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Unsteady rectilinear gas flows with the velocity distribution of homentropic centered simple waves

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In [10] it is found that homogeneous solutions of the Lagrangian equations of motion for the unsteady rectilinear motion of a perfect gas all have the same velocity distribution and the same particle paths in the x, t-plane. When matching two different homogeneous solutions along a common particle path the velocity and pressure could be made continuous across the common path, but discontinuities in density, temperature and entropy had to be accepted. In this paper a more general class of solutions is constructed, which preserves the velocity distribution and contains the homogeneous solutions. It is shown that a generalized flow and a homogeneous flow can be matched along a common particle path without discontinuities in the parameters mentioned.

W pracy [10] stwierdzono, że wszystkie jednorodne rozwiązania równań ruchu Lagrange'a dla nieustalonych prostoliniowych ruchów gazu doskonałego mają te same rozkłady prędkości i te same tory cząsteczek w płaszczyźnie x, t. Dopasowując do siebie dwa różne rozwiązania jednorodne wzdłuż wspólnej trajektorii cząstek można by było uzyskać rozkłady prędkości i ciśnienia ciągłe w kierunku poprzecznym, jednak funkcje gęstości, temperatury i entropii musiałyby być nieciągłe. W niniejszej pracy skonstruowano szerszą klasę rozwiązań zachowującą rozkład prędkości i zawierającą rozwiązania jednorodne. Pokazuje się, że przepływy jednorodne i uogólnione można „skleić" ze sobą wzdłuż wspólnej trajektorii cząstek bez naruszania ciągłości wspomnianych parametrów.

В работе [10] констатировано, что все однородные решения уравнений движения Лагранжа для неустановившихся прямолинейных движений идеального газа имеют те же самые распределения скоростей и те же самые траектории частиц в плоскости x, t. Согласовывая друг к другу два разных однородных решения вдоль общих траекторий частиц, можна было получить распределения скоростей и давления непрерывные в поперечном направлении, однако функции плотности, температуры и энтропии должны быть разрывными. В настоящей работе построен более широкий класс решений, сохраняющий распределение скоростей и содержавший однородные решения. Доказывается, что однородные и обобщенные течения можно „сшить" друг с другом вдоль общей траектории частиц без нарушения непрерывности упомянутых параметров.

Notation

x Cartesian coordinate,
t time,
u velocity,
a speed of sound,
S entropy per unit mass,
p pressure,
V specific volume,
R gas constant per unit mass,
h Lagrangian mass coordinate,
n degree of homogeneity,
$B(\psi)$, $b(\psi)$ entropy functions (Cf. Eqs. (1.1) and (4.1)),

r, s	Riemann invariants (Cf. Eqs. (3.5) and (3.6)),
$A, C, C^*, C_2,$ U, U_∞, u_0, a_0	constants,
$N = \dfrac{n}{(1+\varkappa)(n+\varkappa)}$	= constant (See Eq. (5.10)),
$\gamma = \dfrac{c_p}{c_v}$	, constant ratio of specific heats,
$\varkappa = \dfrac{\gamma-1}{\gamma+1}$	,
$\lambda = \sqrt{\dfrac{\varkappa(1+\varkappa)(n+\varkappa)}{n+1+\varkappa}}$	= constant (See Eq. (5.4)),
ψ	Lagrangian particle coordinate, constant along particle path,
σ	entropy function (See Sect. 7).

1. Introduction

THIS PAPER deals with the unsteady, rectilinear motion of a perfect gas. The term "rectilinear" is preferred to "one-dimensional" since flows with cylindrical or spherical symmetry are outside the scope of the paper.

Unsteady rectilinear motions of a perfect gas have been studied extensively, both theoretically and experimentally. It may be sufficient to mention the word "shock-tube" and books by HADAMARD [1], COURANT and FRIEDRICHS [2], STANIUKOVICH [3], ZELDOVICH and RAIZER [4], WHITHAM [5] and LIGHTHILL [6].

There exist, however, within this body of theory some weak patches where few results are available. One of these areas appears when non-homentropic flows are considered. These are flows where the entropy S is no longer a constant, but varies from one element of gas to another. In that case the well-known homentropic-, isentropic- or Poisson-relation

$$p\varrho^{-\gamma} = pV^\gamma = \exp\left(\frac{S-S_0}{c_v}\right) = \text{const},$$

has to be replaced by

$$p\varrho^{-\gamma} = pV^\gamma = \exp\left(\frac{S(\psi)}{c_v}\right) = B(\psi) = b(\psi)^\gamma, \tag{1.1}$$

where ψ is a suitable Lagrangian particle- or mass-coordinate.

The equivalent functions $S(\psi)$, $B(\psi)$ and $b(\psi)$ are arbitrary functions as far as the equations of motion are concerned. Since in actual experiments and other situations where non-homentropic flows appear these functions cannot be arbitrary, it is of interest to ask how they should be determined.

This question acquires further interest if one considers that non-homentropic flows appear, amongst others, behind non-uniformly travelling shock-waves. By studying non-homentropic flows, one may hope to be in a better position to answer some questions about non-uniformly travelling shock-waves.

Two simple configurations with non-uniformly travelling shock-waves which may serve as a guide are the problems originally studied by FRIEDRICHS [7] and by GOULD [8], and which are illustrated in Figs. 1 and 2. It is convenient to call them the *F*- and *G*-problem. The *F*-problem considers the overtaking of a shock-wave by a rarefaction wave, the *G*-problem the head-on collision of a shock-wave and a rarefaction wave.

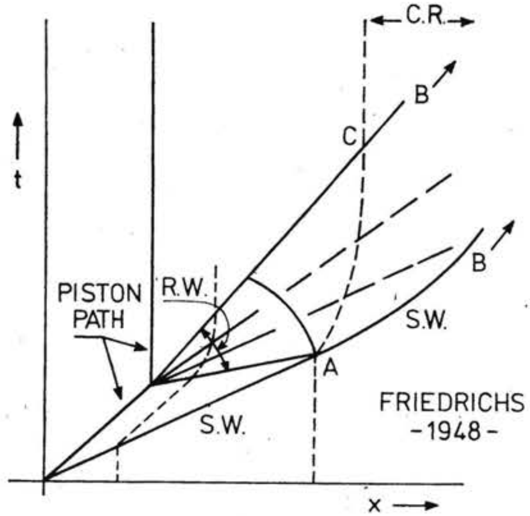

FIG. 1. Shock-wave overtaken by rarefaction wave. *S.W.* — shock-wave trajectory, *R.W.* — rarefaction wave, *C.R.* — contact region.

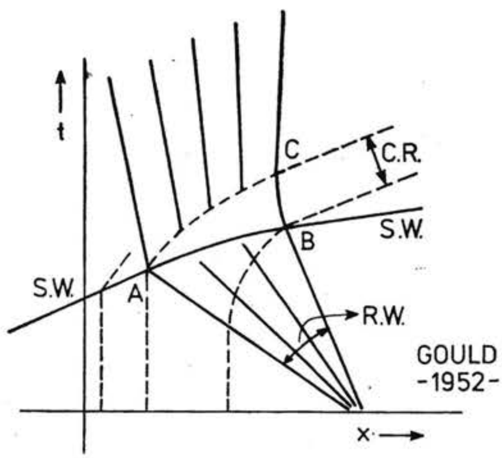

FIG. 2. Head-on collision of shockwave and rarefaction wave. *S.W.* — shock-wave trajectory, *R.W.* — rarefaction wave, *C.R.* — contact region.

Starting from (sectionally) homentropic initial-conditions, most of the segments in which the resulting flow may be decomposed are homentropic. Only the triangular domains *ABC* in the figures and the adjacent contact-regions (*CR*) are non-homentropic.

In their simplest form, when the rarefaction waves initiating the interaction are centered, both problems display some forms of similarity. Keeping this point in mind, I considered some time ago solutions of the Lagrangian equations of motion which are homogeneous in the Lagrangian mass coordinate h and the time t. The Poisson-relation for these flows takes the form

$$p\varrho^{-\gamma} = pV^{\gamma} = B(h) = \bar{B}h^{n(\gamma+1)},$$

with $\bar{B}$ a constant and n the degree of homogeneity.

A summary of these and some other calculations was presented two years ago at the Symposium in Olsztyn, and in Stockholm [9], while the detailed investigation is available as a report [10].

It came as a surprise that all these flows for different n have the same velocity distribution and the same particle paths in the physical x, t-plane. One finds that

$$u(x,t) = (1-\varkappa)\left(\frac{x-x_0}{t} - U\right) + U, \tag{1.2}$$

$$\left(\frac{x-x_0}{t} - U\right)t^{\varkappa} = \frac{u_0}{1-\varkappa}h^{n+\varkappa}, \tag{1.3}$$

with U the constant terminal velocity, x_0 and u_0 constants, while $\varkappa = \dfrac{\gamma-1}{\gamma+1}$. (This combination of γ is denoted by μ^2 in [2]).

In particular, for $n = 0$, the homentropic centered simple wave of the classical theory appears.

Considering that in the F- and G-problems one boundary of the triangular domain ABC is a particle path, one may consider the matching of two homogeneous flows, with different n, along a common particle path.

This problem was considered in some detail in [10]. It turns out to be possible to match two different homogeneous flows along a common particle path, say $\psi = \psi_0$, with ψ a Lagrangian coordinate, not necessarily identical with h, while satisfying the continuity of velocity and pressure along ψ_0, i.e.

$$\begin{aligned} u(\psi_0^+, t) &= u(\psi_0^-, t), \\ p(\psi_0^+, t) &= p(\psi_0^-, t). \end{aligned} \tag{1.4}$$

However, it is found also that discontinuities in density, temperature and entropy have to be accepted along $\psi = \psi_0$. The appearance of these discontinuities makes it seem unlikely that these composite flows will appear in, for example, the F- and G-problem, where the shock-wave seems the only curve where discontinuities in the physical parameters themselves may be expected.

The present investigation was started to find out whether this difficulty could be overcome. Before going into details it may be stated that this is indeed the case. The homogeneous flows of Ref. [10] can be generalized while retaining their velocity distribution, particle paths and entropy. Also a homogeneous flow and a generalized flow can be matched along a particle path $\psi = \psi_0$ in such a way that continuity of velocity, pressure, density, temperature and entropy is preserved across $\psi = \psi_0$.

Two points seem crucial for the analysis to be presented. First, the velocity distribution u is linear in x, the Cartesian coordinate. It follows that $\dfrac{\partial u}{\partial x}$, and from the continuity equation, also the specific expansion rate are functions of the time only and uniform throughout the gas at each instant t.

Secondly, we do not take the speed of sound a as a dependent variable, but its square, $a^2 = \gamma RT$.

The two other dependent variables are the velocity u and the entropy S.

2. The general solutions for the prescribed velocity

The equations of motion which form the starting point are the conservation of mass, momentum and energy in the physical x, t-plane (x — Cartesian coordinate, t — time). Assuming that the gas is an ideal gas with constant specific heats, and omitting effects of viscosity and heat conduction, these equations may be put in the form

$$\frac{\partial a^2}{\partial t}+u\frac{\partial a^2}{\partial x}+(\gamma-1)a^2\frac{\partial u}{\partial x}=0,$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\gamma-1}\frac{\partial a^2}{\partial x}=\frac{a^2}{\gamma R}\frac{\partial S}{\partial x}, \tag{2.1}$$

$$\frac{\partial S}{\partial t}+u\frac{\partial S}{\partial x}=0.$$

The velocity distribution $u(x, t)$ is prescribed to be

$$u = A\frac{x}{t}+C, \tag{2.2}$$

with A, C constants and including the velocity distribution (1.2) found for the homogeneous flows.

Integration of the differential equation

$$\frac{dx}{dt}=u,$$

yields the particle paths in the form

$$t^{1-A}\left(\frac{x}{t}+C^*\right)=\psi, \tag{2.3}$$

with $C = C^*(A-1)$, and ψ a constant along a particle path. Also one requires $A \neq 1$. For $A = 1$ a special case appears, which is not to be discussed here.

Selecting ψ as a Lagrangian particle coordinate and considering that the entropy S is constant along a particle path, we write

$$S(x, t) = S(\psi).$$

Writing the velocity u in the Lagrangian coordinates (ψ, t), one finds

$$u = A\psi t^{A-1}-C^*.$$

Substitution of $u(x, t)$ in Eq. $(2.1)_1$ yields a linear partial differential equation for a^2 of the form

$$\frac{\partial a^2}{\partial t}+\left(A\frac{x}{t}+C\right)\frac{\partial a^2}{\partial x}+(\gamma-1)\frac{A}{t}a^2=0. \tag{2.4}$$

The characteristic equations are

$$\frac{dt}{t} = \frac{dx}{Ax+Ct} = -\frac{da^2}{A(\gamma-1)a^2},$$

and two independent solutions of this system are easily found. The first one is the expression (2.3) for ψ and the second is

$$a^2 t^{(\gamma-1)A} = \text{const} = C_2. \tag{2.5}$$

A general solution of Eq. (2.4) can then be written down.

Since a^2 is needed explicitly in Eq. $(2.1)_2$, the general solution is taken in the form

$$a^2 = t^{-(\gamma-1)A} f(\psi), \tag{2.6}$$

with $f(\psi)$ an arbitrary function of ψ. Taking stock of the situation u, S and a^2 have been determined, while Eqs. $(2.1)_1$ and $(2.1)_3$ are satisfied. It remains to satisfy the momentum equation $(2.1)_2$. Substitution of Eq. (2.2), of $S(\psi)$ and of Eq. (2.6) into Eq. $(2.1)_2$ shows that this equation can indeed be satisfied provided

$$A = \frac{2}{\gamma+1} = 1-\varkappa, \tag{2.7}$$

in agreement with Eq. (1.2) and provided further the following compatibility condition for $f(\psi)$ and $S(\psi)$ is satisfied:

$$f'(\psi) - \frac{1}{c_p} S'(\psi) f(\psi) = 2\varkappa^2 \psi, \tag{2.8}$$

with dashes denoting derivatives to ψ.

Substituting A from Eq. (2.7) in the appropriate places and collecting the results, we found the solutions

$$u = (1-\varkappa)\frac{x}{t} + C = (1-\varkappa)\psi t^{-\varkappa} - C^*,$$

$$a^2 = t^{-2\varkappa} f(\psi),$$

$$S = S(\psi),$$

$$\psi = t^{\varkappa}\left(\frac{x}{t} + C^*\right),$$

while $f(\psi)$ and $S(\psi)$ have to satisfy Eq. (2.8) and $C = -\varkappa C^*$.

Inspection of the velocity u in the above expressions shows that for $t \to \infty$ the velocity of all the fluid elements approaches the constant value $-C^*$ which is the constant terminal velocity. There is some advantage to consider the flow from a reference frame moving with the terminal velocity. Introduction of the required Galilei transformations leaves the equations of motion invariant, while it removes some of the constants from the solutions found.

Omitting the details, the solutions then take the final form

$$(2.9)\qquad \begin{aligned} u &= (1-\varkappa)\frac{x}{t} = (1-\varkappa)\psi t^{-\varkappa},\\ a^2 &= t^{-2\varkappa}f(\psi),\\ S &= S(\psi),\\ \psi &= xt^{-(1-\varkappa)}, \end{aligned}$$

where $f(\psi)$ and $S(\psi)$ have to satisfy Eq. (2.8).

Eliminating t between u and a^2, one finds that in the family of centered flows obtained the relation below exists:

$$(2.10)\qquad (1-\varkappa)^2\psi^2a^2-u^2f(\psi) = 0.$$

In the following section the homentropic flows in the family will be considered.

3. The homentropic solutions

If the flow is homentropic, $S(\psi)$ is a constant and the compatibility condition (2.8) simplifies. One finds for $f(\psi)$ the expression

$$(3.1)\qquad f(\psi) = \varkappa^2\psi^2+f(0),$$

with $f(0)$ an integration constant, and for a^2

$$(3.2)\qquad a^2 = \varkappa^2\psi^2t^{-2\varkappa}+f(0)t^{-2\varkappa} = \varkappa^2\left(\frac{x}{t}\right)^2+f(0)t^{-2\varkappa}.$$

When $f(0)$ is taken equal to zero, the solutions represent the two homentropic centered simple waves of the classical theory, but taken together in such a way that $x = 0$ and $\psi = 0$, represent the vacuum-lines in the x, t-plane, respectively the Lagrangian ψ, t-plane.

When $f(0)$ is taken different from zero, the solutions are no longer simple waves. The solutions which appear will be called Von Mises' flows since they are mentioned in the book of VON MISES ([11], p. 78). Taking $f(0) > 0$, there is no vacuum-line and a vacuum only appears for $t \to \infty$. Taking $f(0) < 0$, the flow has physical significance only if

$$(3.3)\qquad \varkappa^2\psi^2+f(0) \geqslant 0,$$

and we have essentially two flows, separated by a strip with negative temperatures and so without physical significance, in the ψ, t-plane.

To verify these statements we consider in succession the three cases (i) $f(0) = 0$, (ii) $f(0) = \varkappa^2\alpha^2$ and (iii) $f(0) = -\varkappa^2\alpha^2$ where α^2 is a positive constant.

Case (i), $f(0) = 0$

This is the flow with

$$(3.4)\qquad \begin{aligned} u &= (1-\varkappa)\frac{x}{t} = (1-\varkappa)\psi t^{-\varkappa},\\ a &= \pm\varkappa\frac{x}{t} = \pm\varkappa\psi t^{-\varkappa}. \end{aligned}$$

Since a, the speed of sound, cannot be negative, it follows that the +sign in Eq. $(3.4)_2$ applies in the positive halves of the x, t- and ψ, t-plates, while the negative sign applies in the negative half-planes, $x < 0$ and $\psi < 0$. Also the lines $x = 0$ and $\psi = 0$ are the vacuumlines where $a = 0$.

Considering the positive half-planes, one deduces from Eq. (3.4)

$$u-\frac{1-\varkappa}{\varkappa}a = u-\frac{2}{\gamma-1}a = 0,$$

indicating that this flow is the classical simple wave with

$$s = u-\frac{2}{\gamma-1}a = 0. \tag{3.5}$$

In the same way one deduces that the negative half-planes represent the simple wave with

$$r = u+\frac{2}{\gamma-1}a = 0, \tag{3.6}$$

where the Riemann invariants have been denoted by letters r and s.

The above conclusions also easily follow from Eq. (2.10) if it is kept in mind that a cannot be negative, while $u > 0$ in the positive half-planes and $u < 0$ in the negative half-planes. Also the form of the characteristics and other properties of the centered simple waves are easily retrieved.

It will also be clear that in the application of centered simple waves, say in shock-tube flow, only segments of the flows constructed here are needed.

Consider for example the flow in an infinitely long straight tube generated by the instantaneous removal (at $t = 0$) of a membrane (located at $x = 0$) separating a vacuum part ($x > 0$) and a section ($x \leqslant 0$) filled with a uniform gas ($a = a_0$) at rest.

Taking the appropriate Galilei transformation into account $\left(\text{the terminal velocity is } \frac{2a_0}{\gamma-1}\right)$, this flow corresponds with the segment of the solution (3.4) in the interval

$$-\frac{a_0}{\varkappa} \leqslant \frac{x}{t} \leqslant 0,$$

which may easily be verified.

Case (ii), $f(0) = \alpha^2\varkappa^2$

This Von Mises' flow is determined by

$$\begin{aligned} u &= (1-\varkappa)\frac{x}{t} = (1-\varkappa)\psi t^{-\varkappa}, \\ a^2 &= \varkappa^2\left(\frac{x}{t}\right)^2+\varkappa^2\alpha^2 t^{-2\varkappa} = \varkappa^2 t^{-2\varkappa}(\psi^2+\alpha^2). \end{aligned} \tag{3.7}$$

The velocity distribution and the particle paths are the same as those in the simple waves of Case (i) but a^2 and hence the temperature has been raised with a part depending on the time only. As a consequence a temperature zero and the associated vacuum are reached only for $t \to \infty$, but a vacuum-line as in Case (i) is absent. Also the characteristics have changed form and the simple wave property, that one of the Riemann-invariants r (as defined in Eq. (3.6)) or s (as defined in Eq. (3.5)) is constant throughout the flow no longer applies. Something like it is retained if one considers Eq. (2.10), which takes the form

$$(1-\varkappa)^2\psi^2a^2-\varkappa^2(\psi^2+\alpha^2)u^2 = 0$$

and yields in the positive half-plane, where $u > 0$,

$$u\sqrt{1+\frac{\alpha^2}{\psi^2}}-\frac{2a}{\gamma-1} = 0 \tag{3.8}$$

and in the negative half-plane

$$u\sqrt{1+\frac{\alpha^2}{\psi^2}}+\frac{2a}{\gamma-1} = 0. \tag{3.9}$$

The possibility to extend a^2 with a term depending on the time only is due to the simple form of the velocity distribution. Since $\frac{\partial u}{\partial x}$ and the specific expansion rate depend on the time only, the coefficients of a^2 in the third term of Eqs. $(2.1)_1$ and (2.4) no longer contain x and a solution for a^2 depending on t can be superposed without difficulty. In the momentum equation $(2.1)_2$ the R.H.S. remains equal to zero and the L.H.S. is not affected by the change in a^2 since only the x-derivative of a^2 is required.

It should be noted that the first part of the preceding argument applies equally well if Eqs. $(2.1)_1$ and (2.4) are reduced to equations for a instead of a^2. In the momentum equation $(2.1)_2$, however, the nonlinear term

$$\frac{2a}{\gamma-1}\frac{\partial a}{\partial x}$$

then appears and here the argument fails. It suggests that the point is a bit subtle and may not be trivial. It also explains why a^2 was taken as one of the variables and not the speed of sound a itself.

It is of interest to note that several details of the flow can be calculated exactly and in [12] these have been considered. The characteristics have been computed in detail and it is verified that the Riemann invariants r and s (defined by Eqs. (3.6) and (3.5)) are constant along the appropriate characteristics. Since the solution is a general homentropic flow, one also verifies that the expression for a^2 in Eq. (3.7) can be converted to the form

$$t^{2\varkappa} = -\left(\frac{2\alpha}{\gamma+1}\right)^2\frac{1}{r\cdot s}, \tag{3.10}$$

which is a solution of the appropriate Euler-Poisson-Darboux equation

$$\frac{\partial^2 t}{\partial r\,\partial s}+\frac{1}{2\varkappa}\cdot\frac{1}{r-s}\left(\frac{\partial t}{\partial s}-\frac{\partial t}{\partial r}\right) = 0. \tag{3.11}$$

The generation of this flow, or part of it, in an experiment can be achieved by producing the appropriate boundary conditions (for example by moving pistons) and initial conditions. However, they seem artificial and neither simple nor obvious. It is possibly due to these circumstances that the Von Mises' solutions have not been considered further.

Case (iii), $f(0) = -k^2\alpha^2$

This is the second kind of Von Mises' flow determined by

$$(3.12)\quad \begin{aligned} u &= (1-\varkappa)\frac{x}{t} = (1-\varkappa)\psi t^{-\varkappa}, \\ a^2 &= \varkappa^2\left(\frac{x}{t}\right)^2 - \varkappa^2\alpha^2 t^{-2\varkappa} = \varkappa^2 t^{-2\varkappa}(\psi^2-\alpha^2). \end{aligned}$$

Since $a^2 = \gamma RT$ can not be negative, the flow has physical relevance only for $|\psi| \geqslant \alpha$. The vacuum-lines are the particle paths $\psi = \pm\alpha$ and in the x, t- and ψ, t-planes we have essentially two distinct flows. One with $u < 0$ in the negative half-plane for $-\infty \leqslant \psi \leqslant -\alpha$, the other in the positive half-plane with $u > 0$ for $\alpha \leqslant \psi \leqslant +\infty$.

Compared to the simple wave flows of Case (i), the flow takes place at a reduced temperature level and as a consequence the vacuum situation with $T = 0$ already appears at $\psi = \pm\alpha$, instead of at $\psi = 0$. Similar remarks as in Case (ii) apply here and in [12] a number of details are worked out.

We next turn to non-homentropic cases and therefore first consider the compatibility condition (2.8).

4. The compatibility condition

In the solutions (2.9) the two arbitrary functions $f(\psi)$ and $S(\psi)$ appear which have to satisfy the compatibility condition (2.8). This condition may be considered as a non-homogeneous linear differential equation for the determination of $f(\psi)$. The general solution of this differential equation is composed of two parts; the general solution of the homogeneous equation (with R.H.S. equal to zero) and a particular solution of the non-homogeneous equation.

In this light $\varkappa^2\psi^2$ in the preceding section is the particular solution, while the constants $\pm\varkappa^2\alpha^2$ represent the general solution of the homogeneous equation

$$f'(\psi) = 0.$$

Taking only the particular solution, the homentropic centered simple waves were obtained. Admitting also the general solution of the homogeneous equation, the Von Mises' flows were obtained.

The construction of the general solution of Eq. (2.8) for arbitrary $S(\psi)$ is not difficult, but may be simplified further by replacing $S(\psi)$ by another function. Therefore, consider the isentropic relation in the form (1.1). This yields

$$(4.1)\quad S(\psi) = c_v \ln B(\psi) = c_p \ln b(\psi),$$

and substituting $S(\psi)$ in terms of $b(\psi)$, the condition (2.8) may be put in the form

$$(4.2)\quad \frac{d}{d\psi}\left(\frac{f(\psi)}{b(\psi)}\right) = 2\varkappa^2 \frac{\psi}{b(\psi)}.$$

The general solution of $f(\psi)$ now requires a quadrature only and one obtains

$$(4.3)\qquad f(\psi) = 2\varkappa^2 b(\psi)\left\{\int \frac{\psi}{b(\psi)}\,d\psi + C\right\},$$

with C an integration constant.

To express $b(\psi)$ and hence the entropy in terms of $f(\psi)$, Eq. (4.2) may be rewritten in the form

$$(4.4)\qquad \frac{d}{d\psi}\left(\frac{f(\psi)}{b(\psi)}\right) = 2\varkappa^2 \frac{\psi}{f(\psi)}\cdot\frac{f(\psi)}{b(\psi)},$$

yielding upon integration

$$(4.5)\qquad b(\psi) = f(\psi)\exp\left\{-2\varkappa^2 \int \frac{\psi}{f(\psi)}\,d\psi + C\right\},$$

and

$$(4.6)\qquad \frac{1}{c_p} S(\psi) = \ln b(\psi) = \ln f(\psi) - 2\varkappa^2 \int \frac{\psi}{f(\psi)}\,d\psi + C,$$

with C again an integration constant.

Since the present investigation was undertaken to see whether discontinuities could be removed, which appear when two of the homogeneous solutions studied in [10] are matched, the homogeneous flows will be considered in the next section. It will be shown that the homogeneous solutions of [10] are similar to the homentropic simple waves in this sense, that they will appear when for the appropriate entropy distribution $S(\psi)$ only the particular solution for $f(\psi)$ in the compatibility condition is used. By adding to $f(\psi)$ the solution of the homogeneous equation, one has more scope enabling us to remove the discontinuities.

5. The homogeneous solutions

In [10] solutions of the Lagrangian equations of motion are considered, which are homogeneous functions of the Lagrangian mass coordinate h and the time t.

Let us start with the assumption that h and t are positive $(0 \leqslant h \leqslant +\infty,\ 0 \leqslant t \leqslant +\infty)$. For the velocity u, the speed of sound a, and the Cartesian coordinate x the following expressions were obtained:

$$(5.1)\qquad \begin{aligned} u(h,t) &= u_0 h^{n+\varkappa} t^{-\varkappa} + U_\infty,\\ a(h,t) &= a_0 h^{n+\varkappa} t^{-\varkappa},\\ \left(\frac{x-x_0}{t} - U_\infty\right) t_\varkappa &= \frac{u_0}{1-\varkappa} h^{n+\varkappa}. \end{aligned}$$

In these expressions u_0, a_0, U_∞, x_0 and n are constants. The constant U_∞ is the constant terminal velocity of all the fluid elements when $t \to \infty$, x_0 is the point where the entire mass is concentrated at $t = 0$ and n is the degree of homogeneity. Transferring to a reference frame moving with the terminal velocity U_∞ and origin in x_0, the formulae (5.1) simplify after the proper Galilei transformation to

$$
\begin{aligned}
u &= u_0 h^{n+\varkappa} t^{-\varkappa},\\
a &= a_0 h^{n+\varkappa} t^{-\varkappa}, \qquad (5.2)\\
x t^{-(1-\varkappa)} &= \frac{u_0}{1-\varkappa} h^{n+\varkappa}.
\end{aligned}
$$

The constants u_0 and a_0 are related by

$$
(5.3) \qquad a_0 = \pm \frac{u_0}{1-\varkappa} \sqrt{\frac{\varkappa(1+\varkappa)(n+\varkappa)}{n+1+\varkappa}} = \pm \frac{\lambda u_0}{1-\varkappa},
$$

with the constant λ clearly

$$
(5.4) \qquad \lambda = \sqrt{\frac{\varkappa(1+\varkappa)(n+\varkappa)}{n+1+\varkappa}}.
$$

It was found that not all values of n were admissible. For n in the interval

$$
(5.5) \qquad -(1+\varkappa) \leqslant n \leqslant -\varkappa, \quad -1 \leqslant n+\varkappa \leqslant 0,
$$

the pressure p and the specific folume V appeared with opposite signs, which is physically unacceptable.

As a consequence the solutions separate naturally in two classes, those with $n > -\varkappa$ and those with $n < -(1+\varkappa)$. For $n > -\varkappa$ the constant u_0 is positive and the +sign in Eq. (5.3) has to be taken. For $n < -(1+\varkappa)$ the constant u_0 is negative and the $-$sign in Eq. (5.3) applies. Also it was found that the interval $0 \leqslant h \leqslant +\infty$ corresponds with $0 \leqslant x \leqslant +\infty$ for $n > -\varkappa$ and with $-\infty \leqslant x \leqslant 0$ for $n < -(1+\varkappa)$.

It was possible to extend the solutions to negative values of h in such a way that the interval $-\infty \leqslant h < 0$ corresponds with flows obtained from the previous ones i.e., Eqs. (5.2), by reflection with respect to the line $x = 0$ in the x, t-plane, and implying reversing the direction of the velocity.

Introducing $h^* = -h(-\infty \leqslant h \leqslant 0)$, these solutions may be written as

$$
\begin{aligned}
u &= -u_0 (h^*)^{n+\varkappa} t^{-\varkappa},\\
a &= a_0 (h^*)^{n+\varkappa} t^{-\varkappa}, \qquad (5.6)\\
-x t^{-(1-\varkappa)} &= \frac{u_0}{1-\varkappa} (h^*)^{n+\varkappa}.
\end{aligned}
$$

In the following discussion attention will be largely restricted to solutions with positive h.

Comparison of the expressions (2.9) and (5.2) yields

$$
(5.7) \qquad \psi = \frac{u_0}{1-\varkappa} h^{n+\varkappa},
$$

$$
(5.8) \qquad f(\psi) = (1-\varkappa)^2 \left(\frac{a_0}{u_0}\right)^2 \psi^2 = \lambda^2 \psi^2.
$$

Substitution of Eq. (5.8) into the compatibility condition (2.8) then yields

$$\frac{1}{c_p} S'(\psi) = \frac{2(\lambda^2-\varkappa^2)}{\lambda^2}\cdot\frac{1}{\psi} = \frac{2N}{\psi}, \tag{5.9}$$

where a new constant N is introduced by

$$N = 1-\frac{\varkappa^2}{\lambda^2} = \frac{n}{(1+\varkappa)(n+\varkappa)}. \tag{5.10}$$

One may verify that in the N, n-plane Eq. (5.10) represents a hyperbola and further that N cannot exceed the value $+1$ for all admissible values of n.

Integration of Eq. (5.9) yields

$$\frac{1}{c_p} S(\psi) = 2N\ln\psi + \frac{1}{c_p} S(1), \tag{5.11}$$

with $S(1)$ an integration constant. Employing Eqs. (1.1) and (4.1) one may write also

$$b(\psi) = b(1)\psi^{2N}. \tag{5.12}$$

Since all the homogeneous flows have the same velocity distribution and the same particle paths in the x, t-plane, two flows with different values of n can in general be matched along a common particle path. To begin with, it is found that this matching is possible only for two flows with degrees of homogeneity n, both exceeding $-\varkappa$, or both less than $-(1+\varkappa)$. By using the extensions to negative values of h mentioned before this restriction on the n can be dropped.

Along the common particle path the conditions (1.4), assuring continuity of velocity and pressure can be satisfied. It was also found in [10] and [12] and has been mentioned already that discontinuities in density, temperature and entropy, however, have to be accepted.

It is shown in [12] that the discontinuities are of the form

$$\begin{aligned} \frac{\varrho^{(2)}(\psi_0, t)}{\varrho^{(1)}(\psi_0, t)} &= \left(\frac{\lambda^{(1)}}{\lambda^{(2)}}\right)^2, \\ \frac{1}{c_p}\{S^{(2)}(\psi_0)-S^{(1)}(\psi_0)\} &= 2\ln\frac{\lambda^{(2)}}{\lambda^{(1)}}, \\ \frac{T^{(2)}(\psi_0, t)}{T^{(1)}(\psi_0, t)} = \left\{\frac{a^{(2)}(\psi_0, t)}{a^{(1)}(\psi_0, t)}\right\}^2 &= \left(\frac{\lambda^{(2)}}{\lambda^{(1)}}\right)^2, \end{aligned} \tag{5.13}$$

with $\lambda^{(1)}$ and $\lambda^{(2)}$ the values of λ belonging to the degrees of homogeneity n_1 and n_2 of the two flows joined along $\psi = \psi_0$.

Before continuing with the generalized flows one may note that between u and a in the homogeneous flows the linear relations exist

$$\begin{aligned} u-\frac{u_0}{a_0}a &= 0, \quad \text{for} \quad 0 \leqslant h \leqslant +\infty, \\ u+\frac{u_0}{a_0}a &= 0, \quad \text{for} \quad -\infty \leqslant h < 0, \end{aligned} \tag{5.14}$$

reminding one of the homentropic simple waves. For $n = 0$ the relations (5.14) reduce to the classical Riemann invariants (3.5) and (3.6). It was also shown in [10] that for $n = -(1+2\varkappa)$ the relation $(5.14)_1$ reduces to

$$u+\frac{1}{\gamma}a = 0,$$

a relation which had appeared before in a flow problem with a Ludford-Martin-Staniukovich gas.

6. The generalized flows

The flows to be considered here have the same velocity distribution and particle paths as before. The entropy distribution has the form (5.11) obtained for the homogeneous flows. Substitution of this value in the compatibility condition (2.8) yields by means of Eqs. (5.12) and (4.3) or, otherwise, for $f(\psi)$ the solution

$$f(\psi) = \lambda^2\psi^2+C\psi^{2N}. \tag{6.1}$$

For $C = 0$ this reduces to the form (5.8) of the homogeneous flows and the second term in Eq. (6.1) is clearly the general solution of the homogeneous equation obtained from Eq. (2.8) by putting the R.H.S. equal to zero.

Collecting the results we now have

$$\begin{aligned} u &= (1-\varkappa)\frac{x}{t} = (1-\varkappa)\psi t^{-\varkappa}, \\ a^2 &= t^{-2\varkappa}(\lambda^2\psi^2+C\psi^{2N}), \end{aligned} \tag{6.2}$$

$$\frac{1}{c_p}S(\psi) = \frac{1}{c_p}S(1)+2N\ln\psi, \tag{5.11$'$}$$

with C an arbitrary constant, λ defined by Eq. (5.4) and N defined by Eq. (5.10).

If a homogeneous flow is matched along a common particle path $\psi = \psi_0$ to a generalized flow, the matching can be done in such a way that in addition to pressure and velocity, also the other parameters density, temperature and entropy are continuous across $\psi = \psi_0$.

Consider for example the homentropic centered simple wave (homogeneous flow with $n = 0$) with

$$\begin{aligned} u &= (1-\varkappa)\psi t^{-\varkappa}, \\ a^2 &= \varkappa^2 t^{-2\varkappa}\psi^2, \\ S &= S_h = \text{const}, \end{aligned} \tag{6.3}$$

to be present in the interval $0 \leqslant \psi \leqslant \psi_0$, while for $\psi > \psi_0$ the generalized flow given by Eqs. (6.2) and (5.11) is present. It is clear that the velocities are continuous across the path $\psi = \psi_0$.

Making the entropy continuous requires the constants S_κ, $S(1)$ and N to be selected in such a way that

$$\frac{1}{c_p} S_\kappa = \frac{1}{c_p} S(1) + 2N \ln \psi_0, \tag{6.4}$$

while the continuity of a^2 requires C to satisfy

$$\kappa^2 \psi_0^2 = \lambda^2 \psi_0^2 + C\psi_0^{2N}. \tag{6.5}$$

The continuity of a^2 implies the continuity of the temperature, while the relations

$$(\gamma - 1)\ln \varrho = \ln a^2 - \frac{1}{c_v} S - \ln \gamma, \tag{6.6}$$

$$\frac{\gamma - 1}{\gamma} \ln p = \ln a^2 - \frac{1}{c_p} S - \ln \gamma, \tag{6.7}$$

obtained from the thermodynamic relations indicate that also the continuity of density and pressure are now assured. One may also verify that a homogeneous, non-homentropic flow (with $C = 0$), can be matched to Eq. (6.3) along $\psi = \psi_0$ in such a way that p in Eq. (6.7) is continuous across $\psi = \psi_0$ but it is then impossible to satisfy also the continuity of S, ϱ and $a^2 = \gamma RT$ and the relations (5.13) appear.

Substitution of C from Eq. (6.5) in $f(\psi)$ yields

$$f(\psi) = \lambda^2 \psi_0^2 \left\{ \left(\frac{\psi}{\psi_0}\right)^2 - N \left(\frac{\psi}{\psi_0}\right)^{2N} \right\}. \tag{6.8}$$

In [12] some details of this flow are worked out.

In particular, the characteristics can again be calculated in detail.

7. Some related investigations

The literature on non-homentropic flows is not very extensive and a few papers may be singled out that are close to the work reported here.

NAYLOR [13] studies non-homentropic flows by employing a modified Crocco stream-function ψ defined by

$$\psi_x = a^{\frac{1-\kappa}{\kappa}}, \quad \psi_t = -ua^{\frac{1-\kappa}{\kappa}}.$$

Also he finds

$$x_\psi = a^{-\frac{1-\kappa}{\kappa}}, \quad x_t = u. \tag{7.1}$$

At some stage in the analysis Naylor considers "degenerate" solutions of the equation for $x(\psi, t)$

$$x_\psi^{\gamma+1} x_{tt} - x_{\psi\psi} - x_\psi \frac{S(\psi)}{\gamma R} = 0, \tag{7.2}$$

by applying Poisson's method. This implies that there exists also a relation

$$f(x_t, x_\psi) = 0,$$

or in terms of Eq. (7.1) a relation between the speed of sound a and the velocity u. On this basis he deduces the following theorems:

1. *Unsteady "simple wave" solutions, for which the fluid speed and the acoustic speed are functionally related, of the Lagrangian equation of unsteady rectilinear gas flow with non-constant entropy existing only when the distribution of entropy is of the form* $S(\psi) = C\ln\psi$, *C being some constant.*

2. *In the x, t-plane the curves of constant speed and acoustic speed comprise a system of straight lines concurrent through the origin, that is, the flows obtained are simple centered waves.*

ARDAVAN-RHAD [14] studied the Friedrichs problem mentioned in the Introduction. His analysis begins with the construction of an unsteady, rectilinear, non-homentropic flow, which is such that throughout the flow

$$u = \frac{2}{\gamma-1} ah(\sigma)+\text{const}, \tag{7.3}$$

where $h(\sigma)$ represents an arbitrary function of the entropy σ

$$\sigma = -\frac{1}{\gamma(\gamma-1)} \cdot \frac{S-S_0}{c_v} = -\frac{S-S_0}{\gamma R}.$$

The solution found takes the form

$$\begin{aligned} x &= \{u+ah(\sigma)\}t, \\ t &= a^{-\frac{\gamma+1}{\gamma-1}} f(\sigma), \\ f &= (h^2-1)^{-\frac{\gamma+1}{2(\gamma-1)}} \exp\left[\frac{\gamma+1}{2}\int\frac{d\sigma}{h^2-1}\right]. \end{aligned} \tag{7.4}$$

One may verify that this solution coincides with Eq. (2.9) by putting

$$f(\psi) = f^{2\varkappa}(\sigma), \quad h(\sigma) = \frac{\varkappa\psi}{\sqrt{f(\psi)}}.$$

The expression $(7.4)_3$ is the analogue of Eqs. (4.3) and (4.5).

GUNDERSEN [15] develops a systematic perturbation analysis for non-homentropic flows, assuming that the zero-order flow is homentropic. Applying this analysis to the homentropic centered simple wave, it is found that entropy perturbations in the first approach leave the velocity field unchanged. This clearly agrees with the properties of the solutions constructed here.

8. Concluding remarks

The problem of the non-homentropic flow comes down first to the determination of the entropy distribution. Since there exists little information on this point, it is common to assume some form for the entropy distribution consider and the 'resulting flows. As a consequence several investigations, and the present one is no exception, have the character of "solutions looking for a problem", instead of the more usual situation of "a problem looking for its solution".

It is, however, thought that by confronting some of these "solutions looking for a problem" with sufficiently simple actual flows as for example the F- and G-problem, this situation may in due course be changed.

In the present paper special attention has been given to the matching of a homentropic and a non-homentropic centered wave along a common particle path. It is found that continuity of the parameters across the common path is assured provided one of the two flows is generalized. Either the (non-homentropic) homogeneous flow is generalized as indicated in Sect. 6, or what is equally well possible, the homentropic centered simple wave is generalized to a Von Mises flow (Sect. 3) and matched to a homogeneous flow (Sect. 5). In terms of the earlier remarks the situations considered may provide the problems the solutions were looking for.

Considering that in the present class of flows the velocity distribution and the particle paths remain invariant, while $S(\psi)$ and $f(\psi)$ may be modified in accordance with Eq. (2.8), there is clearly some link with the more general mathematical investigations on "Similarity and Group Analysis" associated with the names of Ovsjannikov and Ibragimov in the USSR and Bluman, Cole and Ames in Canada and the USA.

Finally, it may be mentioned that the class of solutions (2.9) is not yet sufficiently flexible to provide a complete solution for the triangular domains ABC in either the F- or G-problem in its simplest configuration. Along BC, the boundary with the contact region, one needs constant velocity and constant pressure, while BC should also be a characteristic. These conditions cannot be satisfied within the class of solutions (2.9). Some extension seems necessary.

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