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On the concept of residual microstresses in plasticity; a more fundamental approach

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A SIMPLE structural model of polycrystalline metals composed of two kinds of cubic grains forming a regular aggregate is considered. Grains of one family begin to deform plastically at a certain level of loading while grains of the other family remain elastic during the deformation process. This model allows us to study the redistribution of microstresses during complex plastic deformation processes. As an illustration the effect of the fading memory is analysed in detail.

Rozpatruje się prosty strukturalny model metali polikrystalicznych, złożony z sześciennych ziaren dwóch rodzajów tworzących regularny układ. Ziarna jednego rodzaju zaczynają deformować się plastycznie przy pewnym poziomie obciążenia, podczas gdy ziarna drugiego rodzaju pozostają w stanie sprężystym. Model umożliwia analizę redystrybucji mikronaprężeń resztkowych w czasie złożonych procesów plastycznego odkształcania. Jako ilustrację zbadano efekt zanikającej pamięci materiału przy obciążaniu plastycznym.

Рассматривается простая структурная модель поликристаллических металлов, состоящая из кубических зерен двух родов, образующих регулярную систему. Зерна одного рода начинают деформироваться пластически при некотором уровне нагрузки, в то время как зерна второго рода остаются в упругом состоянии. Модель дает возможность анализа редистрибуции остаточных микронапряжений во время сложных пластических процессов деформирования. Как иллюстрация исследован эффект исчезающей памяти материала при пластической нагрузке.

1. Introduction

MOST metals subjected to complex two- or three-axial loadings display a very complex behaviour which is usually called the generalized Bauschinger effect. Many attempts have been made in the mathematical theory of plasticity in order to describe this effect in terms of mathematics. One of the basic concepts in the mathematical formulation of the strain-hardening phenomenon including the generalized Bauschinger effect is the concept of so called residual microstresses. This concept has been formulated by J. I. KADASHEVITCH and V. V. NOVOZHILOV in their work published in 1958 [1]. It consitutes the basis of the so called kinematic strain hardening rule of metals (see W. PRAGER [2]). If in these theories the initial yield condition of a virgin non-deformed material is assumed in the form of the Huber-Mises criterion

$$s_{ij}s_{ij} = 2k^2, \tag{1.1}$$

then, after plastic deformation, the yield condition may be written as

$$(s_{ij}-\alpha_{ij})(s_{ij}-\alpha_{ij}) = 2k^2, \tag{1.2}$$

where s_{ij} is the stress deviatoric, k is the initial yield locus in simple shear test, and α_{ij} stands for the internal parameter, which is interpreted as the tensor of residual microstresses. According to such a hardening law the initial yield surface (1.1) in the six-dimensional stress space is shifted as a rigid body. The parameter α_{ij} is represented in this space as the translation vector of the central point of the initial yield surface.

In order to obtain a physical interpretation of the α_{ij} parameter, Kadashevitch and Novozhilov considered a mechanical model composed of a slip element supported by two pairs of springs oriented orthogonally. It is evident, however, that the analysis based upon such a simplified model, although very useful in preliminary investigations, cannot give a deeper insight into the mechanics of rearrangement of internal stresses in metals subjected to complex loadings.

Much more promising are the attempts in which various theoretical models of the internal structure of metals are introduced. In these models aggregates of grains are considered forming the polycrystalline structure of metals. The boundaries between adjacent grains in the aggregate are regarded as surfaces of zero thickness, since in real metals these boundaries have been estimated to be only few atoms thick (see for example C. S. BARRET [3]).

Considering the fact that T. H. LIN [4] has presented a review of various works including attempts to apply the experimental single crystal stress-strain relationship to deduce the laws of plastic flow of a polycrystalline material, we will mention here only the works closely related to the aim of the present paper.

In most papers the aggregates of identical differently oriented single crystals were considered. For example, G. I. Taylor analyzed in 1938 an aggregate of randomly oriented crystals under uniaxial tension. He assumed crystals to be rigid-plastic. T. H. LIN [4] considered a model composed of differently oriented crystals of long square cylinders. Each crystal is assumed to have one slip direction. Six different orientations of these crystals were chosen. The material of crystals was assumed to be elastic-plastic. Thus the residual microstresses and the generalized Bauschinger effect could be analyzed (see [5]). This idea was later extended to the three-dimensional model composed of identical differently oriented elastic-plastic cubes each having one slip plane.

The previous work [7] considered a model in which a special idealized configuration of cracks and voids was assumed. Such a configuration causes local stress concentrations and local plastic yielding of the material. Using this model it was possible to examine the effect of redistribution of residual microstresses on the generalized Bauschinger effect of materials subjected to complex biaxial cyclic loading.

These models represent an idealization of the structure of pure metals in which all grains have the same properties. However, technical polycrystalline alloys are usually composed of grains of different mechanical properties. This work presents a simple model in which cubic elastic-plastic grains of different values of the yield locus form a regular aggregate. This model may be subjected to various three-dimensional states of stresses allowing to study the influence of the stress state on the redistribution of microstresses during the initial stage of the complex deformation of metals.

2. Basic relations for the model

Let us consider a simple model shown in Fig 1. It represents a regular array of cubic grains of two kinds A and B. The elastic properties of grains A and B are assumed to be

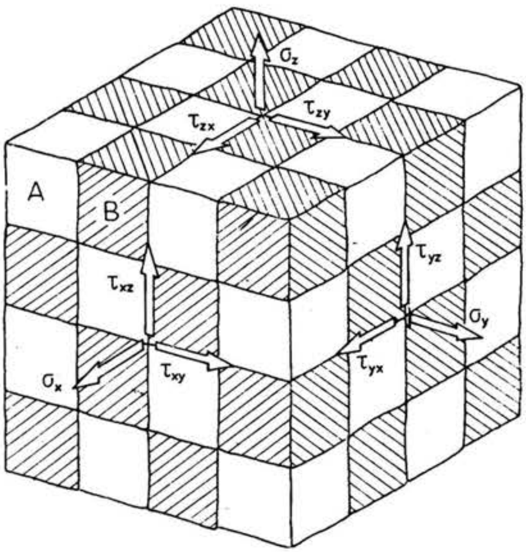


FIG. 1.

identical and are defined by the elastic moduli G or E and the Poisson ratio ν. Grains B are assumed to deform plastically without strain-hardening when stresses in them satisfy the Huber-Mises yield criterion

$$s_{ij}^{B} s_{ij}^{B} = \frac{2}{3} (\sigma_{pl}^{B})^2, \tag{2.1}$$

where s_{ij}^{B} is the stress deviatoric and σ_{pl}^{B} denotes the yield locus of grains B in simple uniaxial tension. Let us assume, moreover, that grains A remain elastic since we will consider here only the initial stage of deformation of the aggregate. Thus the properties of the two kinds of grains in simple tension are as shown in Fig. 2.

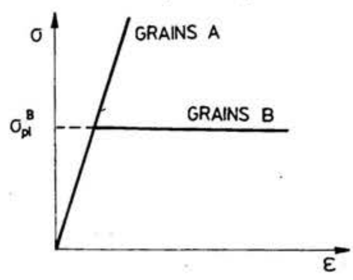


FIG. 2.

The model may be subjected to any arbitrary state of stress σ_{ij} as shown in Fig. 1. As far as grains B remain elastic the stresses σ_{ij}^{A} in grains A and σ_{ij}^{B} in grains B are equal to σ_{ij}. Grains B begin to yield plastically when the external stresses σ_{ij} reach such a level that the condition

$$s_{ij} s_{ij} = \frac{2}{3} (\sigma_{pl}^{B})^2 \tag{2.2}$$

is satisfied.

A further increase of the external stresses σ_{ij} is responsible for the fact that stresses in grains A and B are of different value, this is so because at any level of external loading stresses in grains B must satisfy the yield condition (2.1). For any increment $d\sigma_{ij}$ of the external stresses we must calculate the increments $d\sigma_{ij}^A$ and $d\sigma_{ij}^B$ of stresses in grains A and B, respectively. These increments must satisfy the equations of equilibrium

$$d\sigma_{ij}^A + d\sigma_{ij}^B = 2d\sigma_{ij}. \tag{2.3}$$

Let us assume that the plastic part $(d\varepsilon_{ij}^p)^B$ of strains in grains B is given by

$$(d\varepsilon_{ij}^p)^B = d\lambda s_{ij}^B, \tag{2.4}$$

where s_{ij}^B is the deviatoric part of the stress tensor σ_{ij}^B and $d\lambda$ stands for a proportionality factor.

Elastic strain increments in grains A and B are given by the expressions

$$\begin{aligned} (d\varepsilon_{ij}^e)^A &= \frac{1}{2G}\left(d\sigma_{ij}^A - \frac{\nu}{1+\nu}\, d\sigma_{kk}^A \delta_{ij}\right), \\ (d\varepsilon_{ij}^e)^B &= \frac{1}{2G}\left(d\sigma_{ij}^B - \frac{\nu}{1+\nu}\, d\sigma_{kk}^B \delta_{ij}\right), \end{aligned} \tag{2.5}$$

respectively, where G and ν are elastic constants and δ_{ij} is the Kronecker symbol. The repetition of the subscript k denotes summation from one to three. The total strain increment $d\varepsilon_{ij}^B$ in grains B is then given by

$$d\varepsilon_{ij}^B = (d\varepsilon_{ij}^e)^B + (d\varepsilon_{ij}^p)^B = \frac{1}{2G}\left(d\sigma_{ij}^B - \frac{\nu}{1+\nu}\, d\sigma_{kk}^B \delta_{ij}\right) + d\lambda s_{ij}^B. \tag{2.6}$$

The compatibility of the aggregate requires the equality $d\varepsilon_{ij}^A = d\varepsilon_{ij}^B$ to be satisfied. On substituting Eqs. (2.5) and (2.6) we can write the compatibility condition in the form

$$\frac{1}{2G}\left(d\sigma_{ij}^B - \frac{\nu}{1+\nu}\, d\sigma_{kk}^B \delta_{ij}\right) + d\lambda s_{ij}^B = \frac{1}{2G}\left(d\sigma_{ij}^A - \frac{\nu}{1+\nu}\, d\sigma_{kk}^A \delta_{ij}\right). \tag{2.7}$$

The end point of the stress vector σ_{ij}^B must always be located on the yield surface of the material of grains B. If the particular form of the yield criterion (2.1) is written in the general form $F(\sigma_{ij}^B) = 0$, then the stress increment $d\sigma_{ij}^B$ must satisfy the equation

$$\frac{\partial F}{\partial \sigma_{ij}^B}\, d\sigma_{ij}^B = 0. \tag{2.8}$$

Summing up, we have in the general case a system of thirteen equations (six equations of equilibrium (2.3), six equations of compatibility (2.7) and of the equation (2.8)) with thirteen sought requested magnitudes: six stress increments $d\sigma_{ij}^A$ in grains A, six stress increments $d\sigma_{ij}^B$ in grains B and the proportionality factor $d\lambda$.

This system of equations will be solved for particular cases of external loading with respect to the stress increments $d\sigma_{ij}^A$ and $d\sigma_{ij}^B$. Then, using the small increment technique we may calculate for each increment of the external loading $d\sigma_{ij}$ the corresponding increments $d\sigma_{ij}^A$ and $d\sigma_{ij}^B$. As the starting point of the numerical procedure we must take that value of the external loading σ_{ij} which corresponds to the end point of the initial fully elastic portion of the loading path or, in other words, the value of σ_{ij} which satisfies the yield condition (2.2). Then, assuming the first increment of external loading $d\sigma_{ij}$, we calculate the increments $d\sigma_{ij}^A$, $d\sigma_{ij}^B$ and stresses σ_{ij}^A, σ_{ij}^B. Now the next increment $d\sigma_{ij}$ is assumed and the procedure is repeated until the end point of the prescribed loading path is reached.

3. Residual microstresses

Suppose now that our model loaded up to the prescribed magnitude of the external loading is then fully unloaded. Plastic strain $(\varepsilon_{ij}^p)^B$ in grains B may be expressed as the difference between reversed elastic strains of grains A and B if they are considered separately. Thus we can write

$$2G(\varepsilon_{ij}^p)^B = (\sigma_{ij}^A)_0-(\sigma_{ij}^B)_0-\frac{\nu}{1+\nu}\delta_{ij}[(\sigma_{kk}^A)_0-(\sigma_{kk}^B)_0],$$

where $(\sigma_{ij}^A)_0$ and $(\sigma_{ij}^B)_0$ denote stresses in grains A and B, respectively, at the end point of the loading path. Therefore, in order to satisfy the compatibility conditions of the aggregate after unloading, we must introduce the residual microstresses $(\sigma_{ij}^A)_r$ in grains A and $(\sigma_{ij}^B)_r$ in plastically deformed grains B. Then the compatibity condition may be written as

$$(\sigma_{ij}^A)_r-(\sigma_{ij}^B)_r-\frac{\nu}{1+\nu}\delta_{ij}[(\sigma_{kk}^A)_r-(\sigma_{kk}^B)_r] = (\sigma_{ij}^A)_0-(\sigma_{ij}^B)_0-\frac{\nu}{1+\nu}\delta_{ij}[(\sigma_{kk}^A)_0-(\sigma_{kk}^B)_0]. \tag{3.1}$$

However, the condition of internal equilibrium of the unloaded aggregate requires the following equality to be satisfied:

$$(\sigma_{ij}^A)_r = -(\sigma_{ij}^B)_r. \tag{3.2}$$

Now the compatibility condition (3.1) takes the form

$$2(\sigma_{ij}^A)_r-\frac{2\nu}{1+\nu}(\sigma_{kk}^A)_r\delta_{ij} = (\sigma_{ij}^A)_0-(\sigma_{ij}^B)_0-\frac{\nu}{1+\nu}\delta_{ij}[(\sigma_{kk}^A)_0-(\sigma_{kk}^B)_0]. \tag{3.3}$$

Solving this equation with respect to $(\sigma_{ij}^A)_r$, we obtain a simple formula for residual microstresses:

$$(\sigma_{ij}^A)_r = \frac{1}{2}[(\sigma_{ij}^A)_0-(\sigma_{ij}^B)_0]. \tag{3.4}$$

Suppose now that the prestressed and then unloaded aggregate is reloaded by an external loading σ_{ij}. In the fully elastic state of the aggregate the stresses in grains A and B are

$$\begin{aligned} \sigma_{ij}^A &= \sigma_{ij}+(\sigma_{ij}^A)_r,\\ \sigma_{ij}^B &= \sigma_{ij}+(\sigma_{ij}^B)_r, \end{aligned} \tag{3.5}$$

respectively. Grains B begin to yield plastically when the stresses σ_{ij}^B satisfy the condition (2.1). Thus the yield condition of the prestressed aggregate may be written in the form

$$[s_{ij}+(s_{ij}^B)_r][s_{ij}+(s_{ij}^B)_r] = 2k^2, \tag{3.6}$$

where $(s_{ij}^B)_r$ is the deviatoric of microstresses in grains B. Note that $(s_{ij}^B)_r$ may be identified with the internal parameter α_{ij} in the kinematical hardening law (1.2). However, now the interpretation of the parameter α_{ij} as a tensor connected with residual microstresses is more clear.

The yield condition (3.6) may also be written in the following form:

$$\begin{aligned}\{[\sigma_x+(\sigma_x^B)_r]-[\sigma_y+(\sigma_y^B)_r]\}^2+\{[\sigma_y+(\sigma_y^B)_r]-[\sigma_z+(\sigma_z^B)_r]\}^2 \\ +\{[\sigma_z+(\sigma_z^B)_r]-[\sigma_x+(\sigma_x^B)_r]\}^2 \\ +6\{[\tau_{xy}+(\tau_{xy}^B)_r]^2+[\tau_{yz}+(\tau_{yz}^B)_r]^2+[\tau_{zx}+(\tau_{zx}^B)_r]^2\} = 6k^2,\end{aligned} \tag{3.6'}$$

which may be directly used in the analysis of particular modes of prestressing.

4. Example of uniaxial tension

In this case the only non-zero component of the tensor of external loading is the component σ_x. At each stage of deformation we must calculate the increments of internal stresses $d\sigma_x^A$, $d\sigma_y^A$, $d\sigma_z^A$ and $d\sigma_x^B$, $d\sigma_y^B$, $d\sigma_z^B$. For reasons of symmetry we have $d\sigma_y^A = d\sigma_z^A = d\sigma_t^A$ and $d\sigma_y^B = d\sigma_z^B = d\sigma_t^B$, where the notations $d\sigma_t^A$ and $d\sigma_t^B$ are introduced for the sake of brevity. Increments of shear stresses in grains A and B are equal to zero. Instead of Eq. (2.3) we can write

$$\begin{aligned} d\sigma_x^A+d\sigma_x^B &= 2d\sigma_x, \\ d\sigma_t^A+d\sigma_t^B &= 0. \end{aligned} \tag{4.1}$$

The compatibility equations (2.7) take the form

$$\begin{aligned} d\sigma_x^B-2\nu d\sigma_t^B+Ed\lambda s_x^B &= d\sigma_x^A-2\nu d\sigma_t^A, \\ -\nu d\sigma_x^B+(1-\nu)d\sigma_t^B+Ed\lambda s_t^B &= -\nu d\sigma_x^A+(1-\nu)d\sigma_t^A, \end{aligned} \tag{4.2}$$

where E is the elastic modulus, $s_t^B = s_y^B = s_z^B$ and $s_t^A = s_y^A = s_z^A$ are deviatoric components of internal stresses.

The yield condition (2.1) for grains B may be written as

$$(\sigma_x^B-\sigma_y^B)^2+(\sigma_y^B-\sigma_z^B)^2+(\sigma_z^B-\sigma_x^B)^2 = (\sigma_{pl}^B)^2. \tag{4.3}$$

Condition (2.8) takes the form

$$d\sigma_t^B = -\frac{s_x^B}{2s_t^B}d\sigma_x^B. \tag{4.4}$$

Solving these equations with respect to $d\sigma_x^B$, we obtain

$$d\sigma_x^B = \frac{\nu+\dfrac{s_t^B}{s_x^B}}{2\nu+\dfrac{1-\nu}{2}\dfrac{s_x^B}{s_t^B}+\dfrac{s_t^B}{s_x^B}}d\sigma_x. \tag{4.5}$$

Assuming finite increments of external loading $d\sigma_x$, we may calculate the corresponding increments $d\sigma_x^B$ in grains B and then from Eqs. (4.4) and (4.1) increments $d\sigma_t^B$, $d\sigma_x^A$ and $d\sigma_t^A$. Let us note, however, that our calculations are simplified by the fact that the initial values of the components s_x^B and s_t^B are $s_x^B = \frac{2}{3}\sigma_{pl}^B$ and $s_t^B = -\frac{1}{3}\sigma_{pl}^B$. Thus, from Eq. (4.4) we obtain $d\sigma_t^B = d\sigma_x^B$. It means that in the first step and also in the following steps of calculations the state of stress in grains B increases by a spherical stress increment tensor $d\sigma_x^B = d\sigma_y^B = d\sigma_z^B$ defined by Eq. (4.5). The deviatoric components s_x^B and s_t^B appearing in Eqs. (4.4) and (4.5) do not change, and relationship (4.5) takes the form

$$d\sigma_x^B = \frac{1}{3}d\sigma_x. \tag{4.5'}$$

Thus, for any arbitrary value of the external stress $\sigma_x > \sigma_{pl}^B$ the stresses in grains B are

$$\begin{aligned} \sigma_x^B &= \sigma_{pl}^B + \frac{1}{3}(\sigma_x - \sigma_{pl}^B), \\ \sigma_y^B &= \sigma_z^B = \frac{1}{3}(\sigma_x - \sigma_{pl}^B). \end{aligned} \tag{4.6}$$

Making use of the equations of equilibrium (4.1), we obtain the stresses in grains A:

$$\begin{aligned} \sigma_x^A &= \frac{5}{3}\sigma_x - \frac{2}{3}\sigma_{pl}^B, \\ \sigma_y^A &= \sigma_z^A = -\frac{1}{3}(\sigma_x - \sigma_{pl}^B). \end{aligned} \tag{4.7}$$

Suppose now that the aggregate is prestressed by the tensile stress $(\sigma_x)_0$ and then unloaded. The residual stresses can be calculated using the formulae (3.4) and (4.6)–(4.7). Finally, we obtain for grains A

$$\begin{aligned} (\sigma_x^A)_r &= \frac{2}{3}[(\sigma_x)_0 - \sigma_{pl}^B]. \\ (\sigma_y^A)_r &= (\sigma_z^A)_r = -\frac{1}{3}[(\sigma_x)_0 - \sigma_{pl}^B], \end{aligned} \tag{4.8}$$

and, consequently, for grains B

$$\begin{aligned} (\sigma_x^B)_r &= -\frac{2}{3}[(\sigma_x)_0 - \sigma_{pl}^B], \\ (\sigma_y^B)_r &= (\sigma_z^B)_r = \frac{1}{3}[(\sigma_x)_0 - \sigma_{pl}^B]. \end{aligned} \tag{4.8'}$$

The ratio of residual stress components takes then the form

$$\frac{(\sigma_y^A)_r}{(\sigma_x^A)_r} = -\frac{1}{2}. \tag{4.9}$$

Let the material prestressed uniaxially by stresses $(\sigma_x)_0$ be biaxially reloaded by the arbitrary biaxial combination of the stresses σ_x and σ_y. The remaining components of external loading are equal to zero. Residual microstresses are defined by Eq. (4.8). The shear components of residual microstresses are equal to zero. The yield condition for prestressed material results directly from the general expression (3.6'). After rearrangements we finally obtain

$$[\sigma_x+(\sigma_x^B)_r-(\sigma_z^B)_r]^2-[\sigma_x+(\sigma_x^B)_r-(\sigma_z^B)_r]\sigma_y+\sigma_y^2 = (\sigma_{pl}^B)^2. \tag{4.10}$$

Thus the initial yield ellipse

$$\sigma_x^2-\sigma_x\sigma_y+\sigma_y^2 = (\sigma_{pl}^B)^2$$

is shifted along the σ_x-axis by the segment

$$(\sigma_x^B)_r-(\sigma_y^B)_r = (\sigma_x)_0-\sigma_{pl}^B.$$

5. A theoretical study of the fading memory effect under conditions of plane state of stress

Let us consider a particular case of the plane state of stress in which there exist only two non zero-components σ_x and σ_y of external stresses. The remaining stress components and particularly the τ_{xy} component are assumed to be equal to zero. Moreover, let us assume for simplicity that the Poisson ratio is also equal to zero ($\nu = 0$).

The increments of stresses in grains A and B must satisfy the equations of equilibrium (compare (2.3))

$$\begin{aligned} d\sigma_x^A+d\sigma_x^B &= 2d\sigma_x,\\ d\sigma_y^A+d\sigma_y^B &= 2d\sigma_y,\\ d\sigma_z^A+d\sigma_z^B &= 0. \end{aligned} \tag{5.1}$$

The compatibility equations (compare (2.7)) are

$$\begin{aligned} d\sigma_x^B+Ed\lambda s_x^B &= d\sigma_x^A,\\ d\sigma_y^B+Ed\lambda s_y^B &= d\sigma_y^A,\\ d\sigma_z^B+Ed\lambda s_z^B &= d\sigma_z^A. \end{aligned} \tag{5.2}$$

Condition (2.8) takes now the form

$$s_x^B d\sigma_x^B+s_y^B d\sigma_y^B+s_z^B d\sigma_z^B = 0. \tag{5.3}$$

Solving Eqs. (5.1), (5.2), and (5.3) with respect to stress increments in grains B we finally obtain the following equations:

$$\begin{aligned} \frac{2}{3}(\sigma_{pl}^B)^2 d\sigma_x^B &= [(s_y^B)^2+(s_z^B)^2]d\sigma_x-s_x^B s_y^B d\sigma_y,\\ s_x^B s_y^B d\sigma_y^B &= (s_z^B)^2 d\sigma_x-[(s_x^B)^2+(s_z^B)^2]d\sigma_x^B, \end{aligned} \tag{5.4}$$

from which the stress increments $d\sigma_x^B$ and $d\sigma_y^B$ may be step by step computed by means of the finite difference technique. The third stress increment $d\sigma_z^B$ is then given by Eq. (5.3).

The stress increments $d\sigma_x^A$, $d\sigma_y^A$ and $d\sigma_z^A$ in grains A are then defined by Eqs. (5.1). Having found the stress increments we may calculate at each step of computations the stresses σ_x^B, σ_y^B, σ_z^B and σ_x^A, σ_y^A, σ_z^A corresponding to the respective level of external loading σ_x, σ_y. For any level of external loading we may also calculate from the formulae (3.4) and (3.2) the components of residual microstresses $(\sigma_x^A)_r$, $(\sigma_y^A)_r$, $(\sigma_z^A)_r$ and $(\sigma_x^B)_r$, $(\sigma_y^B)_r$, $(\sigma_z^B)_r$ and then define the position of the yield surface.

The equation defining the position of the shifted yield ellipse in the σ_x, σ_y-plane is obtained by eliminating from the general equation (3.6') the zero components of external loading $\sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$ and the zero components of residual microstresses $(\tau^B_{xy})_r = (\tau^B_{yz})_r = (\tau^B_{zx})_r = 0$ [note that the component $(\sigma^B_z)_r$ is not equal to zero]. Finally, after rearrangements we obtain the equation of the shifted yield ellipse in the form

$$(5.5)\quad [\sigma_x+(\sigma^B_x)_r-(\sigma^B_z)_r]^2-[\sigma_x+(\sigma^B_x)_r-(\sigma^B_z)_r][\sigma_y+(\sigma^B_y)_r-(\sigma^B_z)_r] + [\sigma_y+(\sigma^B_y)_r-(\sigma^B_z)_r]^2 = (\sigma^B_{pl})^2.$$

The position of the central point of the shifted yield ellipse is defined by the radius-vector $\mathbf{r}$ whose components in the σ_x and σ_y directions are

$$(5.5')\qquad \begin{aligned} r_x &= -(\sigma^B_x)_r+(\sigma^B_z)_r,\\ r_y &= -(\sigma^B_y)_r+(\sigma^B_z)_r, \end{aligned}$$

respectively.

In order to study the effect of the fading memory of the model in the process of complex plastic deformation, two particular loading paths were compared. These loading paths are shown in the inset in Fig. 3. According to the programme I the model was first pre-

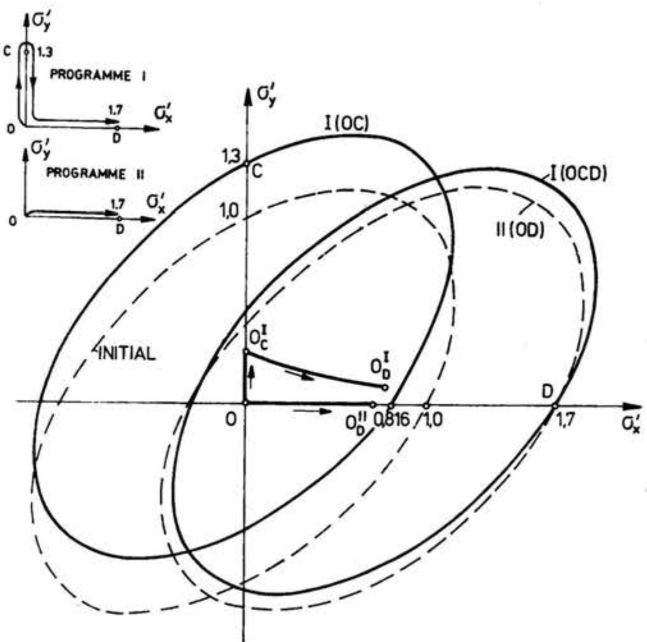
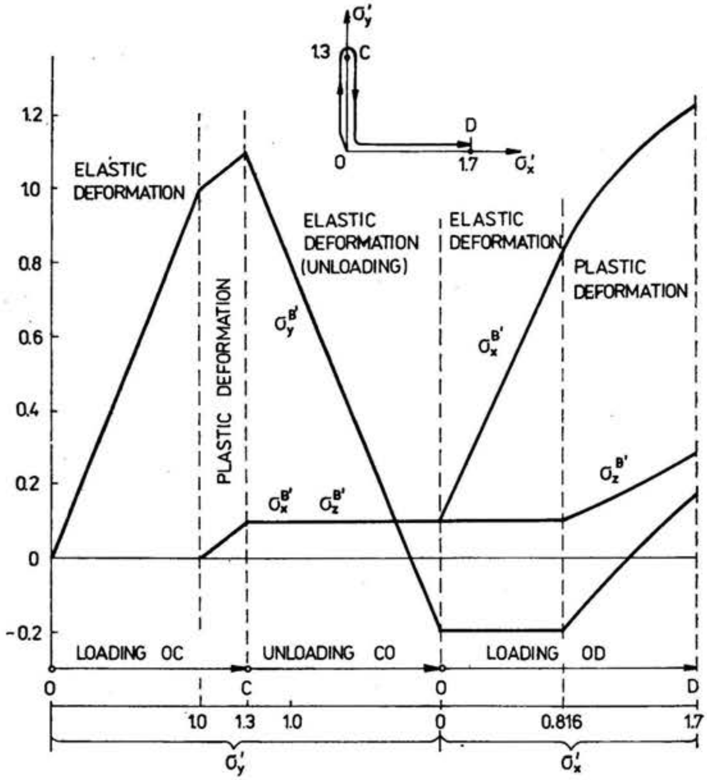


FIG. 3.

stressed by uniaxial tension in the y-direction up to the point C defined by the dimensionless stress $\sigma'_y = \sigma_y/\sigma^B_{pl} = 1.3$, then unloaded and reloaded by uniaxial tension in the x-direction. The loading programme II consists in uniaxial tension in the x-direction only.

The numerically calculated dimensionless stresses $\sigma^{B'}_x = \sigma^B_x/\sigma^B_{pl}$, $\sigma^{B'}_y = \sigma^B_y/\sigma^B_{pl}$, $\sigma^{B'}_z = \sigma^B_z/\sigma^B_{pl}$ in grains B are shown in Fig. 4. In spite of the fact that the external loading



ELASTIC DEFORMATION
PLASTIC DEFORMATION
ELASTIC DEFORMATION (UNLOADING)
ELASTIC DEFORMATION
PLASTIC DEFORMATION
LOADING OC
UNLOADING CO
LOADING OD

FIG. 4.

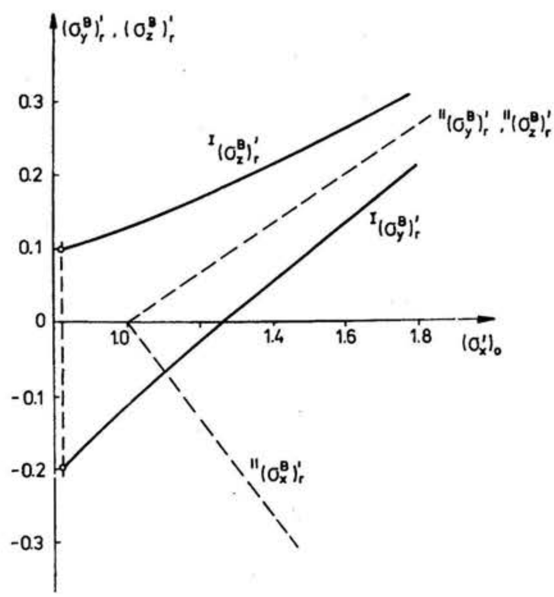


FIG. 5.

programmes consist in uniaxial tensions in various directions, the state of stress inside the material is three-dimensional.

Figure 5 presents residual dimensionless microstresses

$$(\sigma_x^B)'_r = (\sigma_x^B)_r/\sigma_{pl}^B, \quad (\sigma_y^B)'_r = (\sigma_y)_r^B/\sigma_{pl}^B, \quad (\sigma_z^B)'_r = (\sigma_z^B)_r/\sigma_{pl}^B$$

in grains B corresponding to various stages of prestressing along the prestressing programmes I and II. Continuous lines show how the residual microstresses ${}^{\mathrm{I}}(\sigma_y^B)'_r$ and ${}^{\mathrm{I}}(\sigma_z^B)'_r$ change during the final stage (sector OD) of the loading programme I. It is clearly visible that with the increasing length of the loading path along the σ_x axis (sector OD of the loading programme I) these microstresses tend to the common value of residual microstresses ${}^{\mathrm{II}}(\sigma_y^B)'_r$, ${}^{\mathrm{II}}(\sigma_z^B)'_r$ calculated for the prestressing programme II. This means that the model progressively "forgets" the initial portion OC of the prestressing programme I.

This effect of the fading memory is even more clearly visible in Fig. 3 showing the final positions of the shifted yield ellipse for the two prestressing programmes. In the prestressing programme I after loading until the point C ($\sigma'_y = 1.3$) the initial ellipse is shifted along the σ'_y-axis to the position marked by I(OC). The central point of the ellipse is now marked by O_C^{I}. The ellipse did not change its position during unloading from C to O nor even during subsequent loading along the σ'_x-axis to the stress level $\sigma'_x = 0.816$. When the σ_x stresses further increased until the point D ($\sigma'_x = 1.7$), the central point of the ellipse moved along the curvilinear path from O_C^{I} to O_D^{I}. The position I(OCD) of the ellipse corresponds to the end point D of the loading path.

In the prestressing programme II the initial yield ellipse has been simply shifted along the σ'_x-axis by the segment $O-O_D^{\mathrm{II}}$. The central point of the ellipse at the end point of the loading path D is marked by O_D^{II}. The final position of the ellipse is marked by II(OD).

Comparing the final positions of the yield ellipses for the two prestressing programmes we can conclude that the longer the common sector OD of the loading paths in respect to the initial sector OC in the programme I, the smaller the difference between the yield curves resulting from both programmes. This theoretical result may be referred to as the effect of the fading memory of the material. A similar theoretical conclusion was obtained in a previous work [9] on the basis of the kinematical hardening hypothesis.

6. Comparison of the fading memory effect with experimental results

The effect of the fading memory was experimentally investigated in the previous work [9]. Specimens of the aluminium alloy PA-3 (according to Polish standards) were plastically prestressed according to the loading programmes similar to those analyzed in the previous section, and then yield surfaces under general plane state of stress were investigated. To make this paper sufficiently self-contained the main experimental result will be shortly described below.

Let the specimens of the material plastically prestressed according to the loading programme I be reloaded by various combinations of stresses σ_x, σ_y and τ_{xy}, the remaining stress components being equal to zero. The initial yield ellipsoid

$$\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2 = (\sigma_{pl}^B)^2$$

will be, after prestressing, shifted in the σ_x, σ_y-plane to the position described by the equation

$$(6.1) \qquad (\sigma_x+a_x)^2+(\sigma_x+a_x)(\sigma_y+a_y)+(\sigma_y+a_y)^2+3\tau_{xy}^2 = (\sigma_{pl}^B)^2,$$

where [compare (5.5)]

$$a_x = (\sigma_x^B)_r-(\sigma_z^B)_r, \quad a_y = (\sigma_y^B)_r-(\sigma_z^B)_r.$$

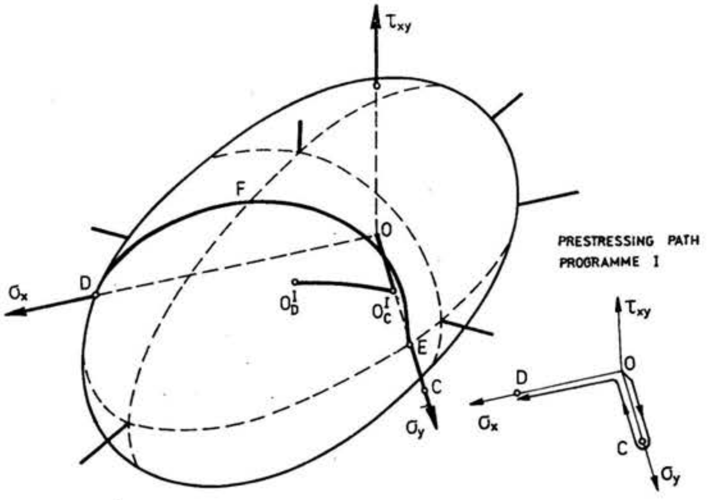


FIG. 6.

The final position of the shifted ellipsoid after prestressing is shown in Fig. 6. The central point was shifted from O to O_D^I. The theoretical line EFD lying on the surface of the shifted ellipsoid corresponds to the states of uniaxial tensions of specimens cut out from the prestressed sheet material in directions forming various angles with the x-axis. In order to obtain the equation of this line in the $\sigma_x, \sigma_y, \tau_{xy}$-space, we must solve Eq. (6.1) together with the relation

$$\sigma_x\sigma_y = \tau_{xy}^2$$

resulting directly from Mohr's circle for uniaxial tension.

In the previous paper [9] the theoretical lines EFD corresponding to the prestressing programmes analogous to the programmes I and II discussed above were found on the basis of the kinematical hardening rule. These lines are compared in Fig. 7 with the experimental curves. The difference between corresponding pairs of theoretical and experimental curves is quite large. Thus it is evident that the kinematical hardening rule and the model shown in Fig. 1, giving very similar results, depart from the real behaviour of metals undergoing plastic deformation. An interesting feature, however, should be noted. In Fig. 7 the distance between theoretical curves for both prestressing programmes is practically the same as the distance between experimental curves. This means that our model predicts very well the length of the second portion OD of the prestressing programme I after which the material "forgets" the initial portion OC of the loading history.

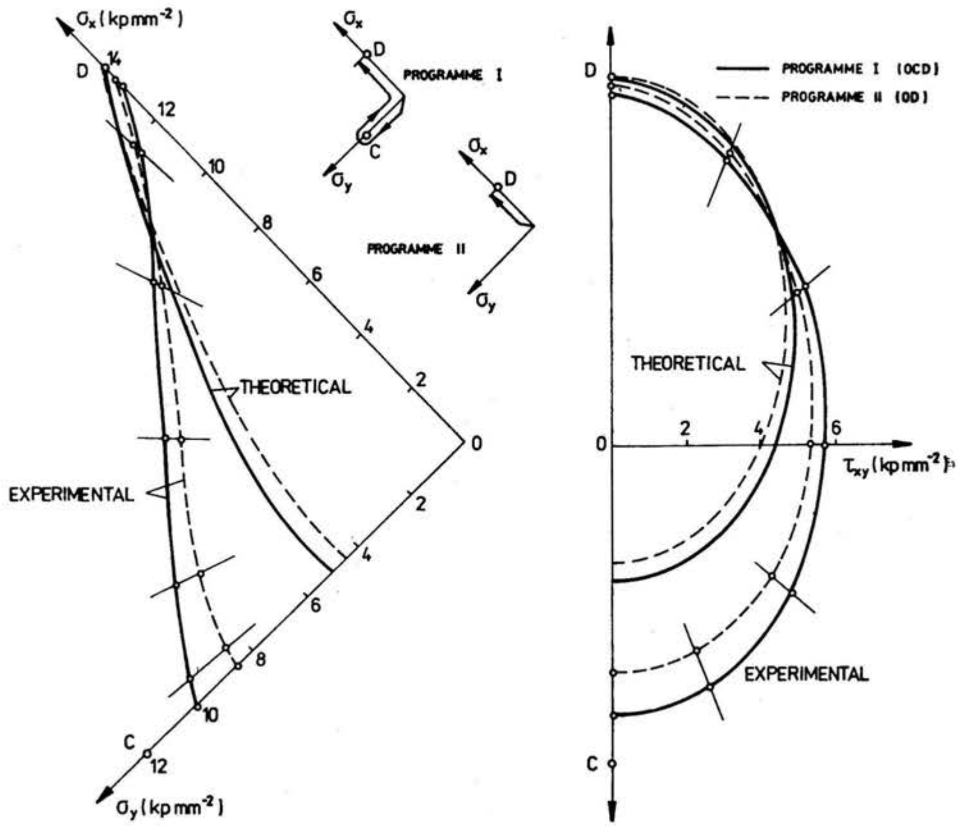


FIG. 7.

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