \tag{4.6}$$

in the isentropic approximation, or as

$$v_H(\sigma) = -a_0\frac{\sigma}{E_0} + \frac{1}{3}\frac{E_1}{E_0^3}a_0\sigma^2 - \frac{7}{18}\frac{E_1^2}{E_0^5}a_0\sigma^3 + \ldots \tag{4.7}$$

in the adiabatic approximation. Moreover, from Eqs. (4.1)–(4.4) it follows that

$$|v_H(\sigma)| > |v_s(\sigma)| \quad \text{for} \quad \sigma \neq 0 \tag{4.8}$$

and

$$\lim_{\sigma\to\pm\infty} v_s'(\sigma) = \lim_{\sigma\to\pm\infty} v_H'(\sigma) = 0. \tag{4.9}$$

Analysis of Eqs. (4.5)–(4.9) yields the conclusion that the origin of the system ($\sigma = 0$, $v = 0$) is the point of second-order tangency of the isentrope $v_s(\sigma)$ and the shock adiabate $v_H(\sigma)$ (not only the first but also the second derivatives coincide) in the isentropic approximation, and the usual point of tangency in the adiabatic approximation.

The curves $v_s(\sigma)$ and $v_H(\sigma)$ have no common points other than the single point mentioned above. Their general qualitative character in both approximations is shown in Fig. 2. In the case of adiabatic approximation the curves intersect each other also at the origin

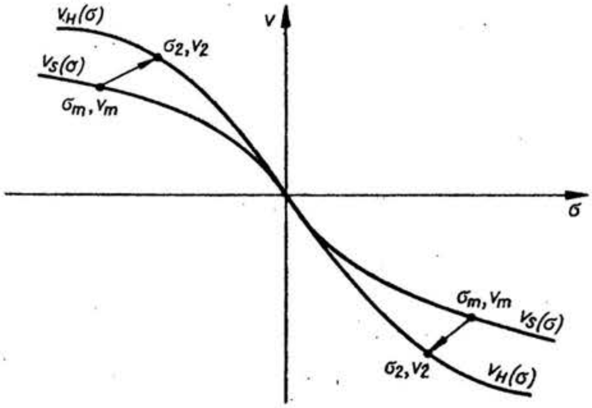

FIG. 2.

of the system, in spite of their first order tangency in the same point, since the change of sign of σ is accompanied by the change of sign of the modulus E_1 (cf. e.g. Eqs. (4.3) and (4.4)).

Once the forms of $v_s(\sigma)$ and $v_H(\sigma)$ are identified, let us pass to the analysis of the wave fronts configuration resulting from the act of cumulation.

5. Analysis of the wave fronts configuration after the act of cumulation in the isentropic approximation (Fig. 3).

Anylysis of the passage from the state σ_m, v_m lying on the isentrope $v_s(\sigma)$ to the state σ_2, v_2 lying on the shock adiabate $v_H(\sigma)$ (cf. Fig. 2) yields the conclusion that this process is physically possible provided $|\sigma_m| > |\sigma_2|$ and $|v_m| < |v_2|$. Other transition paths leading

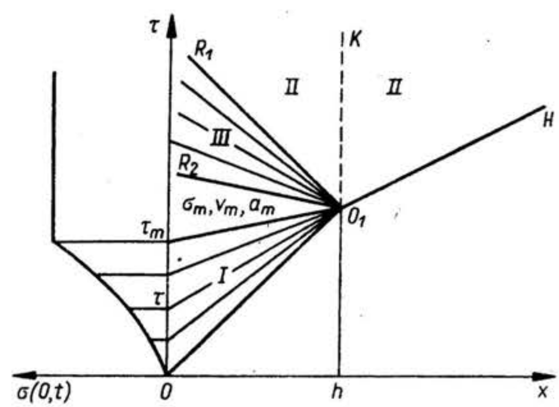

FIG. 3.

to states different from the one given here are physically unrealistic as long as interference of external interactions in the cumulation plane is not taken into account.

According to the inequalities derived above, in the region to the right of the cumulation plane (for $x > h$ and $t > h/a_0$), a shock wave front O_1H(Fig. 3) will be propagated through the unperturbed medium:

$$x-h = D(t-t_h), \quad t_h = h/a_0. \tag{5.1}$$

Behind the shock wave front, the material cumulation plane O_1K moves with the material velocity v_2. It should be observed that in the isentropic approximation the plane does not constitute a contact discontinuity (that is why it is located within the region II in Fig. 3). To the left from that plane a centered unloading wave $O_1R_1-O_1R_2$ (acceleration wave) is propagated reducing the stress σ from the value σ_m to σ_2. Reflection of that wave from the free boundary of the half-space will not be discussed in this paper.

The wave configuration outlined above is shown in Lagrangean coordinates x, t in Fig. 3, and the corresponding qualitative stress variation profiles σ, material velocities v and displacement gradients m are given in Fig. 4.

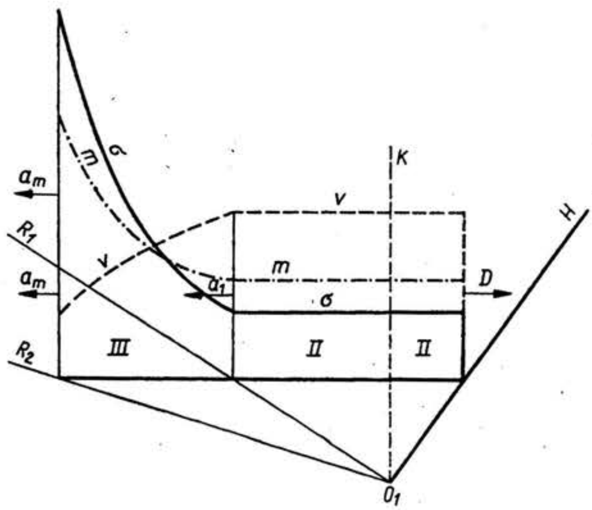

FIG. 4.

Analytical solution of the problem in the individual regions of the phase plane x, t proceeds as follows.

Region I

Parameters of the state and motion of the medium in the zone of cumulation of the incident simple wave are given by the formulae (3.8)–(3.10).

Region II

In the zone contained within the region II (Fig. 3), motion of the medium is stationary, according to the theory of decay of arbitrary discontinuity [1–3]. Hence, making use of the Hugoniot relations at the shock wave front O_1H, (2.11), (2.12) and the constitutive relations (2.14), the parameters of the state and motion of the medium in region II may be expressed as follows:

(5.2) $$\sigma_2 = \frac{E_0^2}{E_1}\frac{D^2}{a_0^2}\left(\frac{D^2}{a_0^2}-1\right),$$

(5.3) $$v_2 = -\frac{E_0 D}{2E_1}\left[\sqrt{4\frac{D^2}{a_0^2}\left(\frac{D^2}{a_0^2}-1\right)+1}-1\right],$$

(5.4) $$m_2 = \frac{E_0}{2E_1}\left[\sqrt{4\frac{D^2}{a_0^2}\left(\frac{D^2}{a_0^2}-1\right)+1}-1\right].$$

It is seen that all the parameters are determined by the velocity D which will be found by solving the problem in the region III.

Region III

In the region III a centered acceleration wave is propagated. Along the negative characteristics

(5.5) $$x-h = -a_3(m_3)\left(t-\frac{h}{a_0}\right),$$

the following relation is satisfied:

(5.6) $$v_3 = F(m_3)+v_m-F_m,$$

where

(5.7) $$F(m_3) = \frac{\varrho_0 a_0^3}{3E_1}\left\{\left[\frac{a_3(m_3)}{a_0}\right]^3-1\right\},$$
$$F_m = \frac{\varrho_0 a_0^3}{3E_1}\left[\left(\frac{a_m}{a_0}\right)^3-1\right],$$

(5.8) $$v_m = -\frac{\varrho_0 a_0^3}{3E_1}\left[\left(\frac{a_m}{a_0}\right)^3-1\right],$$
$$a_m = a_0\left(\frac{h}{h-a_0\tau_m}\right).$$

Time τ_m is shown in Fig. 3.

In addition, from Eq. (5.5) it follows that

(5.9) $$a_3(m_3) = \frac{h-x}{t-h/a_0}.$$

Finally, after using the relations (5.6)–(5.9), the velocity in region III is expressed by the formula

(5.10) $$v_3(a_3) = \frac{E_0}{3E_1}\left[\left(\frac{a_3}{a_0}\right)^3-2\left(\frac{a_m}{a_0}\right)^3+1\right]a_0$$

or

(5.11) $$v_3(x,t) = \frac{E_0}{3E_1}\left\{\left[\frac{h-x}{a_0(t-h/a_0)}\right]^3-2\left(\frac{a_m}{a_0}\right)^3+1\right\}a_0.$$

Then from Eqs. (2.5) and (5.9) we obtain

$$m_3(a_3) = \frac{E_0}{2E_1}\left[\left(\frac{a_3}{a_0}\right)^2 - 1\right] \tag{5.12}$$

or

$$m_3(x, t) = \frac{E_0}{2E_1}\left\{\left[\frac{h-x}{a_0(t-h/a_0)}\right]^2 - 1\right\}. \tag{5.13}$$

Consequently, substitution of the expressions (5.12) and (5.13) into the constitutive equation (2.14) yields

$$\sigma_3(a_3) = \frac{E_0^2}{4E_1}\left[\left(\frac{a_3}{a_0}\right)^4 - 1\right] \tag{5.14}$$

or

$$\sigma_3(x, t) = \frac{E_0^2}{4E_1}\left\{\left[\frac{h-x}{a_0(t-h/a_0)}\right]^4 - 1\right\}. \tag{5.15}$$

In order to render the constructed solution unique, we must determine the velocity of propagation of the shock wave front D and the direction number of the characteristic O_1R_1 denoted by a_1. To this end the condition of continuity of stress σ and material velocity v at the interface between regions II and III (characteristic O_1R_1, Fig. 3) will be used.

Equating the stresses σ_2 and σ_3 given by the formulae (5.2) and (5.14), we obtain

$$\frac{D^2}{a_0^2}\left(\frac{D^2}{a_0^2} - 1\right) = \frac{1}{4}\left(\frac{a_1^4}{a_0^4} - 1\right)$$

or

$$D = \left[\frac{1}{2}\left(\frac{a_1^2}{a_0^2} + 1\right)\right]^{1/2} a_0 \tag{5.16}$$

and for

$$\frac{a_1}{a_0} \gg 1, \tag{5.17}$$

$$D \approx a_1/\sqrt{2}. \tag{5.18}$$

Furthermore, the equality $v_2 = v_3$ combined with Eqs. (5.3), (5.10) and (5.16) yields

$$\left\{1.125\left[\left(\frac{a_1}{a_0}\right)^4 - 1\right]\left[\left(\frac{a_1}{a_0}\right)^2 - 1\right]\right\}^{1/2} = 2\left(\frac{a_m}{a_0}\right)^3 - \left(\frac{a_1}{a_0}\right)^3 - 1. \tag{5.19}$$

This is a transcendental equation which enables the numerical evaluation of the direction number a_1 of the characteristic O_1R_1.

From the analysis of the left and right hand sides of Eq. (5.19) it is found that the equation has a single real root (solid lines in Fig. 5—root $(a_1/a_0)_S$). Inequality (5.17) satisfied, Eq. (5.17) yields the result

$$a_1 \approx 0.99 a_m. \tag{5.20}$$

Thus the problem of disintegration of an arbitrary discontinuity in the isentropic approximation has been solved. Let us now pass to the adiabatic approximation.

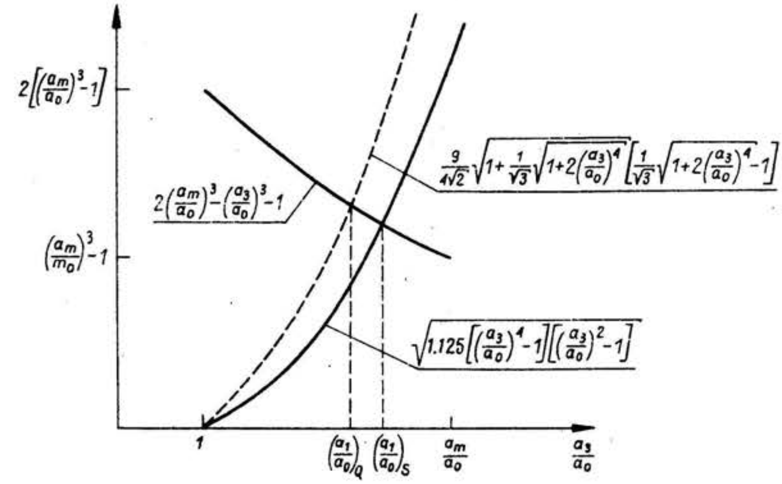

FIG. 5.

6. Analysis of the wave fronts configuration after the act of cumulation in the adiabatic approximation

Since the shapes of the curves $v_S(\sigma)$ and $v_H(\sigma)$ in the case of adiabatic approximation are similar to those of the isentropic approximation (cf. Sect. 4), the wave configuration of the solution after the act of cumulation will also resemble the one determined previously (cf. Figs. 3 and 6). The difference consists in the fact that now in the cumulation plane O_1K a contact discontinuity is generated (jump of the parameter m, Fig. 7). Line O_1K represents now the interface between two different regions.

Here are the solutions of the problem in various regions of the phase plane.

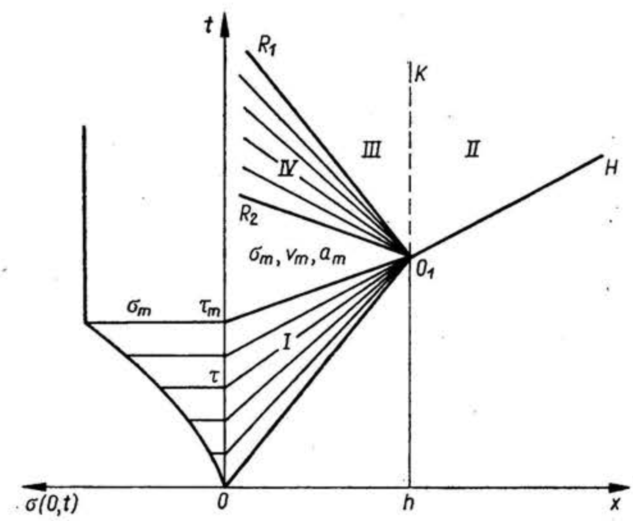

FIG. 6.

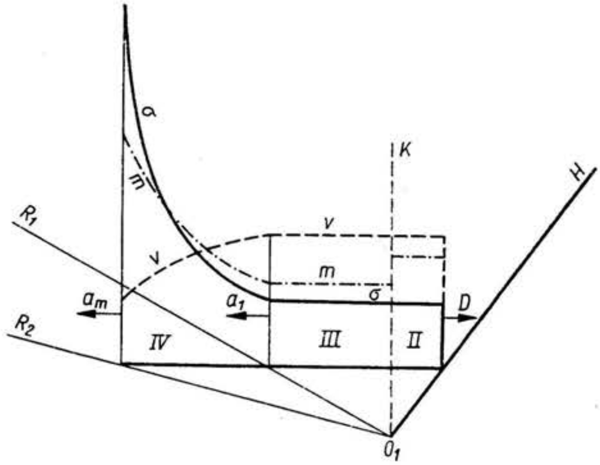

FIG. 7.

Region II

From Eqs. (2.11), (2.12) and (4.4) we obtain

(6.1) $$\sigma_2 = \frac{3}{2}\frac{(\varrho_0 D^2)^2}{E_1}(D^2-a_0^2) = \frac{3}{2}\frac{E_0^2}{E_1}\frac{D^2}{a_0^2}\left(\frac{D^2}{a_0^2}-1\right),$$

(6.2) $$v_2 = -\frac{3}{2}\frac{\varrho_0 D}{E_1}(D^2-a_0^2) = -\frac{3}{2}\frac{E_0}{E_1}\left(\frac{D^2}{a_0^2}-1\right)D,$$

(6.3) $$m_2 = \frac{3}{4}\left[\sqrt{\left(\frac{E_0}{E_1}\right)^2+4\left(\frac{E_0^2}{E_1^2}\right)\frac{D^2}{a_0^2}\left(\frac{D^2}{a_0^2}-1\right)}-\frac{E_0}{E_1}\right].$$

Region III

Here we have

(6.4) $$\sigma_3 = \sigma_2, \quad v_3 = v_2.$$

Next, from the constitutive equation (2.14) and the relation (6.4) it follows that

$$m_3^2+\frac{E_0}{E_1}m_3-\frac{3}{2}\left(\frac{E_0 D}{E_1 a_0}\right)^2\left[\left(\frac{D}{a_0}\right)^2-1\right] = 0$$

or

(6.5)
$$m_3 = \begin{cases} \frac{1}{2}\left\{\sqrt{\left(\frac{E_0}{E_1}\right)^2+6\left(\frac{E_0 D}{E_1 a_0}\right)^2\left[\left(\frac{D}{a_0}\right)^2-1\right]}-\frac{E_0}{E_1}\right\} & \text{if} \quad \sigma_3 > 0, \quad E_1 > 0, \\ -\frac{1}{2}\left\{\sqrt{\left(\frac{E_0}{E_1}\right)^2+6\left(\frac{E_0 D}{E_1 a_0}\right)^2\left[\left(\frac{D}{a_0}\right)^2-1\right]}+\frac{E_0}{E_1}\right\} & \text{if} \quad \sigma_3 < 0, \quad E_1 < 0. \end{cases}$$

Region IV

Proceeding like in Region III of the case considered in the previous section, we obtain

(6.6) $$\sigma_4(a_4) = \frac{E_0^2}{4E_1}\left[\left(\frac{a_4}{a_0}\right)^4-1\right],$$

(6.7) $$v_4(a_4) = -\frac{E_0}{3E_1}\left\{2\left(\frac{a_m}{a_0}\right)^3 - \left[\left(\frac{a_4}{a_0}\right)^3 + 1\right]\right\}a_0,$$

(6.8) $$m_4(a_4) = \frac{E_0}{2E_1}\left[\left(\frac{a_4}{a_0}\right)^2 - 1\right],$$

where

(6.9) $$a_4(x, t) = \frac{h-x}{t-\dfrac{h}{a_0}}.$$

The condition of continuity of stresses and velocities along the characteristic $O_1 R_1$ (Fig. 6) yields

(6.10) $$D = \left\{\frac{1}{2}\left[1+\frac{1}{\sqrt{3}}\sqrt{1+2\left(\frac{a_1}{a_0}\right)^4}\right]\right\}^{1/2} a_0$$

and

(6.11) $$\frac{9}{4}\left\{\frac{1}{2}\left[1+\frac{1}{\sqrt{3}}\sqrt{1+2\left(\frac{a_1}{a_0}\right)^4}\right]\right\}^{1/2}\left[\frac{1}{\sqrt{3}}\sqrt{1+2\left(\frac{a_1}{a_0}\right)^4}-1\right] = 2\left(\frac{a_m}{a_0}\right)^3 - \left(\frac{a_1}{a_0}\right)^3 - 1.$$

Equation (6.11) possesses, similarly to the case of isentropic equation (5.19), a single real root (dashed line in Fig. 5, root $(a_1/a_0)_Q$). Inequality (5.17) fulfilled, Eqs. (6.10) and (6.11) yield the respective results:

(6.12) $$D \approx a_1/\sqrt[4]{6} \approx 0.64a_1,$$

(6.13) $$a_1 \approx \frac{8\sqrt[4]{27}}{9\sqrt[4]{2}+8\sqrt[4]{27}}\, a_m \approx 0.63a_m.$$

Thus the principal aim of this paper has been achieved. To conclude, let us quote the main results following from the presented analysis of the problem.

7. Final conclusions

1. In an isotropic elastic medium centrally-cumulative simple waves of finite deformations may be propagated, and they are cumulated on a single plane. This cumulation is achieved by proper design of the boundary value profile, Eqs. (3.5)–(3.7).

2. At the cumulation plane an unstable thermo-mechanical state of the medium is generated called the arbitrary discontinuity, which disintegrates to form a stable system of weak and strong discontinuity waves.

3. Two approximations of the motion occurring behind the shock wave front have been considered, isentropic and adiabatic. It has been proved that in both approximations disintegration of the arbitrary discontinuity is followed by generation of only one stable wave configuration, shown in Fig. 3 for the isentropic approximation, and in Fig. 6—for

the adiabatic approximation. In a gaseous polytropic medium three wave configurations are possible after the disintegration of the arbitrary discontinuity (cf. author's paper [7]).

4. Solution of the problem is represented in the form of closed formulae. It is one of a very few closed-form solutions known in nonlinear elastodynamics.

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