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Crack problems in nonlocal elasticity

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THE RESULTS given in a recent paper [1] are extended to the cases of plane strain shear, tension and anti-plane shear. It is found that non-existence results of a type proved in [1] also apply to these plane problems. Solutions are, however, constructed for nonlocal moduli which have a short range (delta function) component. These solutions exhibit square root singularities of the type found in the classical elastic crack problem, the coefficient of this singularity will depend in general on the particular nonlocal law assumed.

Wyniki uzyskane w poprzedniej pracy autora [1] uogólniono na przypadki płaskiego stanu odkształcenia przy ścinaniu, rozciąganiu oraz na antypłaski stan odkształcenia. Stwierdzono, że konkluzje dotyczące nieistnienia rozwiązań podobne do wniosków pracy [1] stosują się również do rozpatrzonych tutaj zagadnień dwuwymiarowych. Rozwiązania skonstruowano jednak dla modułów nielokalnych zawierających składnik blisko-zasięgowy typu funkcji delta. Rozwiązania zawierają osobliwości pierwiastkowe tak samo jak klasyczne rozwiązania dla ośrodków sprężystych ze szczelinami, a współczynniki występujące przy tych osobliwościach zależą w ogólności od postaci przyjętego prawa nielokalnego.

Результаты, полученные в предыдущей работе автора [1], обобщены на случаи плоского деформационного состояния при сдвиге, растяжении, а также на антиплоское деформационное состояние. Констатировано, что выводы, касающиеся несуществования решений, аналогичны следствиям работы [1] и применяются тоже к рассматриваемым здесь двумерным задачам. Решения построены однако для нелокальных модулей, содержавших близкодействующую составляющую типа дельта-функции. Решения содержат особенности типа радикала, аналогично как классические решения для упругих сред со щелями, а коэффициенты, выступающие при этих особенностях, в общем зависят от вида принятого нелокального закона.

1. Introduction

IN A RECENT paper [1] we have discussed numerical calculations made by Eringen and coworkers [2 and 3] to evaluate the crack tip stresses in a nonlocal elastic medium. We showed (i) that an approximation scheme they suggested had a non-uniform character and (ii) that the problem formulated (a model problem) may in fact have *no solution* with finite displacements. The purpose of the present paper is to investigate whether these characteristics are also present in the cases of anti-plane strain, plane strain tension and plane strain shear. Our results will show that the approximation scheme they suggest i.e. approximate the crack face displacements by the classical elastic displacement and then evaluate the stresses from this approximation, is *non-uniform*. We illustrate this non-uniformity by solving, with the use of matched asymptotic expansions, the problem of a specified crack displacement. These results can be compared with those given in [1].

Also, in reference [1] the non-existence of solutions to the model problem solved numerically in [2] was demonstrated. In Sect. 4 of this paper we give an elementary non-existence proof for the crack problem with finite displacements and a nonlocal modulus as defined in reference [3]. This proof applies to the cases of anti-plane strain, plane strain shear and plane-strain tension. The plane strain shear case had previously been treated by the approximation scheme described above. We discuss this in Sect. 3 and point out why we believe it is in error.

If our results are to be believed, they suggest that the finite crack tip stress results obtained in references [2] and [3] and earlier papers are *not correct*. This is a pity since these papers do attempt a useful application of a nonlocal theory as a bridge between the lattice theory and a continuum theory. It should be noted that there is no inherent difficulty, of the non-existence kind discussed here, if situations such as stress fields around dislocations are considered. Some of these have been considered by Eringen, (e.g. reference [4] and earlier references). This problem can be represented in terms of a specified displacement discontinuity and then the stresses computed from the nonlocal constitutive equations. No mixed boundary value problem is involved in such a calculation.

Other workers on nonlocal theories of elasticity have also discussed integral constitutive equations such as those given in Eq. (2.2) of the text (e.g. references [5], [6] and [7]). However, in [6] Kröner suggests that the nonlocal moduli should consist of a short range and a long range part. The short range part has a delta function dependence on position and thus gives rise to a term like the classical elastic situation plus a nonlocal term. We do not anticipate any difficulties concerning the existence of solutions in this case (although we have not yet attempted an existence proof) and in Sect. 5 we attempt to analyze the crak problem in such a case. We consider the case where constants in the nonlocal moduli (e.g. the lattice parametr) ~~are~~ small compared to the crack length. We make the assumption that when this small parameter (a/l say, a — lattice parameter, l — half-crack length) tends to zero, the problem has a singular perturbation character. Exploiting this, a solution is obtained using a combination of the method of matched asymptotic expansions and the Wiener-Hopf technique. This solution which holds, in the limit $a/l \to 0$, for fairly general nonlocal moduli exhibits the usual square root stress singularity of classical elasticity. The coefficient of this stress-singularity (c.f. Eq. (5.16)) depends on the proportion of „short-range" modulus in the constitutive relation. As this proportion tends to zero, the stress tends to zero (implying no stress singularity) *but* the corresponding displacement tends to infinity, the product of the two, however, remaining constant. This lends support to our non-existence result discussed earlier, i.e. that there is no solution to the corresponding boundary value problem if the nonlocal moduli have only a long range part and only finite displacements are allowed. Multiplying the crack tip stress and displacement fields does give the classical elastic result in the limit $a/l \to 0$ even though separately they differ from their classical elastic counterparts. This is reminiscent of the result (reference [8]) for couple-stress elasticity. In [8] it is proved that for a medium with couple stresses the energy-release rate does tend to the classical elastic result (as the couple stress length parameter tends to zero) even though the stress intensity factors do not tend to their classical elastic counterparts.

To complete this introduction we would like to admit that in [9] it was stated that it was not surprising a nonlocal theory of elasticity could get rid of the stress singularity at a crack tip. If non-existence results such as those described here, in Sect. 4, are correct, perhaps this statement should be changed to "it *is* surprising that a nonlocal theory *cannot* get rid of the stress singularity at a crack tip".

2. Basic equations of nonlocal elasticity

The equations of linear, homogeneous, isotropic, nonlocal elasticity with vanishing body and inertia forces can be written as

$$t_{k,lk} = 0, \tag{2.1}$$

$$t_{kl} = \int_V \{\lambda^1(|\mathbf{x}-\mathbf{x}^1|)e_{rr}(\mathbf{x}^1)\delta_{kl}+2\mu^1(|\mathbf{x}-\mathbf{x}^1|)e_{kl}(\mathbf{x}^1)\}dV(\mathbf{x}^1) \tag{2.2}$$

with

$$e_{kl} = \frac{1}{2}(u_{k,l}+u_{l,k}). \tag{2.3}$$

From Eq. (2.2) the stress $t_{kl}(\mathbf{x})$ at a point $\mathbf{x}$ depends on the strains $e_{kl}(\mathbf{x}^1)$ at *all* points of the body. The integral in Eq. (2.2) is over the volume V of the body enclosed within a surface ∂V.

ERINGEN and co-workers [1, 3] consider two possible forms for the nonlocal elastic moduli λ^1 and μ^1, i.e.

$$(\lambda^1, \mu^1) = (\lambda, \mu)\alpha(|\mathbf{x}^1-\mathbf{x}|)$$

with

$$\alpha(|\mathbf{x}^1-\mathbf{x}|) = \alpha_0(a-|\mathbf{x}^1-\mathbf{x}|),\ |\mathbf{x}^1-\mathbf{x}| \leqslant a = 0, \quad |\mathbf{x}^1-\mathbf{x}| > a, \tag{2.4}$$

where a is the lattice parameter, λ, μ are Lamé constants and α_0 is a normalisation constant determined from

$$\int_V \alpha(|\mathbf{x}^1-\mathbf{x}|)dV(\mathbf{x}^1) = 1. \tag{2.5}$$

In [3] an alternative expression for $\alpha(|\mathbf{x}^1-\mathbf{x}|)$ is used to simplify the subsequent analysis, this is

$$\alpha(|\mathbf{x}^1-\mathbf{x}|) = \alpha_0\exp\left\{-\left(\frac{\beta}{a}\right)^2(x_k^1-x_k)(x_k^1-x_k)\right\} \tag{2.6}$$

(the summation convention applies to the index k), where β is a constant. Each of these expressions is shown to reasonably approximate the dispersion curves of lattice dynamics (c.f. [2], [3] and the references therein).

Equation (2.2) is rewritten as

$$t_{kl} = \int_V \alpha(|\mathbf{x}^1-\mathbf{x}|)\sigma_{kl}(\mathbf{x}^1)dV(\mathbf{x}^1), \tag{2.7}$$

where

$$\sigma_{kl}(\mathbf{x}^1) = \lambda e_{rr}(\mathbf{x}^1)\delta_{kl}+2\mu e_{kl}(\mathbf{x}^1), \tag{2.8}$$

with $e_{ij} = 1/2\,(u_{i,j}+u_{j,i})$. Substituting for Eq. (2.7) in Eq. (2.1) and using Gauss's theorem gives

$$(2.9)\qquad \int_V \alpha(|\mathbf{x}^1-\mathbf{x}|)\sigma_{kl,k}(\mathbf{x}^1)dV(\mathbf{x}^1) - \int_{\partial V} \alpha(|\mathbf{x}^1-\mathbf{x}|)\sigma_{kl}(\mathbf{x}^1)da_k(\mathbf{x}^1) = 0,$$

where ∂V is the boundary surface of V.

For plane strain conditions and a crack on $|\mathbf{x}_1| \leqslant l$, $\mathbf{x}_2 = 0$, Eq. (2.9) becomes

$$(2.10)\qquad \int_R \alpha(|\mathbf{x}^1-\mathbf{x}|)\sigma_{kl,k}(x_1^1, x_2^1)dx_1^1dx_2^1 - \int_{-l}^{l} \alpha(|x_1^1-x_1|)\{\sigma_{2l}(x_1^1, 0)\}dx_1^1 = 0,$$

where the integral with a slash over the two-dimensional infinite space excluding the crack line ($|x_1| \leqslant l$, $x_2 = 0$) $\{\sigma_{2l}(x_1^1, 0)\}$ indicates the jump in σ_{2l} at the crack line. The contribution from the boundary surface ∂V at infinity is zero if the displacement fields are assumed to tend to zero there. Such a situation is considered in [2] and [3] where tractions are applied to the crack surface. Superposition is used to go from these cases to that of a stress free crack with stresses applied at infinity.

Furthermore, arguments are given in [2] and [3] to show that the solution of Eq. (2.10) is equivalent to the solution of the equations

$$(2.11)\qquad -i\xi\bar{\sigma}_{1j} + \frac{d\bar{\sigma}_{2j}}{dx_2} = 0, \quad j = 1,2 \qquad \text{with} \qquad \{\sigma_{2j}(x_1, 0)\} = 0, \quad j = 1,2,$$

where the bar denotes the Fourier transform over x_1, i.e.

$$(2.12)\qquad \bar{\sigma}(\xi, x_2) = \int_{-\infty}^{\infty} e^{i\xi x_1}\sigma(x_1, x_2)dx_1.$$

The boundary conditions become:

(i) for a crack with a shear t_0 applied to the crack surface,

$$(2.13)\qquad \begin{aligned} &\sigma_{22}(x_1, 0) = 0 \quad \text{for all} \quad x_1,\\ &t_{21}(x_1, 0) = -t_0(x_1)|x_1| < l,\\ &u_1(x_1, 0) = 0, \quad |x_1| > l,\\ &(u_1, u_2) \to 0 \quad \text{as} \quad r \to \infty \quad \text{(c.f. [3])}, \end{aligned}$$

(ii) for a crack with a pressure t_0 applied on the crack surface,

$$(2.14)\qquad \begin{aligned} &\sigma_{12}(x_1, 0) = 0 \quad \text{for all} \quad x_1,\\ &t_{22}(x_1, 0) = -t_0(x_1), \quad |x_1| < l,\\ &u_2(x_1, 0) = 0 \quad |x_1| \geqslant l,\\ &(u_1, u_2) \to 0 \quad \text{as} \quad r \to \infty, \quad r^2 = (x_1^2+x_2^2) \quad \text{(c.f. [2])}. \end{aligned}$$

As a third and slightly simpler example consider the anti-plane strain (mode 3 problem in which the only non-zero stresses are t_{13} and t_{23} and the only non-zero displacemen component is u_3 in the third direction.

In this case the boundary conditions are:

$$(2.15)\qquad \begin{aligned} t_{23}(x_1, 0) &= -t_0(x_1), & |x_1| &< l,\\ u_3(x_1, 0) &= 0, & |x_1| &\geqslant l \end{aligned}$$

and

$$u_3 \to 0 \quad \text{as} \quad r \to \infty .$$

The Fourier transform of the equilibrium equation in terms of displacements is then

$$(2.16)\qquad -\xi^2 \bar{u}_3 + \frac{d^2 \bar{u}_3}{dx_2^2} = 0.$$

In [2] and [3] approximate solutions are given of problems (i) and (ii) with t_0 constant. The approximation consists of replacing the unknown crack face displacements by the classical elastic displacement (i.e. that displacement obtained for the same problem but in an elastic medium) and then calculating the resultant stress field from Eq. (2.7). Numerical evidence is given in [2] and [3] to suggest that this approximation agrees more and more closely with the original boundary conditions (2.14) or (2.13) as a/l tends to zero (a is the lattice parameter). However, we shall show in the next section that this agreement is illusory and hence that the conclusions drawn in [2] and [3] may be misleading.

3. The finite crack problem (specified displacements)

If $\alpha(|\mathbf{x}|)$ in Eq. (2.7) is replaced by $\delta(|\mathbf{x}|)$, then the problem specified by the boundary conditions (2.13) with $t_0(x) = t_0$ (a constant) gives for the jump in displacement across the crack:

$$(3.1)\qquad \Delta u_i(x_i, 0) = 2A_i(l^2 - x_1^2)^{1/2}, \quad |x_1| \leqslant l.$$

The subscripts $i = 1, 2$, and 3 apply to the problems (i), (ii) and (iii), respectively. A_i is a constant:

$$(3.2)\qquad A_1 = A_2 = (1-\nu)\frac{t_0}{\mu} \quad \text{and} \quad A_3 = \frac{t_0}{\mu}.$$

In this section we consider the problems (i), (ii) and (iii) but with the boundary condition on $t_{2i}(x_1, 0)$ $|x_1| \leqslant l$ replaced by

$$(3.3)\qquad u_i(x_1, 0) = A_i(l^2 - x_1^2)^{1/2}. \quad |x_1| \leqslant l$$

the subscript $i = 1$, 2 or 3 applying to boundary conditions (2.13), (2.14) and (2.15), respectively.

In [1] we have considered different expressions for $\alpha(|\mathbf{x}|)$ in the analysis of a model problem. For the sake of brevity, details are given here only for the modulus defined in Eq. (2.6). Further, note that for reasonable crack lengths the ratio $a/\beta l$ should be much less than unity (cf. [1] and [3]).

Hence we define

$$(3.4)\qquad \varepsilon = a/\beta l \ll 1$$

and to investigate behavior near the crack tip $x_1 = l$ define new coordinates (X_1, X_2) by

$$x_j = l\delta_{j1} + \varepsilon l X_j, \quad x_j^1 = l\delta_{j1} + \varepsilon l X_j^1, \quad j = 1, 2, \tag{3.5}$$

where

$$\delta_{j1} = 1, \quad j = 1,$$
$$= 0, \quad j \neq 1.$$

New displacement and stress fields are defined by

$$u_i = (\varepsilon l)^{1/2} U_i, \quad t_{ij} = (\varepsilon l)^{-1/2} T_{ij}. \tag{3.6}$$

In these new coordinates it is straightforward to see that Eq. (2.7) becomes

$$T_{kl} = \int_V \alpha^1(|\mathbf{X}^1 - \mathbf{X}|) C_{kl}(\mathbf{X}^1) dV(\mathbf{X}^1) \tag{3.7}$$

with

$$\begin{aligned} C_{kl}(\mathbf{X}^1) &= \lambda E_{rr}(\mathbf{X}^1) + 2\mu E_{kl}(\mathbf{X}^1), \\ E_{kl}(\mathbf{X}) &= \frac{1}{2}\left(\frac{\partial U_k}{\partial X_l} + \frac{\partial U_l}{\partial X_k}\right) \end{aligned} \tag{3.8}$$

and

$$\alpha^1(|\mathbf{X}^1 - \mathbf{X}|) = \frac{1}{\pi} \exp(-\{\mathbf{X}_1^1 - \mathbf{X}_1)^2 + (\mathbf{X}_2^1 - \mathbf{X}_2)^2\}) \tag{3.9}$$

for the function defined in Eq. (2.6).

The displacement boundary conditions become

$$\begin{aligned} U_i(X_1, 0) &= 0, \quad X_1 > 0, \quad X_1 < \frac{-2}{\varepsilon}, \\ U_i(X_1, 0) &= A_i(-X_1)^{1/2} (2l + \varepsilon l X_1)^{1/2}, \quad \frac{-2}{\varepsilon} < X_1 < 0, \end{aligned} \tag{3.10}$$

where $i = 1$ (denotes plane strain shear), $i = 2$ (plane strain tension) and $i = 3$ (anti-plane strain). In addition to Eq. (3.10), we have the boundary conditions

$$C_{22}(X_1, 0) = 0 \quad \text{for all} \quad X_1 \tag{3.11}$$

(from Eq. (2.13)) for the problem (i) (plane strain shear) and

$$C_{21}(X, 0) = 0 \quad \text{for all} \quad X_1 \tag{3.12}$$

(from Eq. (2.14)) for the problem (ii) (plane strain tension).

Transforming the field equations (2.11) back into (x_1, x_2) space and changing to the new coordinates (X_1, X_2) leads to

$$C_{kl,k} = 0, \tag{3.13}$$

where C_{kl} is defined in Eq. (3.8). It is thus straightforward to solve Eq. (3.13) in terms of displacements, this is done in Appendix 1 to give

Problem (i)

$$\bar{T}_{21}(\xi, 0) = \frac{-2\mu(\lambda+\mu)}{(\lambda+2\mu)} |\xi| F_1(\xi) \bar{U}_1(\xi, 0). \tag{3.14}$$

Problem (ii)

$$\bar{T}_{22}(\xi, 0) = \frac{-2\mu(\lambda+\mu)}{(\lambda+2\mu)} |\xi| F_2(\xi) \bar{U}_2(\xi, 0). \tag{3.15}$$

Problem (iii)

$$\bar{T}_{23}(\xi, 0) = -\mu |\xi| F_3(\xi) \bar{U}_3(\xi, 0) \tag{3.16}$$

(the over-bars denote the Fourier transform with respect to X_1) where

$$F_1(\xi) = \int_0^\infty (1-|\xi| X_2^1) e^{-|\xi| X_2^1} dX_2^1 \int_{-\infty}^{\infty} 2\alpha(|X_0|, |X_2^1|) e^{-i\xi X_0} dX_0, \tag{3.17}$$

$$F_2(\xi) = \int_0^\infty (1+|\xi| X_2^1) e^{-|\xi| X_2^1} \int_{-\infty}^{\infty} 2\alpha(|X_0|, |X_2^1|) e^{-i\xi X_0} dX_0 \tag{3.18}$$

and

$$F_3(\xi) = \int_0^\infty \exp(-|\xi| X_2^1) dX_2^1 \int_{-\infty}^{\infty} 2\alpha(|X_0|, |X_2^1|) e^{-i\xi X_0} dX_0. \tag{3.19}$$

Letting $\varepsilon \to 0$ in the boundary conditions (3.10) and taking a Fourier transform over X_1 give

$$\bar{U}_i(\xi, 0) = -A_i l^{1/2} \left(\frac{\pi}{2}\right)^{1/2} e^{\pi i/4} \xi_-^{-3/2}. \tag{3.20}$$

Substituing for this expresion into Eqs. (3.14), (3.15) or (3.16) gives in each case relations of the form

$$\bar{T}_{2j} = B e^{\pi i/4} \xi_+^{-1/2} F_j(\xi), \quad j = 1, 2 \text{ or } 3, \tag{3.21}$$

where

$$B = l^{1/2} \left(\frac{\pi}{2}\right)^{1/2} t_0.$$

Applying the Fourier inversion theorem to Eq. (3.21) one can write

$$T_{2j}(X_1, 0) = \frac{2^{1/2}}{2\pi} \int_0^\infty dX_2^1 \int_{-\infty}^{\infty} \alpha(|X_0|, |X_2^1|) dX_0 \, G_j(X_2^1, X_0), \tag{3.22}$$

where

$$G_j(X, X_0) = \int_0^\infty (1 - S_j \xi X_2^1) \xi^{-1/2} e^{-\xi X_2^1} \{\cos \xi(X_0 + X_1) + \sin \xi(X_0 + X_1)\} d\xi$$

and

$$S_1 = 1, \quad S_2 = -1, \quad S_3 = 0.$$

To obtain Eq. (3.22) we have evaluated the complex inversion path along the real ξ axis ($\operatorname{Im}\xi = 0$) and interchanged the orders of integration. These formal steps in the analysis can presumably be justified if the resulting integrals in Eq. (3.22) converge. Such is the case for the examples discussed in references [1], [2] and [3]. For these cases it can be easily seen from Eq. (3.22) that $T_{2j}(X_1, 0)$ is a continuous function of X_1 and is finite as X_1 tends to zero. Thus $t_{2j}(0, 0)$ tends to infinity like $(\varepsilon l)^{-1/2}$ as ε tends to zero for x_1 tending to zero through positive or negative values of x_1. *Hence the stress concentration predicted in references* [2] *and* [3] *is present also in the boundary values of* t_{2j} *and thus is probably a function of the* (*non-uniform*) *approximation rather than of the original problem itself.* The reader is referred to reference [1] for more illuminating examples in the simpler model problem considered there.

4. Non-existence results

Having established in the last section the non-uniformity of the approximation scheme used in references [2] and [3], we discuss here characteristics of the original problems specified by boundary conditions in the problems (i), (ii) and (iii) of Sect. 2. To do this we rephrase the problem as an integral equation.

For the problem (i) we use as a Green's function the field of an edge dislocation with its Burgers vector in the x_1 direction. Superposing the fields of a continuous distribution of such dislocations lying on $x_2 = 0$, $|x_1| < l$ leads to the result

$$\sigma_{21}(x_1, x_2) = \frac{\mu(\lambda+\mu)}{\pi(\lambda+2\mu)} \int_{-l}^{l} \frac{(x_1-\eta)\{(x_1-\eta)^2-x_2^2\}f_1(\eta)d\eta}{\{(x_1-\eta)^2+x_2^1\}^2} \tag{4.1}$$

with the jump in displacement across the line $x_2 = 0$ given as

$$\Delta u_1 = \int_{x_1}^{l} f_1(\eta)d\eta \tag{4.2}$$

and

$$\int_{-l}^{l} f(\eta)d\eta = 0.$$

Using Eq. (2.7) one can then write

$$t_{21}(x_1, x_2) = \int_V \alpha(|\mathbf{x}^1|)\sigma_{21}(x_1+x_1^1, x_2+x_2^1)dV(\mathbf{x}^1), \tag{4.3}$$

where σ_{21} is given in Eq. (4.1). It is fairly straightforward to check that the Fourier transform of Eq. (4.3) using Eqs. (4.1) and (4.2) leads to an equation identical in form to Eq. (3.14). The stress and displacement fields (4.1) and (4.2) satisfy all the boundary conditions in Eq. (2.13) except the one involving $t_{21}(x_1, 0)$. In addition the stress field of a dislocation satisfies the equilibrium euqations $\sigma_{ij,j} = 0$ (see [10] for example).

Changing the order of integration in Eq. (4.3) gives on $x_2 = 0$ the expression

$$t_{21}(x_1, 0) = \frac{\mu(\lambda+\mu)}{\pi(\lambda+2\mu)} \int_{-l}^{l} f_1(\eta) d\eta K_1(x_1-\eta), \tag{4.4}$$

where

$$K_1(t) = 2\int_0^\infty dX_2^1 \int_{-\infty}^{\infty} \frac{\alpha(|x_1^1|, |x_2^1|)(x_1^1+t)\{x_1^1+t)^2-x_2^1\}^2 dx_1^1}{\{(x_1^1+t)^2+(x_2^1)^2\}^2}. \tag{4.5}$$

Thus the boundary condition $t_{21}(x_1, 0) = -t_0, |x_1| \leqslant l$ together with Eq. (4.4) leads to an integral equation for the unknown dislocation density $f_1(\eta)$.

Similarly, for the problem (ii) we use a continuous distribution of edge dislocations on $x_2 = 0, |x_1| < l$ with the Burgers vector in the x_2 direction. This gives $\sigma_{21}(x_1, 0) = 0$ and

$$\sigma_{22}(x_1, x_2) = \frac{\mu(\lambda+\mu)}{\pi(\lambda+2\mu)} \int_{-l}^{l} \frac{f_2(\eta)(x_1-\eta)\{(x_1-\eta)^2+3x_2^2\}d\eta}{\{(x_1-\eta)^2+x_2^2\}^2} \tag{4.6}$$

and

$$u_2(x_1, 0\pm) = \pm \int_{x_1}^{l} f_2(\eta)\, d\eta$$

with

$$\int_{-l}^{l} f_2(\eta) d\eta = 0.$$

This leads to the expression

$$t_{22}(x_1, 0) = \frac{\mu(\lambda+\mu)}{\pi(\lambda+2\mu)} \int_{-l}^{l} f_2(\eta) d\eta K_2(x_1-\eta) \tag{4.7}$$

with

$$K_2(t) = 2\int_0^\infty dx_2^1 \int_{-\infty}^{\infty} \frac{\alpha(|x_1^1|, |x_2^1|)(x_1^1+t)\{x_1^1+t)^2+3x_2^{1^2}\}^2 dx_1^1}{\{(x_1^1+t)^2+x_2^{1^2}\}^2}. \tag{4.8}$$

Thus, together with the boundary condition $t_{22}(x_1, 0) = -t_0, |x_1| < l$ Eq. (4.7) gives an integral equation for the unknown density $f_2(\eta)$ for the problem (ii).

Finally, for the problem (iii) using the field of a screw dislocation leads to the expression

$$t_{23}(x, 0) = \frac{\mu}{2\pi} \int_{-l}^{l} f_3(\eta) d\eta K_3(x_1-\eta), \tag{4.9}$$

where

$$K_3(t) = 2\int\limits_0^\infty dx_2^1 \int\limits_{-\infty}^{\infty} \frac{\alpha|x_1^1|, |x_2^1|)(x_1^1+t)dx_1^1}{(x_1^1+t)^2+x_2^{1^2}}. \tag{4.10}$$

Taking Fourier transforms of Eqs. (4.7) and (4.10) and noting the expression for the displacement given by Eq. (4.6) leads to equations identical in form with Eqs. (3.15) and (3.16).

From the above we see that each of our problems has been reduced to an integral equation of the first kind for the unknown dislocation density $f(\eta)$. Moreover, for finite displacements u, $f(\eta)$ is integrable. Whether or not the above integral equations will have solutions depends therefore on the kernels $K_i(t-\eta)$ $(i = 1, 2, 3)$. Clearly, if $\alpha(|x_1^1|, |x_2^1|) = \delta(|x_1^1|)\delta(|x_2^1|)$, then $K_3(t) = \frac{1}{t}$ and Eq. (4.7) leads to a Cauchy integral equation which gives the classical elastic solution. However, for other expressions it is not clear whether the above integral equations will have solutions. Suppose, for example,

$$\alpha(|x_1^1|), |x_2^1|) = \delta(|x_2^1|)\alpha_1(|x_1^1|) \quad \text{then} \quad K_1(t) = K_2(t) = K_3(t) \int\limits_{-\infty}^{\infty} = \frac{\alpha_1(|x_1^1|)dx_1^1}{x_1^1+t}.$$

Integral equation with this kernel occur in [1] where arguments are given to show that the problem when $\alpha(|x_1^1|) = \frac{\beta}{a\sqrt{\pi}} \exp\left\{-\left(\frac{\beta}{a}\right)^2 x_1^{1^2}\right\}$, $a \neq 0$ has *no* solution with integrable $f(\eta)$ for the case when the applied stress is constant. If no solution exists for this problem, we expect *no* solution for the case $\alpha(|\mathbf{x}|) = \alpha_0 \exp\left\{-\left(\frac{\beta}{a}\right)^2 \mathbf{x}\cdot\mathbf{x}\right\}$ by comparison. We attempt to prove this below.

The case

$$\alpha(|\mathbf{x}|) = \alpha_0 \exp\{-\beta_1^2(x_1^2+x_2^2)\}, \quad \beta_1 = \frac{\beta}{a}. \tag{4.11}$$

We need to investigate the properties of the kernels, K_1, K_2 and K_3. To do this substitute for Eq. (4.11) and change the x_1^1 integration to a new integration variable $u = x_1^1+t$. The result is

$$K_1(t) = \exp(-\beta_1^2 t^2)2\alpha_0 \int\limits_0^\infty \exp(-\beta_1^2 x_2^{1^2})dx_2^1 \int\limits_{-\infty}^{\infty} \frac{\exp(-\beta_1^2 u^2+2\beta_1^2 ut)u(u^2-x_2^{1^2})du}{(u^2+x_2^{1^2})^2} \tag{4.12}$$

with similar results for $K_2(t)$ and $K_3(t)$. The inner integral in Eq. (4.12) is convergent for all values of t. The only possible neighborhood in which the integral might not converge is in the vicinity of $x_2^1 = 0$, $u = 0$, and that it is in fact convergent there can be easily checked. Similar remarks apply also to K_2 and K_3.

Hence, if $f_1(\eta)$ is integrable in $(-l, l)$, (i.e. finite displacements of the crack faces), then the right hand side of Eq. (4.4) is an analytic function of x_1 for all real x_1. If we replace x_1 by $z = x_1+iy$ (y real), then the right hand side of Eq. (4.4) is an analytic function of

z (the analytic continuation of Eq. (4.4) on $-l < x_1 < l$). Moreover, from Eq. (4.12) with $t = z-\eta$ we see that $K(t)$ has growth properties like $\exp(\beta_1^2 y^2)$ as $y \to \pm\infty$. However, the boundary condition $t_{21}(x_1, 0) = -t_0$, $|x_1| \leqslant l$ says that the left hand side of Eq. (4.4) is a constant for $|x_1| \leqslant l$ and the analytic continuation of this into the complex z plane is simply $-t_0$ (i.e. constant everywhere). This contradicts the equality sign of Eq. (4.4) and hence *no solution with integrable* $f_1(\eta)$ *exists for the boundary condition* $t_{21}(x_1, 0) = -t_0$ *(constant)*, $|x_1| \leqslant l$.

Of course, solutions for special loadings may still exist. For example, the solution $f_1(\eta) = \delta(\eta)$ is consistent with the loading $t_{21}(x_1, 0) = \dfrac{\mu(\lambda+\mu)}{\pi(\lambda+2\mu)} K_1(x_1)$, $|x| \leqslant l$. Note, however, that this loading has precisely those growth characteristics in the complex z plane that our argument above says are necessary.

Similar conclusions apply also to the plane strain tension and anti-plane strain cases.

5. The finite crack problem in a solid whose nonlocal moduli consist of both short-range and long-range behavior

KRÖNER [5, 6] has suggested that the nonlocal moduli should have both a short range and a long range part. We adopt such a constitutive relation here and investigate its consequences. Only an outline of the analysis will be given. It should apply to any nonlocal law of the general form given below. Further work is necessary to justify any particular law, by reference to lattice theory, for example, before calculating the numerical coefficients. The nonlocal moduli (λ^1, μ^1) are now written as

$$(\lambda^1, \mu^1) = (\lambda, \mu)\{b\delta(|\mathbf{x}^1-\mathbf{x}|)+(1-b)\alpha_1(|\mathbf{x}^1-\mathbf{x}|)\}, \tag{5.1}$$

where b is a constant ($b < 1$). The expression $\alpha_1(|x|)$ is normalized as in Eq. (2.5) and may be defined, for example, by the definitions (2.4) or (2.6).

We consider the finite crack problem specified by the boundary conditions (2.13), (2.14) and (2.15) for the problems (i), (ii) and (iii), respectively. Also, we assume that the crack length to the lattice parameter ratio is such that $\varepsilon = a/\beta l \ll 1$ as in Sect. 3. Strictly speaking, this applies to α_1 as defined in Eq. (2.6). For the definition (2.4) replace β by unity in what follows.

If the limit $a \to 0$ is taken in Eqs. (2.4) or (2.6), then $\alpha_1(|\mathbf{x}|) \to \delta(|\mathbf{x}|)$ and in Eq. (5.1)

$$(\lambda^1, \mu^1) \to (\lambda, \mu)\delta(|\mathbf{x}-\mathbf{x}|). \tag{5.2}$$

The problems (i), (ii) and (iii), with the moduli (5.2), become the classical elastic crack problems. The solution to these problems is well known, in particular the behavior near the crack tip $x_2 = 0$, $x_1 = l$ is

$$\begin{aligned} t_{2i} &= K_i(x_1-l)^{-1/2}(2\pi)^{-1/2}, \quad 0 < |x_1-l| \ll l, \\ \Delta U_i &= (l-x_1)^{1/2}\left(\frac{K_i}{M_i}\right)\left(\frac{8}{\pi}\right)^{1/2}, \end{aligned} \tag{5.3}$$

(when t_0 is constant in Eqs. (2.13), (2.14) or (2.15), then $K_i = (\pi l)^{1/2}\, t_0$), $M_1 = M_2 = \mu/(1-\nu)$, $M_3 = \mu$ with $\nu = \lambda/2(\lambda+\mu)$ and as before the index 1, 2 or 3 refers to the problems (i), (ii) or (iii).

Clearly, if $a \neq 0$, then the solution (5.3) will probably not be a correct solution to the nonlocal problem since, in this case, to give the stress field the „strain" field is integrated over the whole body. Nevertheless, we might expect that the classical elastic solution is valid throughout the body if a is small enough, except perhaps near the crack tips where the full nonlocal character of the medium should be important. To investigate the field in this region we define inner coordinates (X_1, X_2) as in Section 3, i.e.

$$x_j = l\delta_{j1} + \varepsilon l X_j \quad \text{(cf. 3.5)}, \quad j = 1, 2 \tag{5.4}$$

with $\varepsilon = a/\beta l$.

We assume that the „outer solution" obtained by letting $\varepsilon \to 0$ in the original problem is valid at distances $\delta \gg \varepsilon$ from each tip. The influence of these features is then transmitted to the „inner" solution through matching conditions near each tip where $\varepsilon \ll \delta < 1$. Since both inner and outer approximations are valid in these regions, they must be asymptotically equivalent there.

The inner problem

Making the change of variables outlined in Eq. (3.5) and (3.6), the inner problem has boundary conditions derived from Eqs. (2.13), (2.14) or (2.15) as

$$\begin{aligned} &C_{i2}(X_1, 0) = 0 \quad \forall X_1, \\ &T_{2i}(X_1, 0) = -(\varepsilon l)^{1/2} t_0, \quad \frac{-2}{\varepsilon} < X_1 < 0, \\ &U_i(X_1, 0) = 0 \quad -\infty < X_1 < \frac{-2}{\varepsilon}, \quad 0 < X_1 < \infty. \end{aligned} \tag{5.5}$$

The subscript i in these equations takes the value 1, 2 or 3 for the problems (i) (ii) or (iii), respectively, with the exception of the first equation which is not relevant when $i = 3$.

If we now let $\varepsilon \to 0$, the last two of the above equations become

$$\begin{aligned} T_{2i}(X_1, 0) &= 0, & -\infty < X_1 < 0, \\ &= T_+(X_1, 0), & 0 < X_1 < \infty, \\ U_i(X_1, 0) &= 0, & 0 < X_1 < \infty, \\ &= U_-(X_1, 0), & -\infty < X_1 < 0, \end{aligned} \tag{5.6}$$

where the functions $T_+(X_1, 0)$, $U_-(X_1, 0)$ are unknown and the index i has been dropped for convenience. Taking the Fourier transform over X_1 as in Sect. 3 then leads to equations similar in form to those given in Eqs. (3.14), (3.15) and (3.16). Hence we get

(5.7 $$\overline{T}_+(\xi, 0) = -M_i|\xi|F_i(\xi)\overline{U}_-(\xi, 0),$$

where F_i is given in Eqs. (3.17), (3.18) or (3.19), the index $i = 1, 2, 3$ again referring to the problems (i), (ii) and (iii). M_i is a constant depending on λ and μ, compare Eqs. (3.14), (3.15) and (3.16) and Eq. (5.3).

To specify the „inner problem" further we need conditions at infinity. These are obtained by matching the inner and outer solutions. Rewriting the outer solution in inner coordinates and taking the appropriate limit gives from Eq. (5.3) the conditions

$$U_i(X_1, 0) \sim \left(\frac{2}{\pi}\right)^{1/2} \frac{K_i}{M_i} (-X_1)^{1/2} \quad \text{as} \quad X_1 \to -\infty$$

(5.8) and

$$T_{2i}(X_1, 0) \sim \frac{K_i}{(2\pi)^{1/2}} X_1^{-1/2} \quad \text{as} \quad X_1 \to \infty.$$

The transformed variables should thus have the behavior

$$\bar{T}_+ \sim \frac{K_i}{2^{1/2}} e^{\pi i/4} \xi_+^{-1/2} \qquad \text{as} \qquad \xi \to 0$$

(5.9) and

$$\bar{U}_- \sim -\frac{K_i}{M_i 2^{1/2}} e^{\pi i/4} \xi_-^{-3/2} \quad \text{as} \quad \xi \to 0.$$

It remains to solve the functional equation (5.7) subject to the matching conditions (5.9). To do this note that Eq. (5.7) is defined on the real line $\operatorname{Im}\xi = 0$, the transforms $\bar{T}_+$ and $\bar{U}_-$ being (unknown) analytic functions for $\operatorname{Im}\xi > 0$ and $\operatorname{Im}\xi < 0$, respectively. The function $|\xi|$ is defined in the complex ξ plane so as to have a positive real part; it is factorized into the product of a plus and minus function as $|\xi| = \xi_+^{1/2}\xi_-^{1/2}$. The expressions $\xi_+^{1/2}$, and $\xi_-^{1/2}$ have cuts from $i0-$ to $-i\infty$ and from $i0+$ to $+i\infty$, respectively. The next step in the standard approach to functional equations such as Eq. (5.7) is to factorize $F(\xi)$ into the product of plus and minus functions each of which is regular and non-zero in its respective half plane. If we assume that such a factorization, $F = F_+F_-$, has been accomplished, then Eq. (5.7) can be rearranged as

$$J \equiv \frac{\bar{T}_+(\xi, 0)}{\xi_+^{1/2} F_+(\xi)} = -M_i \xi_-^{1/2} F_-(\xi) \bar{U}_-(\xi, 0). \tag{5.10}$$

Recall now the expressions (3.17), (3.18) or (3.19) for $F(\xi)$. For the nonlocal moduli considered in this section (Eq. (5.1)) these expressions apply with $\alpha(|\mathbf{X}|)$ replaced by $b\delta(|\mathbf{X}|) + (1-b)\alpha_1(|\mathbf{X}|)$. The function $\alpha_1(|\mathbf{X}|)$ as described above can be derived from Eqs. (2.4) or (2.6) for example. Consider the situation when $\alpha_1(|\mathbf{X}|)$ is defined as in Eq. (3.9), i.e.

$$\alpha_1(|\mathbf{X}|) = \frac{1}{\pi} \exp(-X_1^2 - X_2^2). \tag{5.11}$$

Then, with $\alpha(|\mathbf{X}|)$ as defined above, Eqs. (3.17), (3.18) and (3.19) give

$$F_1(\xi) = b - (1-b)\left\{\frac{|\xi| e^{-\xi^2/4}}{\pi^{1/2}} - \left(1 + \frac{\xi^2}{2}\right) \operatorname{Erfc}\left(\frac{|\xi|}{2}\right)\right\},$$

(5.12)

$$F_2(\xi) = b + (1-b)\left\{\frac{|\xi|}{\pi^{1/2}} e^{-\xi^2/4} + \left(1 - \frac{\xi^2}{2}\right) \operatorname{Erfc}\left(\frac{|\xi|}{2}\right)\right\}$$

and

$$F_3(\xi) = b+(1-b)\operatorname{Erfc}\left(\frac{|\xi|}{2}\right),$$

where

$$\operatorname{Erfc}(z) = \frac{2}{\pi^{1/2}}\int_z^\infty \exp(-t^2)dt.$$

For all real ξ each of F_1, F_2 and F_3 have one sign and tend to b a $|\xi| \to \infty$ on the real line. The factorization $F = F_+F_-$ can then be done in a standard way by taking logs and using Cauchy's theorem giving

$$(5.13) \qquad F_\pm(\xi) = b^{1/2}\exp\left\{\pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log\{E(s)/b\}}{s-\xi}\right\}ds$$

with $\operatorname{Im}\xi > 0$ for F_+ and $\operatorname{Im}\xi < 0$ for F_-, thus the integral is indented below the real axis for $F_+(\xi)$ and above the real axis for $F_-(\xi)$ (ξ real). From Eq. (5.13) it is follows that $F(\xi) \to b^{1/2}$ as $|\xi| \to \infty$ in their respective half-planes of regularity, and also that $F_+(0) = F_-(0) = 1$. This last result can be seen by evaluating the integral in Eq. (5.13) for $F_+(0)$ and $F_-(0)$ by integrating along the real axis and noting that $F(s)$ is an even function of s; the contribution from $s = 0$ leads to the result.

The function J defined jointly by both sides of Eq. (5.10) is analytic in the whole complex plane, except possibly at $\xi = 0$. Now, if $U(X_1, 0)$ is to be bounded as $X_1 \to 0$, then each side of Eq. (5.10) will tend to zero for large $|\xi|$, also the requirements (5.9) at $\xi = 0$ imply the existence of a simple pole at this point. Liouville's theorem then leads to the result

$$(5.14) \qquad J(s) = \frac{K_i e^{\pi i/4}}{2^{1/2}\xi}.$$

From Eq. (5.10) the behavior of the stress and displacement at the crack tip can now be deduced as

$$(5.15) \qquad \begin{aligned} U_i(X_1, 0) &\sim b^{-1/2}\left(\frac{2}{\pi}\right)^{1/2}\frac{K_i}{M_i}(-X_1)^{1/2} \quad \text{as} \quad X_1 \to 0_-, \\ T_{2i}(X_1, 0) &\sim b^{1/2}\frac{K_i}{(2\pi)^{1/2}}X_1^{-1/2} \quad \text{as} \quad X_1 \to 0_+. \end{aligned}$$

Note again that the subscript i in Eq. (5.15) takes the values 1, 2 and 3 and refers to the problems (i), (ii) and (iii), respectively. In the original variables Eqs. (5.15) become, for the crack tip $x_1 = l$,

$$(5.16) \qquad \begin{aligned} u_i(x_1, 0) &\sim (l-x_1)^{1/2}b^{-1/2}\left(\frac{K_i}{M_i}\right)\left(\frac{2}{\pi}\right)^{1/2}, \\ t_{2i}(x_1, 0) &\sim b^{1/2}K_i(2\pi)^{-1/2}(x_1-l)^{-1/2}, \quad 0 < |x_1-l| \ll l. \end{aligned}$$

In other words, the solution for the crack tip stress and displacement field becomes, in the limit $\varepsilon = a/\beta l \to 0$, the classical crack tip elastic displacement field multiplied by $b^{-1/2}$ and the classical crack tip stress field multiplied by $b^{1/2}$.

Although Eq. (5.16) has been derived for the particular case where $\alpha_1(|\mathbf{X}|)$ is defined as in Eq. (5.11), it seems to be much more generally true. In fact this will be true for any $\alpha_1(|\mathbf{x}|)$ transforming to a function $\alpha_1(|\mathbf{X}|)$, in the scaled variables, which leads to an $F_i(\xi)$ with the properties that (i) $F_i(\xi) > 0$ for all real ξ; (ii) $F_i(\infty) = b$, $F_i(0) = 1$; (iii) $F_i(\xi) = F_i(-\xi)$. Property (i) is obviously true for F_2 and F_3 (compare Eqs. (3.18) and (3.19)). Property (ii) will hold provided the contribution from α_1 to $F_i(\xi)$ is a function which tends to zero as $|\xi| \to \infty$. This is not true if $\alpha_1 = \delta(|\mathbf{X}|)$ but seems to be so for functions which tend to zero more slowly than this. Properties $(\text{ii})_2$ and (iii) hold for all $\alpha(|\mathbf{X}|)$. From property (iii) it follows that if $F(\xi) = F_+(\xi)F_-(\xi)$, then $F_+(-\xi)$ is equal to $F_-(\xi)$ apart from a multiplicative constant. If we choose this constant to be unity, then $\lim_{|\xi| \to \infty} F_\pm(\xi) = b^{1/2}$ and $F_+(0) = F_-(0) = 1$, the limits being taken in the respective half-planes of regularity. The result (5.16) then follows without the need for the specific factorization (5.13), although this would be required, of course, to find other information besides the singular crack tip stresses. It is worth noting that the product of the crack tip stress and displacement given in Eq. (5.16) reproduces the classical elastic result; this implies the continuity of the energy release rate as $\varepsilon \to 0$ (compare ref. [8] for a proof of a corresponding result in couple-stress and micropolar elasticity).

Appendix 1

(i) Plane-strain shear

In this case the Fourier transform of Eq. (3.13) with respect of X_1 gives the equation

$$-i\xi\overline{C}_{1l} + \frac{d\overline{C}_{2l}}{dX_2} = 0, \quad l = 1, 2. \tag{A.1}$$

Now, substituting for the X_1 Fourier transform of Eq. (3.8) gives

$$\begin{aligned} &\mu\overline{U}_{1,22} - (\lambda+2\mu)\xi^2\overline{U}_1 - i\xi(\lambda+\mu)\overline{U}_{2,2} = 0, \\ &-i\xi(\lambda+\mu)\overline{U}_{1,2} + (\lambda+2\mu)\overline{U}_{2,22} - \xi^2\mu\overline{U}_2 = 0, \end{aligned} \tag{A.2}$$

and we remind the reader that the definition (2.12) of the Fourier transform is being used (i.e. $\overline{U}(\xi, X_2) = \int_{-\infty}^{\infty} U(X_1, X_2)e^{i\xi X_1}dX_1$. Also, in Eq. (A.2) a comma denotes differentiation.

The solution of these equations in $X_2 > 0$, with $\overline{U}_1$ and $\overline{U}_2$ both tending to zero as $X_2 \to +\infty$ can be written

$$\begin{aligned} \overline{U}_1 &= -\frac{1}{\xi}\left\{|\xi|A(\xi) + \left(|\xi|X_2 - \frac{\lambda+3\mu}{\lambda+\mu}\right)B(\xi)\right\}\exp(-|\xi|X_2), \\ \overline{U}_2 &= i\{A(\xi) + X_2B(\xi)\}\exp(-\xi|X_2), \end{aligned} \tag{A.3}$$

where A and B are arbitrary functions of ξ.

Hence, to satisfy the boundary condition $C_{22}(X_1, 0) = 0$ for all X_1, the condition

$$B(\xi) = \frac{(\lambda+\mu)}{\mu}|\xi|A(\xi) \tag{5.4}$$

is required.

For the shear problem the following symmetry relations hold:

$$U_1(X_1, -X_2) = -U_1(X_1, X_2),$$
$$U_2(X_1, -X_2) = U_2(X_1, X_2).$$

Using these in Eq. (3.7) gives

$$T_{12}(X_1, X_2) = \int_0^\infty dX_2^1 \int_{-\infty}^{\infty} C_{12}(X_0+X_1, X_2^1)\{\alpha(|X_0|, |X_2^1-X_2|) + \alpha(|X_0|, |X_2^1+X_2|)\}dX_0, \tag{A.6}$$

where the change of variable $X_0 = X_1^1 - X_1$ has been made in the inner integral. Now, taking the Fourier transform with respect to X_1 leads to the relation

$$\overline{T}_{12}(\xi, X_2) = \int_0^\infty \overline{C}_{12}(\xi, X_2^1)G(X_2^1, X_2)dX_2^1, \tag{A.7}$$

where

$$G(X_2^1, X_2) = \int_{-\infty}^{\infty} e^{-i\xi X_0}\{\alpha(|X_0|, |X_2^1-X_2|)+\alpha(|X_0|, |X_2^1+X_2|)\}dX_0.$$

Finally, the Fourier transform of Eq. (3.8) gives

$$C_{12}(\xi, X_2) = \mu\left(\frac{\partial \overline{U}_1}{\partial X_2} - i\xi\overline{U}_2\right)$$

and using Eqs. (A.3) and (A.4), then substituting in Eq. (A.6) leads to the relation

$$\overline{T}_{12}(\xi, 0) = 2(\lambda+\mu)\xi A(\xi)F_1(\xi), \tag{A.7$_1$}$$

where

$$F_1(\xi) = \int_0^\infty (-1+|\xi|X_2^1)e^{-|\xi|X_2^1}dX_2^1 \int_{-\infty}^{\infty} 2\alpha(|X_0|, |X_2^1|)e^{-i\xi X_0}dX_0. \tag{A.8}$$

Since

$$\overline{U}_1(\xi, 0) = +\frac{(\lambda+2\mu)}{\mu}\frac{|\xi|}{\xi}A(\xi) \tag{A.9}$$

from Eqs. (A.3) and (A.4), Eq. (3.14) of the main text follows from Eq. (A.7).

(ii) Plane strain tension

For this case there are such symmetry relations:

$$\begin{aligned} U_1(X_1, -X_2) &= U_1(X_1, X_2),\\ U_2(X_1, -X_2) &= -U_2(X_1, X_2) \end{aligned} \tag{A.10}$$

and the boundary condition $\sigma_{12}(X_1, 0) = 0$ for all X_1 is satisfied by the relation

(A.11) $$B(\xi) = (\lambda+\mu)|\xi|A(\xi)/(\lambda+2\mu).$$

Operations similar to those above give

(A.12) $$\overline{T}_{22}(\xi, X_2) = \int_0^\infty \overline{C}_{22}(\xi, X_2^1)G(X_2^1, X_2)dX_2^1,$$

where $G(X_2^1, X_2)$ is as defined in Eq. (A.7). Finally, substituting for $\overline{C}_{22}(\xi, X_2^1)$ derived from Eq. (3.8) leads to the expression

(A.13) $$\overline{T}_{22}(\xi, 0) = \frac{-2\mu(\lambda+\mu)}{(\lambda+2\mu)} i|\xi|A(\xi)F_2(\xi),$$

where

(A.14) $$F_2(\xi) = \int_0^\infty (1+|\xi|X_2^1)e^{-i|\xi|X_2^1}sdX_2^1 \int_{-\infty}^\infty 2\alpha(|X_0|,|X_2^1|)e^{-i\xi X_0}dX_0.$$

Since

$$\overline{U}_2(\xi, 0) = iA(\xi),$$

Eq. (3.15) of the main text follows from Eq. (A.13).

(iii) Anti-plane strain shear

In this case the symmetry of the problem gives

(A.15) $$U_3(X_1, -X_2) = -U_3(X_1, X_2)$$

and steps similar to those above lead to

$$\overline{T}_{23}(\xi, X_2) = \int_0^\infty \overline{C}_{23}(\xi, X_2^1)G(X_2^1, X_2)dX_2^1$$

($G(X_2^1, X_2)$ is defined in Eq. (A.7)) and

$$\overline{U}_3(\xi, X_2) = B_1(\xi)\exp(-|\xi|X_2), \quad X_2 > 0.$$

This leads to the relation

(A.16) $$\overline{T}_{23}(\xi, 0) = -\mu B(\xi)|\xi|F_3(\xi),$$

where

$$F_3(\xi) = \int_0^\infty \exp(-|\xi|X_2^1)dX_2^1 \int_{-\infty}^\infty 2\alpha(|X_0|, |X_2^1|)e^{-i\xi X_0}dX_0$$

and hence to Eq. (3.16) of the text.

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