Arch. Mech., 32, 4, pp. 491–504, Warszawa 1980

Heat conduction in porous materials

M. HLAVÁČEK (PRAGUE)

A CONTINUUM theory is developed for heat conduction in porous materials. The temperature field in the neighbourhood of a spherical pore is approximately described by three macroscopic scalar functions for which a variational principle yields a system of three differential equations of the second order. Propagation of plane harmonic thermal waves is studied. The method is applicable to other diffusion problems.

Opracowano kontynualną teorię przewodzenia ciepła w materiałach porowatych. Pole temperatury w otoczeniu pojedynczej pustki kulistej opisano w sposób przybliżony za pomocą trzech makroskopowych funkcji skalarnych, dla których, po zastosowaniu zasady wariacyjnej, otrzymuje się układ trzech równań różniczkowych drugiego rzędu. Przeanalizowano problem propagacji płaskich fal termicznych. Metodę tę zastosować można również do innych problemów dyfuzyjnych.

Разработана континуальная теория теплопроводности в пористых материалах. Поле температур в окрестности единичной сферической пустоты описано приближенным образом при помощи трех макроскопических скалярных функций, для которых, после применения вариационного принципа, получается система трех дифференциальных уравнений второго порядка. Проанализирована проблема распространения плоских термических волн. Этот метод можно тоже применять к другим диффузным задачам.

1. Introduction

THERE ARE several continuum models of heat conduction in composite materials. The continuum mixture theory was developed for heat conduction in laminated materials [1–2] and in fiber-reinforced materials [3–4], while a certain variational method for heat conduction in the composites of the inclusion matrix type are presented in [5].

The effective stiffness method was originally used for the continuum model of elastic deformation of laminated materials [6]. In the present work the method used in [6] is applied to heat conduction in a porous material.

The porous material is idealized. Spherical pores of a constant radius are located at random so as to make the material macroscopically homogeneous and isotropic. The pores are void and do not conduct heat which propagates only through the material around the pores. The solution is limited to the macroscopically unidimensional problem. For this case the temperature in the vicinity of the pore is approximated in a certain way. In this approximation the temperature in the vicinity of a pore with a centre in any arbitrary point is determined by the knowledge of three scalar functions. On the basis of a variational principle, differential equations are found for these three scalar functions. Thus the heat conduction in a porous material is described by a system of three differential equantions of the second order.

Further propagation of plane thermal harmonic waves is investigated. If the circular frequency of these waves approaches zero, the effective thermal conductivity of the porous material is obtained. The values thus obtained do not differ practically from the values of effective thermal conductivity obtained by other methods [8]. This agreement of results is in favour of the present model.

Further, a multivalent dependence of the complex wave number on the circular frequency of the wave is found. The real part of the wave number describes the attenuation of the wave, while the imaginary part of the wave number is connected with the phase velocity and the wave length.

2. Model of a porous material

Let us consider a material with a large number of small spherical pores of indentical size distributed in such a way that the material is macroscopically homogeneous and isotropic. We assume that the pores are void and that heat propagates through the material around the pores only. We can imagine the material as being divided into elements each of which consists of one pore and of a certain neighbourhood of thermally conductive material. Let us select these elements so that they have an equal volume and are not substantially different from a sphere. For the sake of simplicity we shall replace them by spherical elements with a spherical pore of a radius r_1 in the centre, whose external radius r_2 is determined by the relation

$$\eta = \frac{r_1}{r_2},$$

where η^3 means porosity, i.e. the volume of pores in a unit volume of the material. It is obvious that the adjacent spherical elements partly overlap and form gaps. However, for the construction of a simple deterministic model which does not require detailed

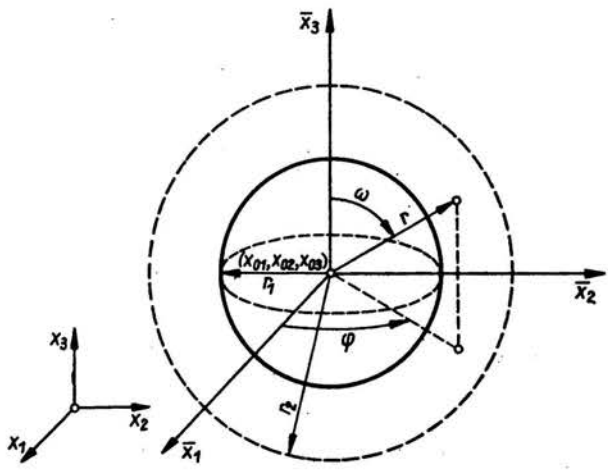

FIG. 1. Coordinate systems.

knowledge of the position of the individual pores, it is necessary to accept some simplifications.

Let x_i denote the global Cartesian coordinates and let us consider a spherical element with a centre in the point x_{0i} (Fig. 1). In the centre of the pore, in the point x_{0i}, let us introduce local rectangular coordinates $\bar{x}_i$ and local spherical coordinates r, φ, ω, while

$$x_i = x_{0i}+\bar{x}_i, \quad i=1,2,3;$$

$$\bar{x}_1 = r\cos\varphi\sin\omega, \quad \bar{x}_2 = r\sin\varphi\sin\omega, \quad \bar{x}_3 = r\cos\omega.$$

Further on we shall confine our considerations to the cases of macroscopically unidimensional heat conduction. This means that the functions describing macroscopic heat conduction will depend on x_3, t only and not on x_1, x_2. Let us consider now two adjacent

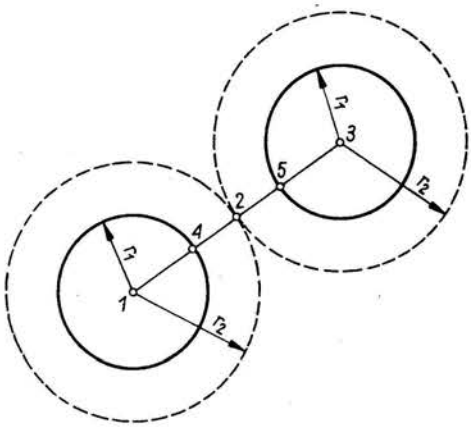

FIG. 2. Two adjoining elements.

spherical elements (Fig. 2), the element with a centre x_{0i} (Point 1 in Fig. 2) touching the element with a centre in Point 3. The point of contact 2 is determined in the local spherical coordinates referred to Point 1, by the coordinates r_2, φ_0, ω_0. Let us assume that the temperature on the line segment $\overline{4\,5}$ (Fig. 2), forming a part of the line connecting the centres 1 and 3 of the adjacent elements, is defined by the relation

$$\vartheta(x_{03}, t; r, \varphi_0, \omega_0) = \vartheta_0|_2+\cos\omega_0[\alpha|_2\sin\xi+\beta|_2(\cos 2\xi-1)], \tag{2.1}$$

where

$$\xi = \frac{\pi}{2}\frac{r-r_2}{r_2-r_1}.$$

$\vartheta_0|_2$, $\alpha|_2$, $\beta|_2$ are so far unknown functions of time t, defined for the time being in Points 2 only, i.e. in the points of contact of adjacent elements. If we put in Eq. (2.1)

$$r = r_2,$$

we obtain

$$\vartheta|_2 = \vartheta_0|_2$$

i.e. $\vartheta_0|_2$ represents the temperature at the boundary of the element in Point 2. The remaining term on the right-hand side of Eq. (2.1) thus represents, for constant φ_0, ω_0, the change

of ϑ in the points of the line segment $\overline{4\,5}$ in comparison with the value of ϑ in Point 2. In Point 4, for

$$r = r_1,$$

it follows from Eq. (2.1) that

$$\left.\frac{\partial\vartheta}{\partial r}\right|_4 = 0$$

and similarly, for Point 5, if we put in Eq. (2.1)

$$r = 2r_2 - r_1,$$

we obtain

$$\left.\frac{\partial\vartheta}{\partial r}\right|_5 = 0.$$

From that it can be seen that the heat flux on the pore surface in Points 4 and 5 equals zero.

Let us note that, for the sake of simplicity, we have assumed that the adjacent spherical elements touch, i.e. that the distance between their centres is $2r_2$. Actually, if the original non-spherical elements differed only little from the sphere, the distance of their centres would be slightly smaller than $2r_2$. Thus, for example, in a plane case of a hexagonal layout of circles — pores of identical radius r_1 with the distance of pore centres $2l$, we can obtain by a simple calculation the radius r_2 of an equivalent circular element

$$r_2 = l\sqrt{\frac{2\sqrt{3}}{\pi}} \approx 1.1l.$$

In this particular plane case the actual distance of the centres of adjacent elements differs from $2r_2$ by about 10%.

The dependence of ϑ on r was selected in Eq. (2.1) in the form of simple goniometric functions satisfaying the condition of zero heat flux on the pore surface. The dependence of ϑ on ω_0 according to the cosine of the angle ω_0 satisfies the condition of the maximum change of ϑ in the element in the direction $\bar{x}_3$ and the minimum (zero) change of ϑ in the directions perpendicular to the axis $\bar{x}_3$.

ϑ in Eq. (2.1) depends on the continuous variables t and r for $r \in \langle r_1, 2r_2 - r_1\rangle$ and, further, on the discrete variable x_{03} (the coordinate of the centres of all elements) and on the discrete variables φ_0, ω_0 which correspond to all points of contact of adjacent elements. The same applies to ϑ_0, α, β which, however, do not depend on r. To obtain a continuum theory, we shall replace, in Eq. (2.1), ϑ, ϑ_0, α, β with smooth functions defined for all x_3, φ and ω. Consequently, every point can be considered as a centre of a certain element (or pore) and every point at the boundary of this element can be considered as the point of contact with the adjacent element. For the case of heat conduction in the direction x_3 we obtain now from Eq. (2.1)

$$\vartheta(x_3, t; r, \omega) = \vartheta_0(x_3 + r_2\cos\omega, t) + \cos\omega[\alpha(x_3 + r_2\cos\omega, t)\sin\xi + \beta(x_3 + r_2\cos\omega, t)\,(\cos 2\xi - 1)]. \tag{2.2}$$

With regard to axial symmetry according to $\bar{x}_3$, ϑ in Eq. (2.2) is independent of φ.

ϑ in the form of Eqs. (2.1) or (2.2) satisfies the condition of zero heat flux on the pore surface and the condition of temperature smoothness in the points of contact of adjacent elements. ϑ in Eqs. (2.1) and (2.2), however, only approximates the actual temperature field in the vicinity of the pores. Equation (2.2) has a relatively simple form and comprises only three unknown independent functions ϑ_0, α, β for which we shall find further on, on the basis of a variational principle, three differential equations.

3. Variational principle for heat conduction

In [10] several variational principles of dynamic coupled thermoelasticity for a nonhomogeneous material were deduced. Section 6 in [10] formulates a variational principle for independent displacement and temperature. For a nonhomogeneous material which conducts heat and does not undergo any deformation (or the effect of temperature and deformation can be separated), this variational principle can be simplified into the form which we shall present in this section.

Let us consider a regular bounded closed region R with a boundary S. Let S_1 and S_2 be parts of S, so that

$$S_1 \cup S_2 = S, \quad S_1 \cap S_2 = 0.$$

Let the thermal properties of the homogeneous material be determined by the specific heat per mass unit c and the thermal conductivity k. If $\vartheta(x_k, t)$ denotes the temperature and ϱ the density of the material, the solution to the boundary — initial value problem of heat conduction is the sufficiently smooth function $\vartheta(x_k, t)$ which satisfies the heat conduction equation

$$k\vartheta_{,ii} - \varrho c\dot{\vartheta} = 0 \quad \text{on} \quad R \times [0, \infty); \tag{3.1}$$

here the comma with an index following denotes the partial derivative with respect to the respective coordinate and the dot surmounting the quantity denotes the derivative with respect to time t; function ϑ satisfies the initial condition

$$\vartheta(x_k, 0) = \tilde{\vartheta}(x_k) \quad \text{on} R \quad \text{for} \quad t = 0 \tag{3.2}$$

as well the boundary condition

$$\vartheta(x_k, t) = \hat{\vartheta}(x_k, t) \quad \text{on} \quad S_1 \times [0, \infty) \tag{3.3}$$

and the boundary condition

$$-k\vartheta_{,i}(x_k, t)n_i = \hat{q}(x_k, t) \quad \text{on} \quad S_2 \times [0, \infty). \tag{3.4}$$

$\tilde{\vartheta}$, $\hat{\vartheta}$ and $\hat{q}$ are the prescribed functions. $\hat{q}$ represents the heat flux through a unit area of the surface S_2, n_i the external unit normal to the surface S_2.

The above mentioned variational principle reads:

Let us define for every $t \geqslant 0$ the functional

$$\mathscr{F}_t(\vartheta) = \int_R \left\{ \frac{1}{2} \varrho c\vartheta * \vartheta + \frac{1}{2} kg * \vartheta_{,i} * \vartheta_{,i} - \varrho c\tilde{\vartheta} * \vartheta \right\} dR + \int_{S_2} g * \hat{q} * \vartheta dS, \tag{3.5}$$

where

$$g(t) = 1 \quad \text{for} \quad t \geqslant 0$$

and $*$ means the convolution

$$f' * f'' = \int_0^t f'(x_k, t-\tau) f''(x_k, \tau) d\tau .$$

The function $\vartheta(x_k, t)$, sufficiently smooth and satisfying the boundary condition (3.3), represents the solution of the boundary-initial value problem of heat conduction if and only if it satisfies, for every $t \geqslant 0$, the condition

$$\delta \mathscr{F}_t(\vartheta) = 0. \tag{3.6}$$

The variation in Eq. (3.6) is considered in the meaning explained in [10]. We shall not carry out the proof of the variational principle, it would be similar to the one in [10].

4. Equation of heat conduction in a porous material

We shall apply the variational principle (3.5)–(3.6) to our model of porous material. In Sect. 2 the porous material has been fictitiously divided into hollow spherical elements. Let us consider a bounded closed region R consisting of a large number of spherical elements. Let S_1 be a surface consisting of those points of the external surfaces of the elements which are the boundary points of the region R. The remaining points of the external surfaces of the elements are thus internal points of the region R. Let S_2 be a set consisting of all points of the internal surfaces of elements. Consequently, S_2 consists of the surface of the pores, i.e. a large number of spherical surfaces of the radius r_1.

If the temperature and the flux in all points of contact of adjacent elements are continuous, the functional $\mathscr{F}_t(t)$ is of the form (3.5) also for our particular case of porous material: As the heat flux on the pore surface equals zero, the surface integral over S_2 in Eq. (3.5) equals zero. The volume integral over R can be written as a sum of the integrals over the individual elements. The actual course of temperature in the elements is unknown; for the purpose of heat conduction in the direction of x_3 it can be replaced by the approximation (2.2) which satisfies both the condition of zero heat flux at the pore boundary and the condition of temperature smoothness in the points of contact of adjacent elements. The fundamental assumption of the method described here will be the replacement of the sum of integrals over the individual elements by the integration in the form

$$\mathscr{F}_t(\vartheta_0, \alpha, \beta) = \frac{1}{\frac{3}{4}\pi r_2^3} \int_R \left\{ \int_0^{2\pi} \int_0^{\pi} \int_{r_1}^{r_2} \left\langle \frac{1}{2} \varrho c \vartheta * \vartheta + \frac{1}{2} k g * \vartheta_{,i} * \vartheta_{,i} \right. \right. \tag{4.1}$$

$$\left. \left. - \varrho c \tilde{\vartheta} * \tilde{\vartheta} \right\rangle r^2 \sin \omega \, dr \, d\omega \, d\varphi \right\} dR .$$

The derivatives in Eq. (4.1) are taken with respect to the local coordinates $\bar{x}_i$. Let us note that the functional (4.1) originated from the functional (3.5) by the replacement of the

integrand with its volume average over the volume of the spherical element. In Eq. (4.1)

$$\tilde{\vartheta}(x_k) = \vartheta(x_k, 0).$$

The functional (4.1) depends on three scalar functions ϑ_0, α, β and defines the homogeneous continuum of a higher order, which approximates the original porous material. The sought equation of heat conduction in a porous material, i.e. the differential equation for ϑ_0, α, β, can be obtained from the condition analogous to Eq. (3.6), viz. from the condition

$$\delta \mathscr{F}_t(\vartheta_0, \alpha, \beta) = 0. \tag{4.2}$$

Using Taylor's expansion, it is possible to write in Eq. (2.2)

$$\vartheta_0(x_3 + r_2\cos\omega, t) = \sum_{n=0}^{\infty} \frac{1}{n!} r_2^n \cos^n\omega \vartheta_0^{(n)}(x_3, t), \tag{4.3}$$

where

$$\vartheta_0^{(n)} = \frac{\partial^n \vartheta_0}{\partial x_3^n} = \vartheta_{0,\underbrace{33\cdots3}_{n}}$$

and similarly for $\alpha(x_3 + r_2\cos\omega, t)$, $\beta(x_3 + r_2\cos\omega, t)$. The substitution of Eqs. (2.2) and (4.3) into Eq. (4.1) yields, after further calculations and integration, the functional $\mathscr{F}_t(\vartheta_0, \alpha, \beta)$ in the form of an integral over the region R, whose integrand is a sum of infinite series. The terms of these series contain the convolutions

$$\vartheta_0^{(n)} * \vartheta_0^{(m)}, \quad \alpha^{(n)} * \alpha^{(m)}, \quad \beta^{(n)} * \beta^{(m)}, \quad \vartheta_0^{(n)} * \alpha^{(m)},$$
$$\vartheta_0^{(n)} * \beta^{(m)}, \quad \alpha^{(n)} * \beta^{(m)}, \quad (n, m = 0, 1, 2, 3, \ldots)$$

and then, analogously, the convolutions of the functions $\tilde{\vartheta}_0^{(n)}$, $\tilde{\alpha}^{(n)}$, $\tilde{\beta}^{(n)} \cdot \alpha^{(n)}$, $\beta^{(n)}$ are defined analogously to $\vartheta_0^{(n)} \cdot \tilde{\vartheta}_0^{(n)}$, $\tilde{\alpha}^{(n)}$, $\tilde{\beta}^{(n)}$ are the values of $\vartheta_0^{(n)}$, $\alpha^{(n)}$, $\beta^{(n)}$ in time $t=0$, n, m acquiring the values of all non-negative integers. For the sake of brevity we do not present here the complicated form of $\mathscr{F}_t(\vartheta_0, \alpha, \beta)$. As we shall not be concerned with the formulation of boundary conditions, we shall consider, in the calculation of $\delta\mathscr{F}_t$, the quantities $\delta\vartheta_0$, $\delta\alpha$ and $\delta\beta$ on S_1 as equal to zero. The condition (4.2) results in the sought system of differential equations for the functions ϑ_0, α, β in the form

$$\begin{aligned} &k(c_1\vartheta_{0,33} - c_2\alpha_{,3} - c_3\beta_{,3}) - \varrho c d_1 \dot{\vartheta}_0 = 0, \\ &k(-c_2\vartheta_{0,3} + c_4\alpha - c_5\alpha_{,33} + c_6\beta - c_7\beta_{,33}) + \varrho c(d_2\dot{\alpha} + d_3\dot{\beta}) = 0, \\ &k(-c_3\vartheta_{0,3} + c_6\alpha - c_7\alpha_{,33} + c_8\beta - c_9\beta_{,33}) + \varrho c(d_3\dot{\alpha} + d_4\dot{\beta}) = 0 \end{aligned} \tag{4.4}$$

where

$$c_1 = 2\eta', \qquad c_2 = \frac{4\eta'}{\pi r_2},$$

$$c_3 = \frac{2\eta'}{r_2}, \qquad c_4 = \frac{1}{4r_2^2}\left\langle \frac{\pi^2}{6\eta'}(1+\eta+\eta^2) + 5 - 3\eta \right\rangle,$$

$$c_5 = \frac{\eta'}{5}, \qquad c_6 = \frac{2}{3r_2^2}\left\langle \frac{\pi}{\eta'} - \frac{8}{3}\eta + \frac{16}{9}\frac{\eta'}{\pi}\right\rangle,$$

$$c_7 = \frac{16}{15}\frac{\eta'}{\pi}, \qquad c_8 = \frac{1}{4r_2^2}\left\langle \frac{2\pi^2}{3\eta'}(1+\eta+\eta^2)+11\eta'\right\rangle,$$

$$c_9 = \frac{3}{5}\eta', \tag{4.5}$$

$$d_1 = 1-\eta^3, \quad d_2 = \frac{1}{6}(1-\eta^3) - \frac{\eta'^2}{\pi^2}(1+\eta),$$

$$d_3 = \frac{8}{3}\frac{\eta'}{\pi}\left\langle 1 - \frac{14}{3}\frac{\eta\eta'}{\pi} - \frac{80}{9}\frac{\eta'^2}{\pi^2}\right\rangle,$$

$$d_4 = \frac{1}{2}\left\langle 1-\eta^3+\frac{\eta'^2}{2\pi^2}[\eta'-16(1+\eta)]\right\rangle,$$

$$\eta = \frac{r_1}{r_2}, \quad \eta' = 1-\eta.$$

In the system of Eq. (4.4) we have the first derivatives of ϑ_0, α, β with respect to time and maximally the second derivatives of ϑ_0, α, β with respect to x_3. The coefficients of derivatives of ϑ_0, α, β with respect to x_3 of orders higher than 2 were found to be equal to zero. Thus we have obtained for ϑ_0, α, β a system of differential equations of the second order.

5. Thermal wave propagation in a porous material

Let us examine now the propagation of plane harmonic thermal waves in a porous material. Let us seek the solution of the system of Eq. (4.4) in the form

$$\begin{aligned} \vartheta_0 &= Te^{-\lambda x_3 + i\omega t}, \\ \alpha &= Ae^{-\lambda x_3 + i\omega t}, \\ \beta &= Be^{-\lambda x_3 + i\omega t}, \end{aligned} \tag{5.1}$$

where T, A, B are constants, ω is the circular frequency and λ the complex wave number

$$\lambda = \lambda_1 + i\lambda_2.$$

$\lambda_1 > 0$ describes the attenuation of the wave. The wave length is

$$L = \frac{2\pi}{\lambda_2}$$

and the phase velocity of the wave

$$c = \frac{\omega}{\lambda_2}.$$

Let us introduce Eq. (5.1) into Eq. (4.4). We obtain a linear homogeneous system of

equations for T, A, B which has a nontrivial solution if and only if the determimant of the system equals zero. This condition has the form

$$\begin{vmatrix} \lambda^2 kc_1 - i\omega\varrho c d_1, & \lambda k c_2, & \lambda k c_3 \\ \lambda k c_2, & k c_4 - \lambda^2 k c_5 + i\omega\varrho c d_2, & k c_6 - \lambda^2 k c_7 + i\omega\varrho c d_3 \\ \lambda k c_3, & k c_6 - \lambda^2 k c_7 + i\omega\varrho c d_3, & k c_8 - \lambda^2 k c_9 + i\omega\varrho c d_4 \end{vmatrix} = 0. \tag{5.2}$$

We shall replace λ, ω by the dimensionless variables $\overline{\lambda}$, $\overline{\omega}$

$$\overline{\lambda} = \lambda r_2, \quad \overline{\omega} = \frac{\omega}{\overline{\varkappa}}, \tag{5.3}$$

where

$$\overline{\varkappa} = \frac{\varkappa}{r_2^2}, \quad \varkappa = \frac{k}{\varrho c}.$$

Further, we shall denote

$$z = i\overline{\omega}. \tag{5.4}$$

By developing the determinant in Eq. (5.2), we obtain, using the dimensionless variables $\overline{\lambda}$, z, the condition (5.2) in the form

$$\overline{\lambda}^6 a_6' + \overline{\lambda}^4 (a_4' + z b_4') + \overline{\lambda}^2 (a_2' + z b_2' + z^2 c_2') + (z b_0' + z^2 c_0' + z^3 d_0') = 0, \tag{5.5}$$

where

$$\begin{aligned} a_6' &= \bar{c}_1 g_9, \\ a_4' &= -(\bar{c}_1 g_1 + \bar{c}_5 g_7 + \bar{c}_7 g_6 + \bar{c}_9 g_8), \\ b_4' &= -(\bar{c}_1 g_2 + d_1 g_9), \\ a_2' &= \bar{c}_1 g_3 + \bar{c}_4 g_7 + \bar{c}_6 g_6 + \bar{c}_8 g_8, \\ b_2' &= \bar{c}_1 g_5 + d_1 g_1 + d_2 g_7 + d_3 g_6 + d_4 g_8, \\ c_2' &= -\bar{c}_1 g_4 + d_1 g_2, \\ b_0' &= -d_1 g_3, \quad c_0 = -d_1 g_5, \quad d_0 = d_1 g_4. \end{aligned} \tag{5.6}$$

In Eq. (5.6) we have introduced

$$\begin{aligned} g_1 &= \bar{c}_5 \bar{c}_8 + \bar{c}_4 \bar{c}_9 - 2\bar{c}_6 \bar{c}_7, \\ g_2 &= \bar{c}_5 d_4 + \bar{c}_9 d_2 - 2\bar{c}_7 d_3, \\ g_3 &= \bar{c}_4 \bar{c}_8 - \bar{c}_6^2, \quad g_4 = d_3^2 - d_2 d_4, \\ g_5 &= \bar{c}_8 d_2 + \bar{c}_4 d_4 - 2\bar{c}_6 d_3, \\ g_6 &= 2\bar{c}_2 \bar{c}_3, \quad g_7 = -\bar{c}_3^2, \\ g_8 &= -\bar{c}_2^2, \quad g_9 = \bar{c}_5 \bar{c}_9 - \bar{c}_2^2. \end{aligned} \tag{5.7}$$

In Eqs. (5.6) and (5.7)

$$\bar{c}_1 = 2\eta', \quad \bar{c}_2 = \frac{4\eta'}{\pi}, \quad \bar{c}_3 = \bar{c}_1,$$

$$\bar{c}_4 = \frac{1}{4}\left\langle \frac{\pi^2}{6\eta'}(1+\eta+\eta^2)+5-3\eta \right\rangle,$$

$$\bar{c}_5 = \frac{\eta'}{5}, \qquad \bar{c}_6 = \frac{2}{3}\left\langle \frac{\pi}{\eta'} - \frac{8}{3}\eta + \frac{16}{9}\frac{\eta'}{\pi} \right\rangle, \tag{5.8}$$

$$\bar{c}_7 = \frac{16}{15}\frac{\eta'}{\pi}, \quad \bar{c}_8 = \frac{1}{4}\left\langle \frac{2\pi^2}{3\eta'}(1+\eta+\eta^2)+11\eta' \right\rangle,$$

$$\bar{c}_9 = \frac{3}{5}\eta',$$

$d_1, d_2, d_3, d_4; \eta, \eta'$ having been defined in Eq. (4.5).

Equation (5.5) defines the multivalent dependence of $\bar{\lambda}$ on z or on $\bar{\omega}$. The left-hand side of Eq. (5.5) is an even polynomial of the sixth order in $\bar{\lambda}$. Thus, for every real $\bar{\omega}$ there are three values of $\bar{\lambda}^2$ and, consequently, three values of $\bar{\lambda}$, for which $\mathrm{Re}\,\bar{\lambda} \geqslant 0$. For the given frequency ω, consequently, there are, in the material, three waves of different wave lengths and different attenuation. We shall investigate the multivalent function $\bar{\lambda}(\bar{\omega})$ numerically in the next section.

Let us investigate now $\bar{\lambda}^2(z)$ for small z. Let us develop $\bar{\lambda}^2(z)$ into a power series with respect to z and consider the first three terms only, viz.

$$\bar{\lambda}^2(z) = A_0 + A_1 z + A_2 z^2. \tag{5.9}$$

The substitution of Eq. (5.9) into Eq. (5.5) yields for A_0, A_1, A_2 the equations

$$A_0^3 a_6' + A_0^2 a_4' + A_0 a_2' = 0,$$

$$A_1 = -\frac{1}{C}(A_0^2 b_4' + A_0 b_2' + b_0'),$$

$$A_2 = -\frac{1}{C}\langle A_1^2(3A_0 a_6' + a_4') + A_1(2A_0 b_4' + b_2') + c_0' \rangle, \tag{5.10}$$

$$C = 3A_0^2 a_6' + 2A_0 a_4' + a_2'.$$

From Eq. $(5.10)_1$ we obtain three roots for A_0, viz.

$$A_0' = 0,$$

$$A_0'' = \frac{1}{2a_6'}\langle -a_4' - \sqrt{a_4'^2 - 4a_2' a_6'} \rangle, \tag{5.11}$$

$$A_0''' = \frac{1}{2a_6'}\langle -a_4' + \sqrt{a_4'^2 - 4a_2' a_6'} \rangle.$$

It can be proved that for $\eta \in (0, 1)$ the following inequalities hold:

$$a_6' > 0, \quad a_4' < 0, \quad a_2' > 0,$$
$$a_4'^2 - 4a_2' a_6' > 0. \tag{5.12}$$

It follows that A_0', A_0'', A_0''' are real and it holds that

$$A_0''' > A_0'' > A_0' = 0.$$

Consequently, also the respective A_1', A_1'', A_1''', A_2', A_2'', A_2''' which we obtain from Eqs. $(5.10)_2$ and $(5.10)_3$ are real.

If we consider, apart from Eq. (5.9), also the three-term expansion

$$\bar{\lambda}(z) = B_0 + B_1 z + B_2 z^2, \tag{5.13}$$

we obtain the real B_0, B_1, B_2 in the form

$$B_0 = \pm\sqrt{A_0}, \quad B_1 = \pm\frac{A_1}{\sqrt{A_0}}, \quad B_2 = \pm\frac{1}{2\sqrt{A_0}}\left(A_2 - \frac{A_1^2}{4A_0}\right). \tag{5.14}$$

If we write the equation of heat conduction in the direction of x_3 in a certain homogeneous (non-porous) material in the form

$$k_{ef}\vartheta_{,33} - (\varrho c)_{ef}\dot{\vartheta} = 0,$$

we obtain for the thermal wave

$$\vartheta = Te^{-\lambda x_3 + i\omega t}$$

the relation

$$\lambda^2 = i\omega\frac{(\varrho c)_{ef}}{k_{ef}}. \tag{5.15}$$

Let us write Eq. (5.9) for A_0', A_0'', A_2' and neglect the term with z^2. The substitution of λ, ω instead of $\bar{\lambda}$, $\bar{\omega}$ in Eq. (5.9) yields

$$\lambda^2 = i\omega\frac{\varrho c}{k}A_1'. \tag{5.16}$$

From Eqs. $(5.11)_1$, $(5.10)_2$ and (5.6)—(5.7) it follows that

$$A_1' = \frac{d_1(\bar{c}_4\bar{c}_8 - \bar{c}_6^2)}{\bar{c}_1(\bar{c}_4\bar{c}_8 - \bar{c}_6^2) + 2\bar{c}_2\bar{c}_3\bar{c}_6 - \bar{c}_2^2\bar{c}_8 - \bar{c}_3^2\bar{c}_4}. \tag{5.17}$$

If we put

$$(\varrho c)_{ef} = d_1\varrho c = (1-\eta^3)\varrho c, \tag{5.18}$$

$$k_{ef} = k\left\langle \bar{c}_1 + \frac{2\bar{c}_2\bar{c}_3\bar{c}_6 - \bar{c}_2^2\bar{c}_8 - \bar{c}_3^2\bar{c}_4}{\bar{c}_4\bar{c}_8 - \bar{c}_6^2} \right\rangle, \tag{5.19}$$

Eq. (5.16) acquires the form of Eq. (5.15).

Consequently, for very small ω, it is possible to obtain approximately that branch of $\bar{\lambda}^2(z)$ for which

$$A_0' = 0,$$

i.e. for which the implication applies that

$$\omega \to 0 \Rightarrow \lambda \to 0,$$

by the investigation of waves propagating in a homogeneous (non-porous) material with the so-called effective characteristics k_{ef}, $(\varrho c)_{ef}$, defined in Eqs. (5.18) and (5.19). The other branches, however, cannot be obtained in this way.

6. Numerical results

To check the suitability of the assumption (2.2) for the temperature field in the vicinity of the pore, we shall compare k_{ef} from Eq. (5.19) with the effective thermal conductivity obtained by other methods.

In [8] the accurate value of the effective dielectric constant for the arrangement of spherical inclusions in the matrix, called in literature the composite sphere assemblage, was obtained. The problem of fining the effective dielectric constant is mathematically identical with the problem of finding the effective thermal conductivity.

If the inclusions are pores with zero thermal conductivity, the accurate effective thermal conductivity $\overline{k}_{ef}$ for this special pore arrangement according to [8], is determined by the formula

$$\overline{k}_{ef} = k\left\langle 1 - \frac{3\eta^3}{\eta^3+2}\right\rangle, \tag{6.1}$$

where k is the thermal conductivity of the matrix and η^3 is the volume porosity. The table below shows the values of k_{ef} from Eq. (6.1) and k_{ef} from Eq. (5.19) for various η^3. The table shows that k_{ef} are only slightly higher than $\overline{k}_{ef}$. Let us note that $\overline{k}_{ef}$ from Eq. (6.1) represents the upper bound of the effective thermal conductivity of a macroscopically homogeneous and isotropic porous material of any shape and microscopic arrangement of pores (see [9], concerned with effective magnetic premeability, which is, once again, a mathematically identical problem).

The practical identification of k_{ef} according to Eq. (5.19) and $\overline{k}_{ef}$ according to Eq. (6.1) is in favour of the presented model.

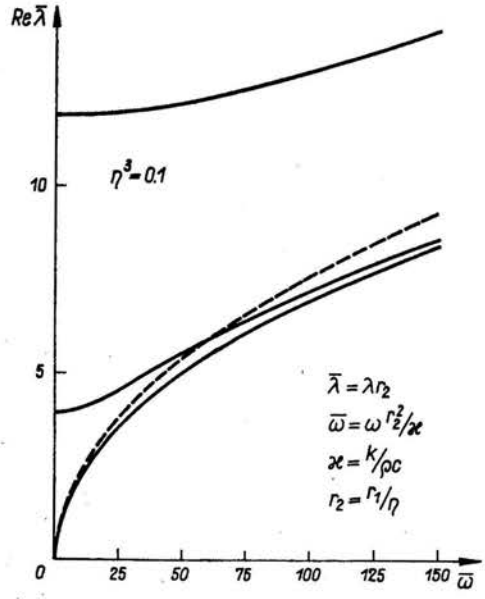

FIG. 3. Thermal waves. Wave number versus frequency. $\eta^3 = 0.1$.

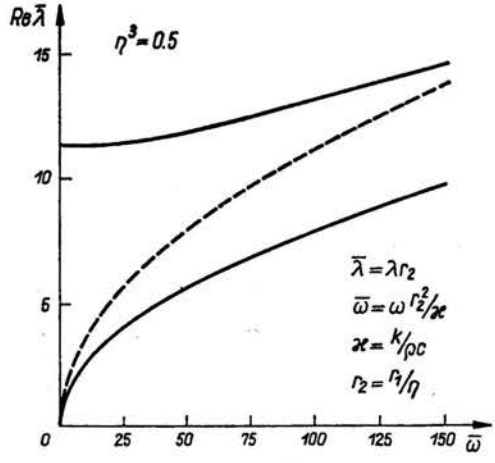

FIG. 4. Thermal waves. Wave number versus frequency. $\eta^3 = 0.5$.

The multivalent dependence of $\overline{\lambda}$ on $\overline{\omega}$ defined in Eq. (5.5) was calculated numerically for the porosities of $\eta^3 = 0.1$ and 0.5. Figures 3—4 show in solid lines the values of $\mathrm{Re}\,\overline{\lambda}$ plotted against $\overline{\omega}$. Figure 3 shows all three branches, Fig. 4 the two lowest branches only. The dashed curves in Figs. 3—4 were calculated from Eqs. (5.15), (5.18) and (6.1) and correspond to the homogeneous (non-porous) material of effective thermal conductivity $\overline{k}_{ef}$ from Eq. (6.1). These effective curves correspond with the lowest branches calculated from Eq. (5.5) and are always higher than these lowest branches. The difference increases with the growing porosity η^3 for a constant $\overline{\omega}$ and with the growing $\overline{\omega}$ for a constant η^3.

7. Conclusion

The work presents a continuum model for heat conduction in porous materials. For the macroscopically unidimensional problem the temperature field is approximated by three scalar functions for which, on the basis of a variational principle, three differential equations of the second order were obtained. The influence of the microstructure is considered in the model by making the coefficients of these equations depend on the pore size and porosity. In the classical approach, when the porous material is replaced with a homogeneous material of effective thermal conductivity, only porosity is taken into account, while the influence of the pore size is not considered.

Table 1.

η^3	$\overline{k}_{ef}$ in (6.1)	k_{ef} in (5.19)
0	$1.0000\,k$	$1.0086\,k$
0.1	$8.5714 \cdot 10^{-1}\,k$	$8.5786 \cdot 10^{-1}\,k$
0.2	$7.2727 \cdot 10^{-1}\,k$	$7.2745 \cdot 10^{-1}\,k$
0.3	$6.0870 \cdot 10^{-1}\,k$	$6.0874 \cdot 10^{-1}\,k$
0.4	$5.0000 \cdot 10^{-1}\,k$	$5.0002 \cdot 10^{-1}\,k$
0.5	$4.0000 \cdot 10^{-1}\,k$	$4.0001 \cdot 10^{-1}\,k$
0.6	$3.0769 \cdot 10^{-1}\,k$	$3.0769 \cdot 10^{-1}\,k$
0.7	$2.2222 \cdot 10^{-1}\,k$	$2.2222 \cdot 10^{-1}\,k$
0.8	$1.4286 \cdot 10^{-1}\,k$	$1.4286 \cdot 10^{-1}\,k$
0.9	$6.8966 \cdot 10^{-2}\,k$	$6.8966 \cdot 10^{-2}\,k$
1.0	0	0

When applied to the propagation of plane harmonic thermal waves, the model with the effective thermal conductivity yields only one (effective) branch of the dependence of the complex wave number with the positive real part (wave attenuation) on frequency. The continuum model presented in this work yields three approximate branches of the dependence of the complex wave number on frequency. The effective branch corresponds to the lowest branch (i.e. the branch with the lowest real part of the wave number). The mutual distance between two corresponding branches grows with the growing frequency and porosity. If the frequency approaches zero, both branches for a fixed porosity coincide.

The present continuum model can be considered as a more accurate version of the classical homogeneous model with effective thermal conductivity. In the application to propagation of thermal harmonic waves, the classical model can be used for very low frequencies only, while the present model is applicable to higher frequencies as well.

References

1. A. H. NAYFEH, *A continuum mixture theory of heat conduction in laminated composites*, J. Appl. Mech., **42**, 2, 399–404, 1975.
2. A. MAEWAL, T. C. BACHE, G. A. HEGEMIER, *A continuum model for diffusion in laminated composite media*, J. Heat Transfer, **98**, 1, 133–138, 1976.
3. A. MAEWAL, G. A. GURTMAN, G. A. HEGEMIER, *A mixture theory for quasi-one-dimensional diffusion in fibre-reinforced composites*, J. Heat Transfer, **100**, 1, 128–133, 1978.
4. H. MURAKAMI, G. HEGEMIER, A. MAEWAL, *A mixture theory for thermal diffusion in unidirectional composities with cylindrical fibres of arbitrary cross section*, Int. J. Solids Struct., **14**, 9, 723–737, 1978.
5. M. BEN-AMOZ, *Heat conduction theory for composite materials*, ZAMP, **27**, 3, 335–345, 1976.
6. C. T. SUN, J. D. ACHENBACH, G. HERRMANN, *Continuum theory for a laminated medium*, J. Appl. Mech., **35**, 3, 467–475, 1968.
7. M. HLAVÁČEK, *A continuum theory for isotropic two-phase elastic composites*, Int. J. Solids Struct., **11**, 10, 1137–1144, 1975.
8. R. LANDAUER, *The electrical resistance of binary metallic mixtures*, J. Appl. Phys., **23**, 779, 1952.
9. Z. HASHIN, S. SHTRIKMAN, *A variational approach to the theory of the effective magnetic permeability of multiphase materials*, J. Appl. Phys., **33**, 3125, 1962.
10. R. E. NICKEL, J. L. SACKMANN, *Variational principles for linear coupled thermoelasticity*, Quart. Appl. Math., **26**, 1, 11–26, 1968.

CZECHOSLOVAK ACADEMY OF SCIENCES,
INSTITUTE OF THEORETICAL AND APPLIED MECHANICS.

Received July 31, 1979.