Arch. Mech., 32, 2, pp. 261–271, Warszawa 1980

Decomposition of non-stationary crack into discontinuity waves

Z. WESOŁOWSKI (WARSZAWA)

THE IDEA that each elementary fracture produces a discontinuity wave was introduced and exposed in [1]. It was shown that in stationary case this idea leads to the already known formulae. In the present paper the non-stationary case is considered. It is shown that the method allows to consider cracks in finite regions. The procedure follows closely to that given by ESHELBY in [2].

Rozważanie ogranicza się do antypłaskiego stanu odkształcenia. Zakłada się, że każde elementarne pęknięcie powoduje powstanie fali nieciągłości. Fale te nakładają się na siebie dając całkowite przemieszczenie. Odpowiednie rozważania dla przypadku stacjonarnego podano w [1]. W niniejszej pracy pokazuje się, że również w przypadku niestacjonarnym otrzymuje się właściwe rezultaty.

Рассуждения ограничиваются антиплоским деформационным состоянием. Предполагается, что каждая элементарная трещина вызывает возникновение волны разрыва. Эти волны накладываются друг на друга, давая полное перемещение. Соответствующие рассуждения для стационарного случая приведены в [1]. В настоящей работе оказывается, что тоже в нестационарном случае получаются правильные результаты.

1. Elementary wave

CONSIDER the propagation of a plane crack into a linear isotropic elastic medium. In the fixed Cartesian coordinate system (x, y, z), the crack occupies the half-plane, Fig. 1

(1.1) $$x < \eta(t), \quad y = 0,$$

where η is a function of time t.

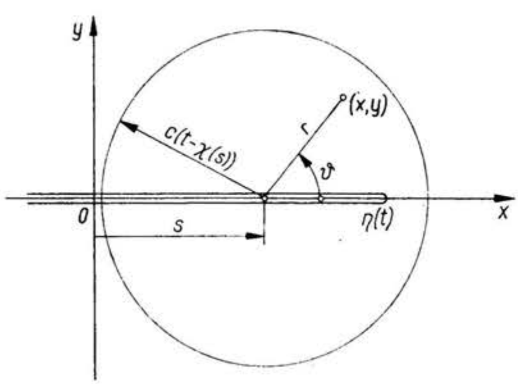

FIG. 1.

The speed of the crack tip is

$$p = \dot{\eta}, \tag{1.2}$$

where a dot denotes the time derivative.

The function inverse to $\eta(t)$ will be denoted by $\chi(x)$, i.e.

$$t = \chi(x), \quad \chi(\eta(t)) \equiv t. \tag{1.3}$$

Let us confine ourselves to the case for which the displacement vector is parallel to the z-axis and does not depend on z. Denoting the corresponding component by u we have

$$u = u(x, y, t), \tag{1.4}$$

$$u_{,xx} + u_{,yy} = \frac{1}{c^2} u_{,tt}, \tag{1.5}$$

$$c^2 = \mu/\varrho,$$

where μ and ϱ are the shear modulus and density of mass, respectively, and c is the propagation speed of transverse waves.

Further calculations are based on the following assumption: Each elementary fracture at $(s, 0)$ of the length ds produces a discontinuity wave centered at $x = s$, $y = 0$. This wave starts at the instant $\chi(s)$. Due to the isotropy of the material, its front is a cylinder of a radius $c(t-\chi(s))$.

In order to find the displacement du of the elementary wave, we write the equation of motion (1.5) in the cylindrical coordinate system (r, ϑ, z) (cf. Fig. 1)

$$(du)_{,rr} + \frac{1}{r}(du)_{,r} + \frac{1}{r^2}(du)_{,\vartheta\vartheta} = \frac{1}{c^2}(du)_{,tt}.$$

It has a solution

$$du = \frac{B}{\sqrt{r}} f[r - c(t-\chi(s))] \sin\vartheta/2, \tag{1.6}$$

where f is an arbitrary function and B a constant. Further, we shall prove that the special case of Eq. (1.6), namely

$$du = \begin{cases} \dfrac{B ds}{\sqrt{r}} \sin\vartheta/2 & \text{for} \quad r \leqslant c(t-\chi(s)), \\ 0 & \text{for} \quad r > c(t-\chi(s)), \end{cases} \tag{1.7}$$

is the discontinuity wave produced by elementary fracture of the length ds. Figure 2 (courtesy of dr. J. F. Kalthoff, Freiburg) provides experimental evidence of elementary waves in steel. For details concerning experimental technique see for example [3].

In Cartesian coordinates Eq. (1.7) reads

$$du = \begin{cases} h(x, y, s) ds & \text{for} \quad r \leqslant c(t-\chi(s)), \\ 0 & \text{for} \quad r > c(t-\chi(s)), \end{cases} \tag{1.8}$$

$$h = B \operatorname{sign} y \frac{\sqrt{\sqrt{(x-s)^2+y^2} - (x-s)}}{\sqrt{(x-s)^2+y^2}}. \tag{1.9}$$

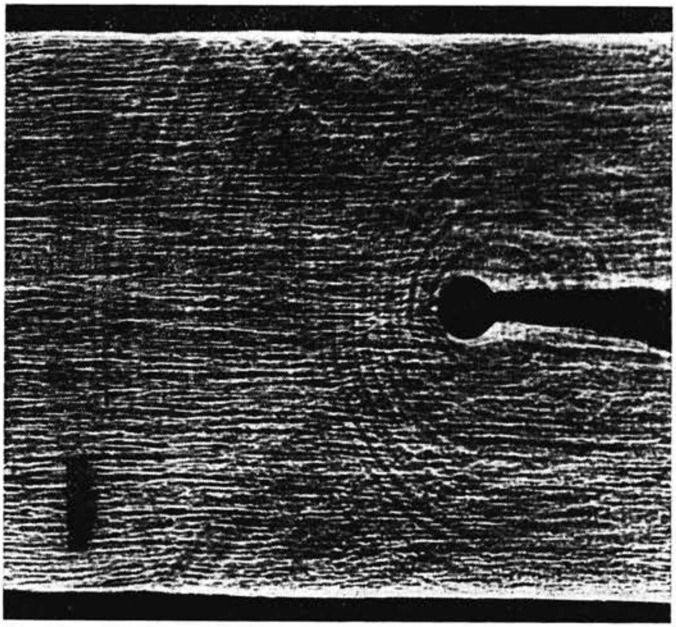

FIG. 2.

Equation (1.7) and Eq. (1.8) equivalent to it are the basic solutions for crack propagation in anti-plane strain. The quantity

$$(du)_{,y} = \frac{\partial}{\partial y} du \tag{1.10}$$

is proportional to the stress $d\tau_{yz}$. For $y = 0$

$$\begin{aligned} du &= 0 \quad \text{for} \quad x-s > 0, \\ (du)_{,y} &= 0 \quad \text{for} \quad x-s < 0. \end{aligned} \tag{1.11}$$

2. Total displacement

All the elementary waves add together and result in the total displacement. The displacement due to the fracture from $x = a$ to $x = \eta(t)$ is

$$u(x, y, t) = \int_a^{\eta(t)} du(x, y, t, s). \tag{2.1}$$

Let us confine the calculations to the subsonic crack

$$0 \leqslant \dot{\eta} < c. \tag{2.2}$$

In this case the wave fronts of the elementary waves produced at points $s_1, s_2, s_3, \ldots$ do not intersect each other and have the shape given in Fig. 3. It is seen that the waves produced near $(\eta(t), 0)$ do not contribute to $u(x, y, t)$. Denoting by $\bar{s}$ the point where the latest wave contributing to u was produced, we have

$$\sqrt{(x-\bar{s})^2+y^2} = c\big(t-\chi(\bar{s})\big). \tag{2.3}$$

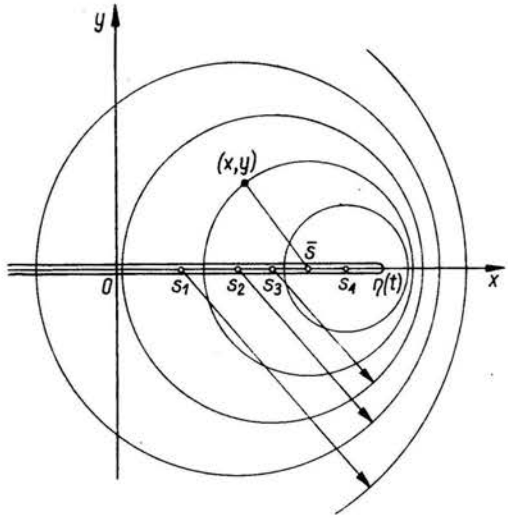

FIG. 3.

Once the function $\chi(s)$ is given, this equation may be solved for $\bar{s}(x, y, t)$. If $\chi(s)$ is continuous, $\bar{s}$ is continuous, too. Note that for $y = 0$ there is

$$x < \bar{s} < \eta(t) \quad \text{for} \quad x < \eta(t); \qquad (2.4)$$
$$\bar{s} < \eta(t) \quad \text{for} \quad x > \eta(t).$$

The inequalities follow either from Eq. (2.2) or directly from Fig. 2 (the wave fronts do not intersect each other).

The formula (2.1) in accordance with Eq. (1.8) may be replaced by

$$u(x, y, t) = \int_a^{\bar{s}(xyt)} h(x, y, t, s)ds, \qquad (2.5)$$

$$h = B(s)\frac{\sqrt{\sqrt{(x-s)^2+y^2}-(x-s)}}{\sqrt{2}\sqrt{(x-s)^2+y^2}}\operatorname{sign}y. \qquad (2.6)$$

We pass to the proof that u as given by Eq. (2.5) satisfies the equation of motion (1.5). For the special case $\bar{s} = \text{const}$, Eq. (1.5) is satisfied automatically because the integrand of Eq. (2.5) satisfies Eq. (1.5). For $\bar{s} = \bar{s}(xyt)$ we proceed as follows. In accordance with Eq. (2.5),

$$u_{,xx} = \int_a^{\bar{s}} h_{,xx}ds + 2h_{,x}\bar{s}_{,x} + h\bar{s}_{,xx},$$

$$u_{,yy} = \int_a^{\bar{s}} h_{,yy}ds + 2h_{,y}\bar{s}_{,y} + h\bar{s}_{,yy}, \qquad (2.7)$$

$$u_{,tt} = \int_a^{\bar{s}} h_{,tt}ds + 2h_{,t}\bar{s}_{,t} + h\bar{s}_{,tt},$$

$$(2.8)\qquad u_{,xx}+u_{,yy}-\frac{1}{c^2}u_{,tt}=\int\limits_a^{\bar{s}}\left(h_{,xx}+h_{,yy}-\frac{1}{c^2}h_{,tt}\right)ds$$
$$+2\left(h_{,x}\bar{s}_{,x}+h_{,y}\bar{s}_{,y}-\frac{1}{c^2}h_{,t}\bar{s}_{,t}\right)+h\left((\bar{s}_{,xx}+\bar{s}_{,yy}-\frac{1}{c^2}\bar{s}_{,tt}\right).$$

The first term on the right hand side equals zero because h satisfies Eq. (1.5). In order to calculate the remaining terms differentiate Eq. (2.3) in turn with respect to x, y and t to obtain

$$(2.9)\qquad \begin{aligned} \bar{s}_{,x}&=\frac{x-\bar{s}}{x-\bar{s}-\chi' cr},\\ \bar{s}_{,y}&=\frac{y}{x-\bar{s}-\chi' cr},\\ \bar{s}_{,z}&=-\frac{cr}{x-\bar{s}-\chi' cr},\qquad r=\sqrt{(x-\bar{s})^2+y^2}. \end{aligned}$$

From Eq. (1.3) it follows $\chi'\dot{\eta}=1$. Therefore the denominator in Eq. (2.9) may be written in the form

$$Q=r\left[\frac{x-\bar{s}}{r}-\frac{c}{\dot{\eta}}\right].$$

It follows from Eq. (2.2) that it equals zero only for $r=0$.

Further differentiation of Eq. (2.9) gives

$$(2.10)\qquad \begin{aligned} \bar{s}_{,xx}&=\frac{1}{Q^2}\left\{-\chi' cr+\chi' c\frac{(x-\bar{s})^2}{r}+\bar{s}_{,x}\left[\chi' cr-\chi' c\frac{(x-\bar{s})^2}{r}+\chi'' c(x-\bar{s})r\right]\right\},\\ \bar{s}_{,yy}&=\frac{1}{Q^2}\left\{(x-\bar{s})-\chi' c\frac{(x-\bar{s})^2}{r}+\bar{s}_{,y}\left[y-\chi' c\frac{y(x-\bar{s})}{r}+\chi'' cr\right]\right\},\\ \bar{s}_{,tt}&=\frac{1}{Q^2}\left\{s_{,t}\left[c\frac{(x-\bar{s})^2}{r}-cr+\chi'' c^2 r\right]\right\}. \end{aligned}$$

Taking into account the above relations and Eq. (1.8), we obtain

$$(2.11)\qquad \begin{aligned} 2\left(h_{,x}\bar{s}_{,x}+h_{,y}\bar{s}_{,y}-\frac{1}{c^2}h_{,t}\bar{s}_{,t}\right)&=-\frac{\sqrt{r-(x-\bar{s})}}{rQ},\\ h\left(\bar{s}_{,xx}+\bar{s}_{,yy}-\frac{1}{c^2}\bar{s}_{,tt}\right)&=\frac{\sqrt{r-(x-\bar{s})}}{rQ} \end{aligned}$$

and, in accordance with Eq. (2.8),

$$(2.12)\qquad u_{,xx}+u_{,yy}-\frac{1}{c^2}u_{,tt}=0.$$

This proves that u as calculated from Eq. (2.1) satisfies the equations of motion.

3. Stationary motion in infinite medium

Consider first stationary motion. The crack tip moves with constant speed $p < c$. The amplitude B does not change in time and the function $\chi(s)$ assumes the special form

$$\chi(s) = \frac{s}{p}. \tag{3.1}$$

Equation (2.3) may be solved to give

$$\bar{s} = -\frac{p^2}{c^2-p^2}\left[x-\frac{c^2}{p}t+\frac{c}{p}\sqrt{(x-pt)^2+\left(1-\frac{p^2}{c^2}\right)y^2}\right]. \tag{3.2}$$

The integration in Eq. (2.5) is elementary and leads to the expression

$$u = B\sqrt{2}\,\mathrm{sign}\,y\sqrt{\sqrt{(x-s)^2+y^2}-(x-s)}\Big|_{-\infty}^{\bar{s}}. \tag{3.3}$$

The lower integration limit does not influence the function u. The upper limit, in accord with Eq. (3.2), gives

$$u = C\sqrt{2}\,\mathrm{sign}\,y\sqrt{\sqrt{(x-pt)^2+(1-p^2/c^2)y^2}-(x-pt)}, \tag{3.4}$$

where

$$C = \frac{B}{\sqrt{1+p/c}}. \tag{3.5}$$

The stresses τ_{xz} and τ_{yz} are proportional to the derivatives $u_{,x}$ and $u_{,y}$. Differentiating u as given by Eq. (3.4), we obtain

$$\tau_{xz} = \mu u_{,x}, \quad \tau_{yz} = \mu u_{,y}, \tag{3.6}$$

$$\tau_{xz} = -\frac{\mu u}{2\sqrt{(x-pt)^2+(1-p^2/c^2)y^2}},$$
$$\tau_{yz} = \frac{\mu y(1-p^2/c^2)}{2u\sqrt{(x-pt)^2+(1-p^2/c^2)y^2}}. \tag{3.7}$$

Note that the displacement jump $[\![u]\!]$ on the crack $x < pt$, $y = 0$ and the stress τ_{yz} in front of the crack tip $x > pt$, $y = 0$ are

$$\tau_{yz} = C\sqrt{1-p^2/c^2}\,/\sqrt{d},$$
$$[\![u]\!] = 2C\sqrt{d}, \tag{3.8}$$

where d is the distance from the crack tip. In each case, when the displacement jump and stress τ_{yz} satisfy (even locally) Eq. (3.8), the coefficient C will be called the crack strength.

Let us pass to the calculation of the crack speed p. Consider the strip shown in Fig. 4. The work done by the external forces in time δt equals (stress $\to 0$ for $|x| \to \infty$)

$$L = \lim_{h\to 0} 2\int_{-\infty}^{\infty} dx\left(\mu\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial t}\,\delta t\right)\Big|_{y=h} = C^2 p\pi\mu\delta t\sqrt{1-p^2/c^2}, \tag{3.9}$$

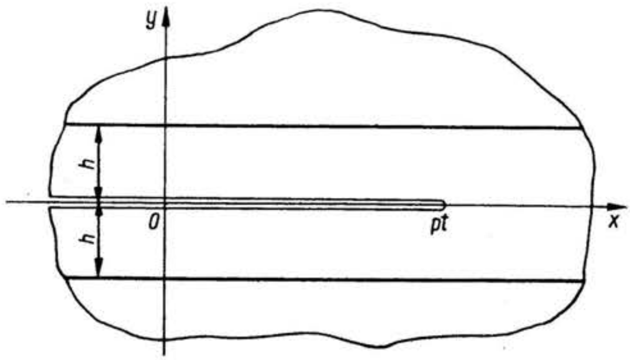

FIG. 4.

is used to produce a crack of length $p\delta t$. If γ denotes the energy necessary to produce a unit area of crack, the energy needed in δt equals $\gamma p \delta t$. The balance equation leads to the formula for the propagation speed

$$\frac{p^2}{c^2} = 1 - \frac{\gamma^2}{\pi^2 \mu^2 C}. \tag{3.10}$$

The propagation speed increases if the crack strength increases. The maximum propagation speed equals the sound speed c and corresponds to infinite crack strength. There exists minimum crack strength setting the crack in motion, namely

$$C_{min} = \sqrt{\frac{\gamma}{\pi\mu}}. \tag{3.11}$$

To the static case there corresponds $p = 0$. Denoting the displacement by u_0, in accord with Eqs. (3.4) and (3.7), we have (for the crack tip situated at $x = 0$)

$$u_0 = C_0 \sqrt{2}\,\mathrm{sign}\,y \sqrt{\sqrt{x^2+y^2}-x}, \tag{3.12}$$

$$\tau_{yz} = \begin{cases} 0 & \text{for} \quad x < 0, \quad y = 0, \\ C_0/\sqrt{d} & \text{for} \quad x > 0, \quad y = 0. \end{cases} \tag{3.13}$$

The full expressions for τ_{xz} and τ_{yz} may be obtained form Eqs. (3.6) and (3.7) by setting $t = 0$.

4. Approximate theory of the crack motion in finite regions

Due to the time changes of the boundary conditions and the reflections of waves produced by fracture, the crack speed p is a function of time t, $p = p(t)$. In order to find an approximate solution, assume that $p(t)$ is a piecewise constant function of time. Denote by $s_1, s_2, \ldots, s_n \ldots$ fixed points on the x-axis, and by $t_1, t_2, \ldots, t_n, \ldots$ the instants at which the crack tip is situated at $s_1, s_2, \ldots, s_n, \ldots$. Assume

$$s_0 \equiv 0 < s_1 < s_2 < \ldots < s_{n-1} < s_n < \ldots, \tag{4.1}$$

(4.2) $$t_0 \equiv 0 < t_1 < t_2 < \ldots < t_{n-1} < t_n < \ldots,$$

(4.3) $$p_n = \frac{s_n - s_{n-1}}{t_n - t_{n-1}} = \text{const} < c.$$

Denote by $\tilde{u}_m$ the actual displacement field at t_m and by u_m the additional displacement due to the fracture from s_{m-1} to s_m. In accordance with Eq. (2.5),

(4.4) $$u_m = \int\limits_{s_{m-1}}^{\bar{s}_m} ds \frac{B_m}{\sqrt{2}} \operatorname{sign} y \frac{\sqrt{\sqrt{(x-s)^2+y^2}-(x-s)}}{\sqrt{(x-s)^2+y^2}},$$

where B_m is a constant. Integration of Eq. (4.4) leads to

(4.5) $$u_m = B_m \sqrt{2} \operatorname{sign} y \sqrt{\sqrt{(x-s)^2+y^2-(x-s)}}\Big|_{s_{m-1}}^{\bar{s}_m}.$$

The latest wave contributing to u_n is centered at $\bar{s}_n$. The algebraic equation for $\bar{s}_m$

(4.6) $$\sqrt{(x-\bar{s}_m)^2+y^2} = c\left(t - t_{m-1} - \frac{\bar{s}_m - s_{m-1}}{p_m}\right)$$

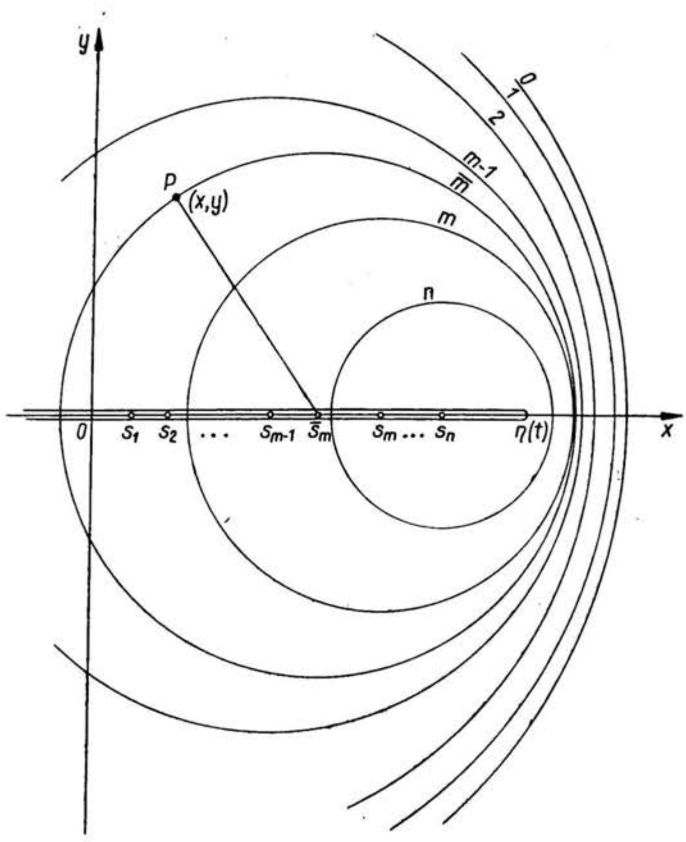

FIG. 5.

(cf. Eqs. (2.3) and (3.2)) has the solution

$$(4.7)\qquad \bar{s}_m - s_{m-1} = -\frac{p_m^2}{c^2-p_m^2}\left[(x-s_{m-1}) - \frac{c^2}{p_m}(t-t_{m-1}) + \frac{c}{p_m}\sqrt{[(x-s_{m-1})-p(t-t_{m-1})]^2+\left(1-\frac{p_m^2}{c^2}\right)y^2}\right].$$

If $\bar{s}_m$ as calculated from the above formula is larger than s_m, $\bar{s}_m > s_m$, it means that all the waves produced in the n-th interval contribute to u_m and in Eq. (4.6) $\bar{s}_m = s_m$ should be taken.

Figure 5 shows the fronts of elementary waves produced at $0, s_1, \ldots, s_n, \ldots$. At the typical point P the additional displacement due to the fracture from s_0 to s_n equals

$$(4.8)\qquad u(P) = u_1+u_2+\ldots+u_m, \quad m \leqslant n.$$

The displacement $u(P)$ is influenced by all the waves produced in the intervals $1, 2, \ldots$ $\ldots, m-1$; therefore

$$(4.9)\qquad \bar{s}_1 = s_1, \quad \bar{s}_2 = s_2, \ldots, \quad \bar{s}_{m-1} = s_{m-1}.$$

The latest wave contributing to $u(P)$ is centered at $\bar{s}_m$ given by the formula (4.7). In accordance with the above, the contributions from the intervals $1, 2, \ldots, m-1, m$ are the following:

$$(4.10)\qquad u_k = B_k\sqrt{2}\,\mathrm{sign}\,y\sqrt{\sqrt{(x-s_k)^2+y^2}-(x-s_k)} - B_k\sqrt{2}\,\mathrm{sign}\,y\sqrt{\sqrt{(x-s_{k-1})^2+y^2}-(x-s_{k-1})},$$
$$k < m,$$

$$(4.11)\qquad u_m = \frac{B_m\sqrt{2}}{\sqrt{1+p_m/c}}\,\mathrm{sign}\,y\sqrt{\sqrt{\xi^2+(1-p_m^2/c^2)y^2}-\xi} - B\sqrt{2}\,\mathrm{sign}\,y\sqrt{\sqrt{(x-s_{m-1})^2+y^2}-(x-s_{m-1})},$$
$$\xi = x-s_{n-1}-p_n(t-t_{n-1}).$$

In both cases (4.10) and (4.11) the stresses τ_{xz} and τ_{yz} may be calculated from Eq. (3.6).

Of particular importance is the stress τ_{yz} corresponding to points on the x-axis in front of the crack tip. At the instant t_{m-1} the crack tip is situated at $x = s_{m-1}$ and the stress τ_{yz} for $x > s_{m-1}$, $y = 0$ is (cf. e.g. [4])

$$(4.12)\qquad \tau_{yz} = \frac{D_{m-1}}{\sqrt{x-s_{m-1}}} + O(|x-s_{m-1}|).$$

The coefficient D_{m-1} is influenced by u_0, all the waves produced between 0 and s_{m-1}, reflections and refractions of these waves, double reflections etc. Finally, it is influenced by changes of the boundary loads, possessing again the form of the waves. Only computational difficulties are involved when calculating D_{m-1}. For $t > t_{m-1}$ to the stress (4.12) are added the stresses due to the waves produced in the interval $s_{m-1} < s < \eta(t)$. Figure 6 shows the wave fronts for $t_{m-1} < t < t_m$. The displacement u_m at point P' is

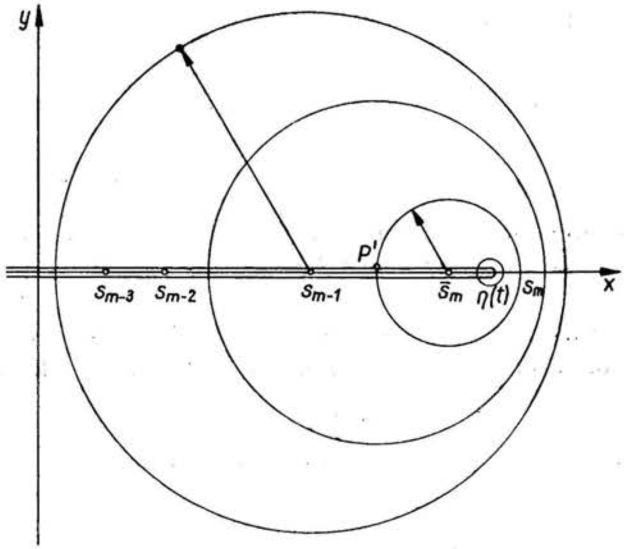

FIG. 6.

given by the formula (4.11). The first term does not influence the stress τ_{yz} at P'. The second term gives

$$\tau_{yz} = -\frac{B_m}{\sqrt{x-s_{m-1}}}. \tag{4.13}$$

The crack is stress free, therefore the stress (4.13) must annihilate the already existing stress (4.12). It follows that the coefficients B_m and D_m are connected by the formula

$$B_m = D_{m-1}. \tag{4.14}$$

In order to complete the solution, the formula must be obtained for p_m. Guided by the procedure exposed in Sect. 3, calculate first the work L_m done by the external forces in time t, provided the crack tip is situated between s_{m-1} and s_m

$$L_m = \lim_{h\to 0} 2\int_{-\infty}^{\infty} dx \left(\mu\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial t}\delta t\right)\Bigg|_{y=h} \tag{4.15}$$

due to the fact that

$$\lim_{h\to 0}\frac{\partial u}{\partial y}\Bigg|_{y=h,\, x<\eta(t)} = 0,$$
$$\lim_{h\to 0}\frac{\partial u}{\partial t}\Bigg|_{y=h,\, x>\eta(t)} = 0, \tag{4.16}$$

the work L is given by the formula, exactly following the formula (3.9)

$$L_m = \frac{B_m^2}{1+p_m/c}p_m\pi\mu\delta t\sqrt{1-p_m^2/c^2}. \tag{4.17}$$

This work has to be equal to $\gamma p_m \delta t$ where γ is the Griffith energy. Therefore,

$$\frac{p_m}{c} = \frac{B_m^4 \mu^2 \pi^2 - \gamma^2}{B_m^4 \mu^2 \pi^2 + \gamma^2}. \tag{4.18}$$

From the above analysis an easy approximate treatment of the dynamic crack propagation follows. The treatment is based on the following steps:

1. Start with the known solution $\overset{(n-1)}{u}(x, y, t)$.
2. Find from Eq. (4.12) the crack strength.
3. Find from Eq. (4.14) the coefficient B_m and from Eq. (4.18) the crack speed p_m.
4. Find from Eq. (4.11) the additional displacement u_n.
5. Take into account reflections and refractions of $u_1, u_2, \dots, u_n$ and calculate $\overset{(n)}{u}$ according to the formula $u = u_0 + u_1 + 1 \dots + u_n +$ reflected, refracted, doubly reflected ... waves produced between 0 and s_n.
6. Repeat points 1–5 for n replaced by $n+1$.

References

1. WESOŁOWSKI, *Brittle fracture as a wave*, Arch. Mech. Stos., **29**, 3, 1977.
2. J. D. ESHELBY, *The elastic field of a crack extending non-uniformly under general anti-plane loading*, J. Mech. Phys. Solids, **17**, 3, 1969.
3. J. F. KALTHOFF, J. BEINERT, S. WINKLER, *Measurements of dynamic stress intensity factors for fast running and arresting cracks in double-cantilevar-beam specimens*, Special Techn. Publ. 627, American Society for Testing and Materials.
4. M. SOKOŁOWSKI, *Stress concentration in an elastic wedge*, IFTR Reports 31, 1978.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received March 4, 1979.