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An exact solution of the problem of unsteady fully-developed viscous flow in slightly curved porous tube

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THE FULLY-DOVELOPED flow in a pipe of circular cross-section which is coiled in a circle is studied, the presure gradient along the pipe varying sinusoidally in time with the frequency ω. The nature of curved-tube viscous fluid motion, as compared with simple straight-tube parabolic flow, causes a higher critical Reynolds number for transition to turbulent flow, relatively high average heat-transfer and masstransfer rates per unit axial pressure drop and significant peripheral distributions of the transport rates. In order to simplify the problem, the radius of curvature of the pipe is assumed large in relation to its own radius. Of special interest is the secondary flow generated by centrifugal effects in the plane of the cross-section of the pipe. A closed form analytic solution is derived for arbitrary values of the frequency parameter α. The secondary flow is found to consist of a steady component and a component oscillatory at the frequency 2ω.

Zbadano w pełni rozwinięty przepływ przez rurę o przekroju kołowym zwiniętą spiralnie, przy czym gradient ciśnienia wzdłuż rury zmienia się sinusoidalnie w czasie z częstością ω. Właściwości ruchu płynu lepkiego w zakrzywionej rurze w porównaniu z prostym przepływem parabolicznym w rurze prostej prowadzą do wyższych liczb Reynoldsa przejścia do przekroju burzliwego, do stosunkowo wysokich wartości średnich prędkości przenoszenia ciepła i masy na jednostkę różnicy ciśnienia osiowego oraz do istotnych różnic w obwodowych rozkładach prędkości transportu. Dla uproszczenia zagadnienia przyjęto, że promień zakrzywienia rury jest dużo większy od promienia jej przekroju. Szczególnie interesujący jest przepływ wtórny wywołany efektami sił odśrodkowych w płaszczyźnie przekroju rury. Otrzymano zamknięte rozwiązanie analityczne przy dowolnej wartości parametru częstości α. Stwierdzono, że przepływ wtórny zawiera składnik stały i składnik oscylacyjny o częstości 2ω.

Исследовано вполне развернутое течение через трубу, с круговым сечением, свернутую в спираль, причем градиент давления вдоль трубы изменяется синусоидальным образом во времени с частотой ω. Свойства движения вязкой жидкости в искривленной трубе, по сравнению с простым параболическим течением в простой трубе, приводят к высшим числам Рейнольдса перехода в турбулентное течение, к сравнительно высоким значениям средних скоростей переноса тепла и массы на единицу разницы осевого давления и к существенным разницам в периметрических распределениях скорости переноса. Для упрощения задачи принимается, что радиус искривления трубы много больше радиуса ее сечения. Особенно интересным является вторичное течение, вызванное эффектами центробежных сил в плоскости сечения трубы. Получено замкнутое аналитическое решение при произвольном значении параметра частоты α. Констатировано, что вторичное течение содержит постоянную составляющую и составляющую осциллирующую с частотой 2ω.

1. Introduction

CURVED tubes or pipe bands are used extensively in industrial equipment such as helical coils or spiral heat exchanges, trombone, coolers chemical, reactors and various heat engines.

Toroidal flow is the limiting case of helical flow with a zero pitch. The mode of fluid flow in a curved tube is characterized by a secondary flow field which is superimposed upon the axial-velocity flow field. The nature of toroidal viscous fluid motion, as compared with simple straight-tube parabolic flow, causes relatively high average heat-transfer and mass-transfer rates per unit axial pressure drop and significant peripheral distribution of the transport rates.

This is why the flow of a fluid in a curved tube of circular cross-section which is coiled in a circle has been widely studied both experimentally [1, 2] and theoretically [3—10]. DEAN [3] found, to the first approximation, that relation between the pressure gradient and the rate of steady flow is not dependent on curvature. In order to show its dependence he modified the analysis [4] by including terms of higher order and was able to show that the reduction in flow due to curvature depends on a single variable $K = \frac{2a}{L}\mathrm{Re}^2$, where a is the radius of the tube, L is the radius of curvature of the pipe axis and Re is the Reinolds' number.

However, Dean's analysis of the secondary flow and the consequent increase in friction factors were restricted to small values of K. BARUA [5] considered fully-developed motion for large K and obtained an approximation solution using the Kármán-Pohlhausen momentum integral method. MCCONALOQUE and SRIVASTAVA [6] extended Dean's work and adopted the parameter $D = 4\mathrm{Re}\sqrt{\frac{2\alpha}{L}}$. Physically this parameter can be considered as the ratio of the centrifugal force induced by the circular motion of the fluid to the viscous force. Their numerical solutions were given over the range $D = 96$ to 605.72, the value $D = 96$ corresponding to the upper limit $K = 576$ of Dean's work.

TRUSDEL and ADLER [7] have obtained results up to $D = 3578$ and GREESPAN [8] has centered his interest on the following range of

$$0 \leqslant D \leqslant 576.$$

In contrast to the case of steady flow, the problem of unsteady flow in a toroidal tube has been, for the most part, ignored.

Recently, LYNE [9] has considered the unsteady flow in a toroidal pipe with circular cross-section, the pressure gradient along the pipe varying sinusoidally in time with the frequency ω. In order to simplify the problem, the radius of curvature of the pipe was assumed large in relation to its own radius.

An asymptotic theory was developed for small values of the frequency parameter $\beta = \sqrt{\frac{2\nu}{\omega a^2}}$, where ν is the kinematic viscosity of the fluid. For sufficiently small values of β it was found that the secondary flow in the interior of the pipe is in the opposite sense to that predicted for a steady pressure gradient.

ZALOSH and NELSON [10] have also treated laminar fully-developed flow in a curved tube of circular cross-section under the influence of a pressure gradient oscillating sinusoidally in time. A solution involving numerical evaluation of finite Hankel transforms was obtained for arbitrary values of the frequency parameter $\alpha = a\sqrt{\frac{\omega}{\nu}}$.

The numerical solution of the ordinary differential equation produced consistent results at low and moderate frequency of oscillation, but difficulties were encountered at high frequencies. These difficulties were due to the lack of the Hankel transforms.

In the present paper we shall study the unsteady motion of a viscous fluid in a porous curved tube with circular cross-section. The assumptions we shall make are a large radius of curvature of the tube in relation to its own radius and fully-developed laminar flow under the influence of a pressure gradient oscillating sinusoidally in time. As opposed to Zalosh and Nelson's work, we shall apply another method and obtain an exact solution of the problem. It is worth noting that Zalosh and Nelson's problem is a particular case of our own problem.

2. Formulation of the problem

Consider an unsteady hydrodynamically, fully-developed laminar flow of a viscous fluid in a porous pipe of circular cross-section coiled in the form of a circle. Figure 1 shows the system of toroidal coordinates (r', φ, θ) for considering the motion of the fluid through

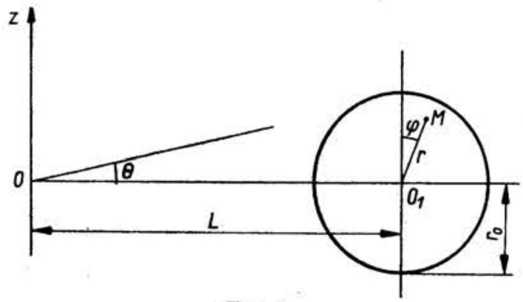

FIG. 1.

the pipe. The distance down the pipe is measured by $L\theta$, where θ is the angle which the cross-section makes with some fixed axial plane containing OZ. Let (u', v', w') denote the corresponding velocity components in the (r', φ, θ) directions at time t.

The equation of continuity is

$$\frac{\partial}{\partial r'}(Lr'u')+\frac{\partial v'}{\partial \varphi}+\frac{u'\sin\varphi}{L+r'\sin\varphi}-\frac{v'\cos\varphi}{L+r'\sin\varphi}=0. \tag{2.1}$$

For a slightly curved tube $(L \gg a)$ we shall have the approximation

$$\frac{\partial}{\partial r'}(L'r'u')+\frac{\partial}{\partial \varphi}(Lv')=0. \tag{2.2}$$

The equation can be identically satisfied through the introduction of a stream function ψ' such that

$$u'=\frac{1}{Lr'}\frac{\partial \psi'}{\partial \varphi}, \qquad v'=-\frac{1}{L}\frac{\partial \psi'}{\partial r'}. \tag{2.3}$$

The unsteady Navier-Stokes equations in this coordinate for ψ' and the axial velocity function $\Omega = W'L$ have the form

$$r'(L+r'\sin\varphi)^2 \frac{\partial}{\partial t}(D^2\psi')+2\Omega\left(r'\cos\varphi\frac{\partial\Omega}{\partial r'}-\sin\varphi\frac{\partial\Omega}{\partial\varphi}\right)-\left[\frac{\partial\psi'}{\partial r'}\frac{\partial}{\partial\varphi}(D^2\psi') - \frac{\partial\psi'}{\partial\varphi}\frac{\partial}{\partial r'}(D^2\psi')\right](L+r'\sin\varphi)+2D^2\psi'\left(r'\cos\varphi\frac{\partial\psi'}{\partial r'}-\sin\varphi\frac{\partial\psi'}{\partial\varphi}\right) = \nu r'(L+r'\sin\varphi)^2 D^4\psi', \tag{2.4}$$

$$\frac{\partial\Omega}{\partial t'}+\frac{1}{(L+r'\sin\varphi)r'}\left[\frac{\partial\psi'}{\partial\varphi}\frac{\partial\Omega}{\partial r'}-\frac{\partial\psi'}{\partial r'}\frac{\partial\Omega}{\partial\varphi}\right] = -\frac{1}{\varrho}\frac{\partial P}{\partial\theta}+\nu D^2\Omega, \tag{2.5}$$

where P denotes pressure, ϱ the density, ν the kinematic viscosity and

$$D^2 = \frac{L+r'\sin\varphi}{r'}\left[\frac{\partial}{\partial r'}\left(\frac{r'}{L+r'\sin\varphi}\frac{\partial}{\partial r'}\right)+\frac{\partial}{\partial\varphi}\left(\frac{1}{r'(L+r'\sin\varphi)}\frac{\partial}{\partial\varphi}\right)\right].$$

When $L \gg a$, we have the approximation

$$L+r'\sin\varphi \approx L \quad \text{and} \quad D^2 \approx \frac{\partial^2}{\partial r'^2}+\frac{1}{r'}\frac{\partial}{\partial r'}+\frac{1}{r'^2}\frac{\partial}{\partial\varphi^2} = \nabla^2.$$

We now impose a single sinusoidal pressure gradient along the tube

$$-\frac{1}{\varrho L}\frac{\partial P}{\partial\theta} = G\cos\omega t', \tag{2.6}$$

where the amplitude G is a prescribed constant.

It is convenient to introduce the following nondimensional notation:

$$r = \frac{r'}{a}, \quad t = \omega t', \quad \psi = \frac{\psi'}{a\nu}, \quad w = \frac{a\Omega}{L\nu} = \frac{aw'}{\nu}.$$

Equations (2.4) and (2.5) now become

$$\alpha^2 r\frac{\partial}{\partial t}(\nabla^2\psi)+2w\left(r\cos\varphi\frac{\partial w}{\partial r}-\sin\varphi\frac{\partial w}{\partial\varphi}\right)-\frac{a}{L}\left[\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\varphi}(\nabla^2\psi) - \frac{\partial\psi}{\partial\varphi}\frac{\partial}{\partial r}(\nabla^2\psi)+2\left(\frac{a}{L}\right)^2\left(r\cos\varphi\frac{\partial\psi}{\partial r}-\sin\varphi\frac{\partial\psi}{\partial\varphi}\right)\nabla^2\psi = r\nabla^4\psi \tag{2.7}$$

and

$$\alpha^2\frac{\partial w}{\partial t}+\frac{a}{L}\frac{1}{r}\left[\frac{\partial\psi}{\partial\varphi}\frac{\partial w}{\partial r}-\frac{\partial\psi}{\partial r}\frac{\partial w}{\partial\varphi}\right]-\nabla^2 w = \frac{Ga^3}{\nu^2}\cos t, \tag{2.8}$$

where $\alpha^2 = \dfrac{\omega a^2}{\nu}$ is the frequency parameter.

Since the ratio $a/L = \delta$ has been assumed to be small, we can linearize the governing equations (2.7) and (2.8). Substituting ψ and w by

$$\psi = \psi_0+\delta\psi_1+\delta^2\psi_2+\ldots, \tag{2.9}$$

$$w = w_0+\delta w_1+\delta^2 w_2+\ldots \tag{2.10}$$

into Eqs. (2.7) and (2.8), we obtain the following equations:

$$\alpha^2 \frac{\partial w_0}{\partial t} - \nabla^2 w_0 = \frac{Ga^3}{\nu^2} \cos t, \tag{2.11}$$

$$\alpha^2 \frac{\partial}{\partial t}(\nabla^2 \psi_0) - \nabla^4 \psi_0 = -\frac{2w_0}{r}\left[r\cos\varphi \frac{\partial w_0}{\partial r} - \sin\varphi \frac{\partial w_0}{\partial \varphi}\right], \tag{2.12}$$

$$\alpha^2 \frac{\partial w_1}{\partial t} - \nabla^2 w_1 = -\frac{1}{r}\left[\frac{\partial \psi_0}{\partial \varphi}\frac{\partial w_0}{\partial r} - \frac{\partial \psi_0}{\partial r}\frac{\partial w_0}{\partial \varphi}\right]. \tag{2.13}$$

Let us assume that the wall porosity of the toroidal tube is described by the following boundary conditions [11, 12]:

(i) steady boundary conditions

$$-w_0 = 0, \qquad w_1 = 0,$$
$$\frac{\partial \psi_0}{\partial r} = 0, \quad \frac{\partial \psi_0}{\partial \varphi} = -\left[\frac{Ga}{\omega\nu}\right]^2 \sin\varphi \quad \text{at} \quad r = 1,$$

(ii) unsteady boundary conditions

$$-w_0 = 0, \qquad w_1 = 0,$$
$$\frac{\partial \psi_0}{\partial r} = 0, \quad \frac{\partial \psi_0}{\partial \varphi} = -\left[\frac{Ga}{\omega\nu}\right]^2 \sin\varphi\cos 2t \quad \text{at} \quad r = 1.$$

We shall also require that the solution be nonsingular within the tube.

3. Solution of the problem

It is well known [13] that the solution of Eq. (2.11) with the boundary condition $w_0 = 0$ at $r = 1$ is

$$w_0(r, t) = \left[\frac{Ga}{\omega\nu}\right][B\cos t + (1-A)\sin t], \tag{3.1}$$

where

$$A = D\,\mathrm{bei}(\alpha r) + C\,\mathrm{ber}(\alpha r), \qquad C = \frac{\mathrm{ber}(\alpha)}{\mathrm{ber}^2(\alpha) + \mathrm{bei}^2(\alpha)},$$

$$B = D\,\mathrm{ber}(\alpha r) - C\,\mathrm{bei}(\alpha r), \qquad D = \frac{\mathrm{bei}(\alpha)}{\mathrm{ber}^2(\alpha) + \mathrm{bei}^2(\alpha)}.$$

If we substitute Eq. (3.1) into Eq. (2.12), we obtain

$$-\left[\alpha^2 \frac{\partial}{\partial t}(\nabla^2 \psi_0) - \nabla^4 \psi_0\right] = \left[\frac{Ga}{\omega\nu}\right]^2 \cos\varphi\Big\{B\frac{dB}{dr} - (1-A)\frac{dA}{dr} + \cos 2t\left[B\frac{dB}{dr} + (1-A)\frac{dA}{dr}\right] + i\sin 2t\left[B\frac{dA}{dr} - (1-A)\frac{dB}{dr}\right]\Big\}. \tag{3.2}$$

Since we expect ψ_0 to contain terms independent of time we write

$$\psi_0 = \psi_{00} + \psi_{02} e^{2it}. \tag{3.3}$$

The governing equations and the boundary conditions for ψ_{00} and ψ_{0L} are

$$\text{(3.4)} \qquad \nabla^4\psi_{00} = \left[\frac{Ga}{\omega\nu}\right]^2 \left[B\frac{dB}{dr} - (1-A)\frac{dA}{dr}\right]\cos\varphi,$$

(i) $$\left.\frac{\partial\psi_{00}}{\partial r}\right|_{r=1} = 0, \quad \left.\frac{\partial\psi_{00}}{\partial\varphi}\right|_{r=1} = -\left[\frac{Ga}{\omega\nu}\right]^2 \sin\varphi,$$

(ii) $$\left.\frac{\partial\psi_{00}}{\partial r}\right|_{r=1} = 1, \quad \left.\frac{\partial\psi_{00}}{\partial\varphi}\right|_{r=1} = 0,$$

$$\text{(3.5)} \quad \nabla^4\psi_{02} - 2i\alpha^2\nabla^2\psi_{02} = \left[\frac{Ga}{\omega\nu}\right]^2 \left\{B\frac{dB}{dr} + (1-A)\frac{dA}{dr} + i\left[B\frac{dA}{dr} - (1-A)\frac{dB}{dr}\right]\right\}\cos\varphi,$$

(i) $$\left.\frac{\partial\psi_{02}}{\partial r}\right|_{r=1} = 0, \quad \left.\frac{\partial\psi_{02}}{\partial\varphi}\right|_{r=1} = 0,$$

(ii) $$\left.\frac{\partial\psi_{02}}{\partial r}\right|_{r=1} = 0, \quad \left.\frac{\partial\psi_{02}}{\partial\varphi}\right|_{r=1} = -\left[\frac{Ga}{\omega\nu}\right]^2 \sin\varphi\cos 2t.$$

In contrast to Zalosh and Nelson's method of solution of the problem, where some difficulties were encountered at high ferquencies, we shall apply another method.

If we seek the solutions of Eqs. (3.4) and (3.5) of the forms

$$\text{(3.6)} \qquad \psi_{00} = \left[\frac{Ga}{\omega\nu}\right]^2 F_0(r)\cos\varphi,$$

$$\text{(3.7)} \qquad \psi_{02} = \left[\frac{Ga}{\omega\nu}\right]^2 F_2(r)\cos\varphi,$$

we obtain

$$\text{(3.8)} \qquad Q^2F_0 = g'(r),$$

$$\text{(3.9)} \qquad Q^2F_2 - 2i\alpha^2 QF_2 = P'(r),$$

where

$$Q \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2},$$

$$g(r) = \frac{B^2 + (1-A)^2}{2}, \quad P(r) = \frac{B^2 - (1-A)^2}{2} + iB(A-1).$$

The function F_0 and F_2 should satisfy the following boundary condition:

(i) $$F_0(1) = 1, \quad F_0'(1) = 0, \qquad F_2(1) = 0, \quad F_2'(1) = 0,$$

(ii) $$F_0(1) = 0, \quad F_0'(1) = 0, \qquad F_2(1) = 1, \quad F_2'(1) = 0.$$

The general solution of the equation

$$\text{(3.10)} \qquad Q^2F_0 = 0$$

is

$$F_0 = C_1 r + C_2 r\ln r + C_3 \frac{1}{r} + C_4 r^3,$$

where C_i $(i = 1, 2, 3, 4)$ are constants.

Now it is easy to find the general solution of Eq. (3.8)

$$F_0 = \phi_0 + D_1 r + D_2 r\ln r + D_3 \frac{1}{r} + D_4 r^3,$$

where D_i $(i = 1, 2, 3, 4)$ are constants and

$$\phi_0 = \left(\frac{3}{2}\ln r - \frac{1}{4}\right) r \int_0^r r^5 g dr - \frac{3}{2} r \int_0^r r^5 \ln r g dr + \frac{1}{2r} \int_0^r r^7 g dr - \frac{r^3}{4} \int_0^r r^3 g dr.$$

Since we require our solution be nonsingular within the tube, i.e

$$\phi_0(0) + D_1 + D_2(1 + ln r)|_{r=0} - \left.\frac{D_3}{r^2}\right|_{r=0} < \infty,$$

we have $D_2 = D_3 = 0$.

In view of boundary conditions we find

$$(3.11)\quad \begin{aligned}\phi_0 = {} & P_1(y)\cos y\,\mathrm{ch}\,y + P_2(y)\sin y\,\mathrm{sh}\,y + P_3(y)\,(\cos y\,\mathrm{sh}\,y + \sin y\,\mathrm{ch}\,y) \\ & + P_4(y)\,(\cos y\,\mathrm{sh}\,y - \sin y\,\mathrm{ch}\,y) + P_5(z)\,(\cos z\,\mathrm{sh}\,z + \sin z\,\mathrm{ch}\,z) + P_6(z)\sin z\,\mathrm{sh}\,z \\ & + P_7(z)\,(\cos z\,\mathrm{sh}\,z - \sin z\,\mathrm{ch}\,z) + P_8(z)\cos z\,\mathrm{ch}\,z - 9Cy\ln y \\ & + \frac{3C}{2} y - \frac{C^2 + D^2}{384} y^5 + \frac{75}{4} C + \frac{30D}{y} - g\big(CL_1(0) + DL_2(0)\big),\end{aligned}$$

where $z = 2y$ and

$$(3.12)\quad \begin{aligned} P_1(y) &= \frac{5D}{2} y^3 + 30Cy - \frac{30D}{y}, \\ P_2(y) &= \frac{75}{4} y^5 D + \frac{29}{4} Cy^3 - 30Dy + \frac{30C}{y}, \\ P_3(y) &= \frac{C}{2} y^4 + \frac{73}{4} Dy^2 - 30C, \\ P_4(y) &= -\left(\frac{15}{4} Dy^4 + \frac{7}{4} Cy^2 - 30D\right), \\ P_5(z) &= -\frac{C^2 + D^2}{5} 15z^4, \\ P_6(z) &= 15\frac{C^2 + D^2}{256}\frac{1}{z}, \\ P_7(z) &= \frac{C^2 + D^2}{256} z^2, \\ P_8(z) &= 3\frac{C^2 + D^2}{64} z. \end{aligned}$$

Here $L_1(0)$ and $L_2(0)$ can be found from the recursion formulae:

$$L_1(0) = \ln\cos y \operatorname{ch} y - \frac{1}{2y}(\cos y \operatorname{sh} y + \sin y \operatorname{ch} y) - \frac{1}{2} J_2(-2),$$

$$L_2(0) = \ln\sin y \operatorname{sh} y - \frac{1}{2y}(\cos y \operatorname{sh} y - \sin y \operatorname{ch} y) - \frac{1}{2} J_1(-2),$$

$$J_1(m) = y^m \cos y \operatorname{ch} y - \frac{m}{2} y^{m-1}(\cos y \operatorname{sh} y + \sin y \operatorname{ch} y) + \frac{m(m-1)}{2} J_2(m-2),$$

$$J_2(m) = y^m \sin y \operatorname{sh} y + \frac{m}{2} y^{m-1}(\cos y \operatorname{sh} y - \sin y \operatorname{ch} y) + \frac{m(m-1)}{2} J_1(m-2),$$

where

$$L_1(m) = \int_0^y y^m \ln y(\cos y \operatorname{sh} y - \sin y \operatorname{ch} y)\, dy,$$

$$L_2(m) = \int_0^y y^m \ln y(\cos y \operatorname{sh} y + \sin y \operatorname{sh} y) dy,$$

$$J_1(m) = \int_0^y y^m(\cos y \operatorname{sh} y - \sin y \operatorname{ch} y) dy,$$

$$J_2(m) = \int_0^y y^m(\cos y \operatorname{sh} y + \sin y \operatorname{ch} y) dy.$$

We will mark that as $y \to 0$

$$\frac{30D}{y} \cos y \operatorname{ch} y - \frac{30D}{y} = \frac{30D}{y}(\cos y \operatorname{ch} y - 1) \xrightarrow[y \to 0]{} 0.$$

Now we have to solve the equation

$$Q(QF_2 - 2i\alpha^2 F_2) = P' \tag{3.13}$$

or

$$r^4 Q\phi = r^4 P', \tag{3.14}$$

where

$$QF_2 - 2i\alpha^2 F_2 = \phi. \tag{3.15}$$

The general solution of Eq. (3.14) is

$$\phi_2 = -\left(2r \int_0^r r^3 P dr + \frac{3}{r} \int_0^r r^5 P dr\right) + D_1 r + \frac{D_2}{r},$$

where D_1 and D_2 are constants.

Let $y = pr$,

$$p = \frac{\alpha}{\sqrt{2}} \quad \text{and} \quad \phi = 2y \int_0^y y^3 p dy + \frac{3}{y} \int_0^y y^5 P dy,$$

then

$$p^5\phi_2 = -\phi + D_1 y + \frac{D_2}{y},$$

where

$$(3.16)\quad \phi(y) = \frac{i(D+iC)^2}{32}\Big\{z^2(\cos z\,\mathrm{sh}\,z+\sin z\,\mathrm{ch}\,z)-2z\sin z\,\mathrm{sh}\,z - \frac{3}{2}(\cos z\,\mathrm{sh}\,z-\sin z\,\mathrm{ch}\,z)+\frac{3}{2}\frac{\cos z\,\mathrm{ch}\,z}{z}\Big\} - \frac{iCD}{32}\Big\{-z^2(\cos z\,\mathrm{sh}\,z-\sin z\,\mathrm{ch}\,z) + 2z\cos z\,\mathrm{ch}\,z - \frac{3}{2}(\cos z\,\mathrm{sh}\,z+\sin z\,\mathrm{ch}\,z)+\frac{3}{2}\frac{\sin z\,\mathrm{sh}\,z}{z}\Big\} + \frac{D+iC}{2} \times\Big\{-i\Big(2y^3-7iy+\frac{6}{y}\Big)\cos(i-1)y+(3y^2+6i)J_0[(i-1)y]\Big\}.$$

Therefore,

$$QF_2 - 2i\alpha^2 F_2 = p^5\phi_2(y)$$

or

$$(3.17)\qquad r^2\frac{d^2F_2}{dr^2}+r\frac{dF_2}{dr}-F_2-2i\alpha^2r^2F_2 = r^2\phi_2.$$

Substitution of $\tau = \alpha r\sqrt{2i} = \alpha r(1+i)$ into Eq. (3.17) gives

$$(3.18)\qquad \tau^2\frac{d^2F_2}{d\tau^2}+\tau\frac{dF_2}{d\tau}-(1+\tau^2)F_2 = -\frac{i\tau^2}{2\alpha^2}\phi(\tau)+D_1\tau+\frac{D_2}{\tau} = \chi(\tau).$$

The general solution of the homogeneous equation is

$$g_0 = C_1 I_1(\tau)+C_2K_1(\tau),$$

where C_1 and C_2 are constants and I_1 and K_1 are the first and second solutions of the Bessel function of first order, respectively. Let us seek a solution of Eq. (3.18) in the form

$$F_2(\tau) = C_1(\tau)I_1(\tau)+C_2(\tau)K_1(\tau).$$

Making use of algebra we find

$$(3.19)\quad F_2(\tau) = [D_3+D_1\delta_1(\tau)+D_2\alpha_1(\tau)+\gamma_1(\tau)]I_1(\tau)+[D_4+D_1\delta_2(\tau) + D_2\alpha_2(\tau)+\gamma_2]K_1(\tau),$$

where

$$\gamma_1(\tau) = -\frac{i}{2\alpha^2}\int_0^\tau \tau\phi K_{11}\,d\tau,\qquad \gamma_2(\tau) = \frac{i}{2\alpha^2}\int_0^\tau \tau^2\phi I_{11}\,d\tau,$$

$$\delta_1(\tau) = \int_0^\tau \tau K_{11}\,d\tau,\qquad \delta_2(\tau) = \int_0^\tau \tau I_{11}\,d\tau,$$

$$\alpha_1(\tau) = \int_0^\tau \frac{K_{11}}{\tau}\,d\tau,\qquad \alpha_2(\tau) = \int_0^\tau \frac{I_{11}}{\tau}\,d\tau$$

and

$$I_{11} = \frac{I_1(\tau)}{I_1'(\tau)K_1(\tau)-I_1(\tau)K_1'(\tau)}, \quad K_{11} = \frac{K_1(\sigma\tau)}{I_1'(\tau)K_1(\tau)-I_1(\tau)-I_1(\tau)K_1'(\tau)}.$$

Since for $r = 0$ the function $K_1(\tau)$ becomes infinite, and the velocity on the axis of the tube must be finite, the constants D_1 and D_2 are necessarily equal to zero. From Eq. (3.19) it follows that

$$F_2(\tau) = [D_3+\gamma_1(\tau)]I_1(\tau)+[D_4+\gamma_2(\tau)]K_1(\tau). \tag{3.20}$$

Unknown constants in this formula are determined by the boundary conditions (i) and (ii)

$$D_3^S = \frac{1}{I_1} - \frac{\gamma_1' I_1 K_1 - \gamma_2' K_1^2}{K_1 I_1' - K_1' I_1} - \gamma_1,$$

$$D_4^S = \frac{\gamma_1' I_1^2 - \gamma_2' I_1 K_1}{K_1 I_1' - K_1' I_1} - \gamma_2,$$

$$D_3^N = \frac{1}{I_1} - \frac{I_1' K_1}{I_1} - \frac{\gamma_1' I_1 K_1 - \gamma_2' K_1^2}{K_1 I_1' - K_1' I_1} - \gamma_1,$$

$$D_4^N = I_1' + \frac{\gamma_1' I_1^2 - \gamma_2' I_1 K_1}{K_1 I_1' - K_1' I_1} - \gamma_2,$$

where the functions $K_1(\tau)$, $I_1(\tau)$, $K_1'(\tau)$, $I_1'(\tau)$, $\gamma_1(\tau)$, and $\gamma_2(\tau)$ are calculated at the wall: $\tau|_{r=1} = (1+i)\alpha$. Then the final results are

$$\begin{aligned} F_2^S(\tau) &= [D_3^S+\gamma_1(\tau)]I_1(\tau)+[D_4^S+\gamma_2(\tau)]K_1(\tau), \\ F_2^N(\tau) &= [D_3^N+\gamma_1(\tau)]I_1(\tau)+[D_4^N+\gamma_2(\tau)]K_1(\tau). \end{aligned} \tag{3.21}$$

Therefore the expressions for ψ_0^s and ψ_0^N are

$$\begin{aligned} \psi_0^S &= \left[\frac{Ga}{\omega\nu}\right]^2 [F_0^N + \mathrm{Real}(F_2^S e^{2it})]\cos\varphi, \\ \psi_0^N &= \left[\frac{Ga}{\omega\nu}\right]^2 [F_0^N + \mathrm{Real}(F_2^N e^{2it})]\cos\varphi. \end{aligned} \tag{3.22}$$

It is possible now to write the stream function ψ_0 of Zalosh and Nelson's problem (unsteady fully-developed flow of a viscous fluid in a slightly curved tube with a wall which is not porous).

$$\psi_0 = \left[\frac{Ga}{\omega\nu}\right]^2 [F_0^N + \mathrm{Real}(F_2^S e^{2it})]\cos\varphi.$$

Furthermore, Eqs. (3.1) and (3.22) can be used to evaluate the right-hand side of Eq. (2.13):

$$\nabla^2 w_1 - \alpha^2 \frac{\partial w_1}{\partial t} = \left[\frac{Ga}{\omega\nu}\right]^3 \mathrm{Real}\left((2F_0+F_2)\frac{T_0(\alpha r i\sqrt{i})}{T_0(\alpha i\sqrt{i})}(e^{it}+e^{3it})\right\}\sin\varphi. \tag{3.23}$$

The right-hand side of this equation suggests a solution of the form

$$w_1 = \frac{1}{2}\left[\frac{Ga}{\omega\nu}\right]^3 \sin\varphi\,\mathrm{Real}\{G_1(r)e^{it}+G_3(r)e^{3it}\}. \tag{3.24}$$

Substituting Eq. (3.24) into Eq. (3.23), two linear ordinary differential equations for G_1 and G_3 are obtained:

$$QG_1-\alpha^2 iG_1 = \left[2F_0\frac{dB}{dr}+F_{2\mathrm{Real}}\frac{dB}{dr}+F_{2\mathrm{Im}}\frac{dA}{dr}\right] + i\left[2F_0\frac{dA}{dr}-F_{2\mathrm{Real}}\frac{dA}{dr}+F_{2\mathrm{Im}}\frac{dB}{dr}\right],$$

$$QG_3-3\alpha^2 iG_3 = \left[F_{2\mathrm{Real}}\frac{dB}{dr}-F_{2\mathrm{Im}}\frac{dA}{dr}\right]+i\left[F_{2\mathrm{Real}}\frac{dA}{dr}+F_{2\mathrm{Im}}\frac{dB}{dr}\right],$$

or

$$\frac{d^2G_1}{dr^2}+\frac{1}{r}\frac{dG_1}{dr}-\left(\alpha^2 i+\frac{1}{r^2}G_1 = (2F_0+F_2)\frac{d}{dr}(B-iA),\right.$$

$$\frac{d^2G_3}{dr^2}+\frac{1}{r}\frac{dG_3}{dr}-\left(3\alpha^2 i+\frac{1}{r^2}\right)G_3 = F_2\frac{d}{dr}(B-iA).$$

The boundary conditions for G_1 and G_3 are $G_1(1)=0$ and $G_3(1)=0$. We shall also require that $G_1(r)$ and $G_3(r)$ be nonsingular within the tube.

It is easy to find that

$$B-iA = \frac{1}{\sin(\alpha i\sqrt{i})}\,\frac{\sin(\alpha ri\sqrt{i})}{r},$$

$$\frac{d}{dr}(B-iA) = \frac{1}{\sin(\alpha i\sqrt{i})}\,\frac{\alpha ri\sqrt{i}\cos(\alpha ri\sqrt{i})-\sin(\alpha ri\sqrt{i})}{r^2}.$$

Let $\tau_1=\alpha r\sqrt{i}$ and $\tau_3=\alpha r\sqrt{3i}$; then we shall have for G_1 and G_3 the equations

$$\tau_1^2\frac{d^2G_1}{d\tau_1^2}+\tau_1\frac{dG_1}{d\tau_1}-(\tau_1^2+1)G_1=\beta_1(r), \tag{3.25}$$

$$\tau_2^2\cdot\frac{d^2G_3}{d\tau_2^2}+\tau_2\frac{dG_3}{d\tau_2}-(\tau_2^2+1)G_3=\beta_2(r) \tag{3.26}$$

and the boundary conditions $G_1(\alpha\sqrt{it})=0$ and $G_3(\alpha\sqrt{3it})=0$. Here

$$\beta_1(r)=(2F_0^\beta+F_2^\beta)\frac{d}{dr}(B-iA)\ \text{and}\ \beta_2(r)=F_2^\beta\frac{d}{dr}(B-iA)$$

have different values for cases of the porous and not porous wall.

After some algebraic transformations one can find

$$G_1^\beta(\tau_1)=D_1^\beta(r)I_1(\tau_1)+D_2^\beta(r)K_1(\tau_1),$$
$$G_4^\beta(\tau_2)=D_3^\beta(r)I_1(\tau_2)+D_4^\beta(r)K_1(\tau_2),$$

where

$$D_1^\beta(r)=\int_0^r\beta_1(r)\frac{K_1'}{I_1'}dr-\int_0^r\beta_1(r)\frac{K_1}{I_1}dr+C_1^\beta,$$

$$D_2^\beta(r)=\int_0^r\beta_1(r)dr-\int_0^r\beta_1(r)\frac{I_1K_1'}{K_1I_1'}dr+C_2^\beta,$$

$$D_3^\beta(r) = \int_0^r \beta_2(r)\frac{K_1'}{I_1}dr + \int_0^r \beta_2(r)\frac{K_1}{I_1}dr + C_3^\beta,$$

$$D_4^\beta(r) = \int_0^r \beta_2(r)dr + \int_0^r \beta_2(r)\frac{I_1K_1'}{K_1I_1'}dr + C_4^\beta.$$

Using boundary conditions for $G_1(\tau_1)$ and $G_3(\tau_2)$, we find $C_2^\beta = C_4^\beta = 0$

$$C_1^\beta = -\frac{K_1(\alpha\sqrt{i})}{I_1(\alpha\sqrt{i})} + \int_0^1 \beta_1(r)\left(\frac{K_1(\alpha r\sqrt{i})}{I_1(\alpha r\sqrt{i})} - \frac{K_1'(\alpha r\sqrt{it})}{I_1'(\alpha r\sqrt{it})}\right)dr,$$

$$C_3^\beta = -\frac{K_1(\alpha\sqrt{3i})}{I_1(\alpha\sqrt{3i})} + \int_0^1 \beta_2(r)\left(\frac{K_1(\alpha r\sqrt{3i})}{I_1(\alpha r\sqrt{3i})} - \frac{K_1'(\alpha r\sqrt{3it})}{I_1'(\alpha r\sqrt{3it})}\right)dr.$$

Results

Velocity profiles across the radii $\varphi = \frac{\pi}{2}$ of the steady part of the secondary flow, lying in the plane of a cross-section, are illustrated in Figs. 2–4, where $\frac{Ga}{\omega\nu} = 0.13$, $\alpha = 1, 5, 10$.

For low frequency and $\frac{Ga}{\omega\nu}$ less than about 0.13 there is no region of negative radial velocity. Thus the centrifugal force gradient drives the fluid towards the outer wall. When $\frac{Ga}{\omega\nu} > 0.13$ and the frequency parameter α is moderate and high, there is a region where the steady part of the flow directed towards the inner wall along a horizontal diameter.

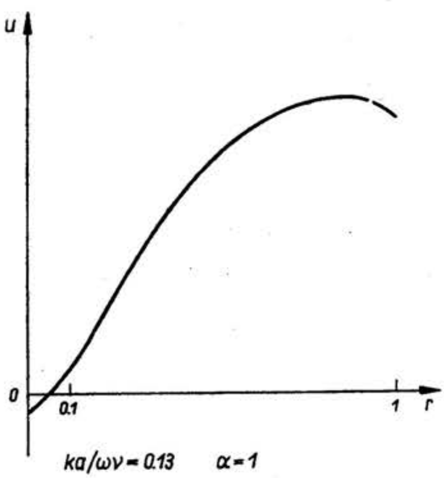

FIG. 2.

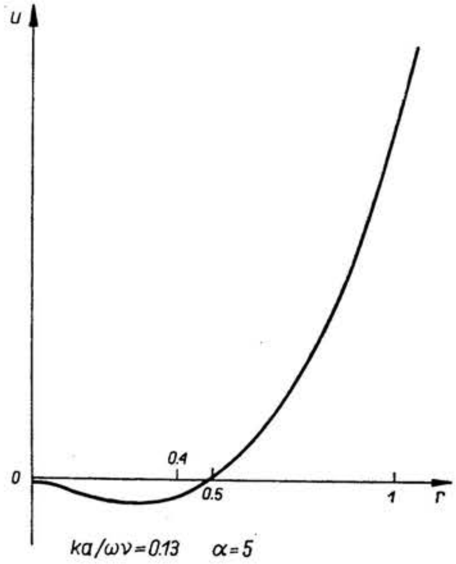

FIG. 3.

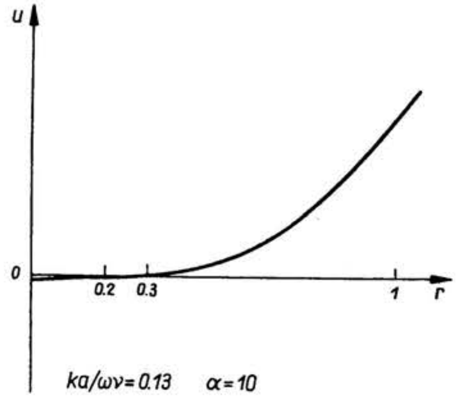

FIG. 4.

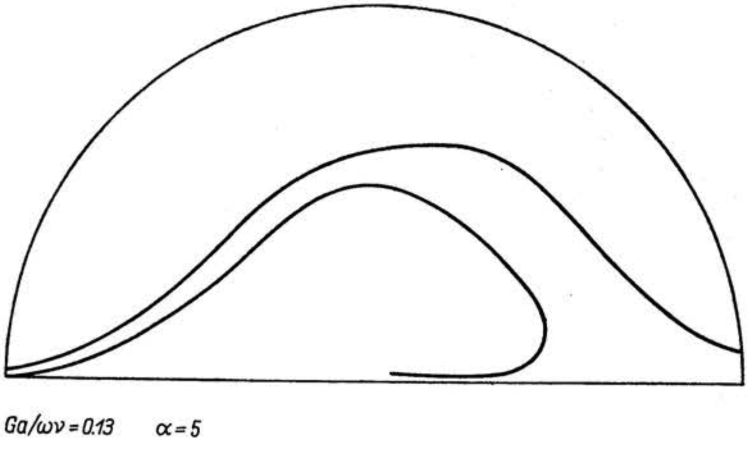

FIG. 5.

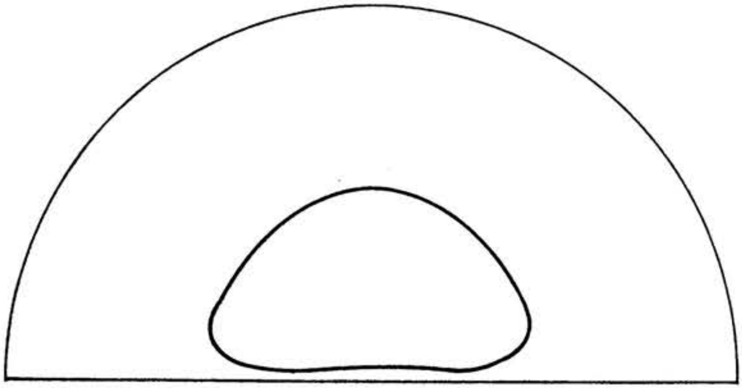

FIG. 6.

The velocity distribution in the secondary flow plane is best displayed by means of the stream function. The results reveal that the secondary flow is composed of a steady component and a component oscillating at the second harmonic of the applied pressure lines of constant stream-function values; they represent the secondary motion which is superimposed on the axial-velocity flow field. Typical contours of the steady part of this function are shown in Figs. 5 and 6 for steady and unsteady blowing, respectively. Here $\frac{Ga}{\omega\nu} = 0.13$, $\alpha = 5$.

References

1. J. GRINDLEY, L. GIBSON, Proc. Roy. Soc., **A.80**, 1908.
2. J. EUSTICE, Proc. Roy. Soc., **A.84**, 1910.
3. W. DEAN, Phil. Mag., Ser. **7**, 4, 1927.
4. W. DEAN, Phil. Mag., Ser. **7**, 5, 1928.
5. S. BARUA, Quort. J. Mech. Appl. Math., **16**, 1963.
6. A. MCCONALOQUE and C. SRIVASTAVA, Proc. Roy. Soc., **A 307**, 1968.
7. C. TRUSDEL and A. ADLER, I. Ch.E.Jl., **16**, 1970.
8. D. GREESPAN, J. Fluid Mech., **57**, 1973.
9. W. LYNE, J. Fluid Mech., **45**, 1970.
10. K. ZALOSH and W. NELSON, J. Fluid Mech., **59**, 1973.
11. R. GUPTA, ZAMM, **54**, 1974.
12. A. BERMAN, J. Appl. Physics, **24**, 9, 1953.
13. H. SCHLICHTING; *Boundary layer theory*, 1960.

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