Arch. Mech., 41, 5, pp. 641-649, Warszawa 1989

On the equilibrium theory of second grade fluids (*)

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THE CLASSICAL theory of fluids is not sufficiently general to include phenomena such as capilla6 rity and phase transitions. For this reason, among many, fluids of second grade were introduced; such a fluid is defined by a constitutive equation in which the stored energy depends not only on the density, but also on the density gradient. In this note we use the principle of virtual work to develop a consistent set of field equations and boundary conditions for such fluids. The theory shows the presence of surface tensions on the external boundary and gives a physically acceptable interpretation of the "hypertraction".

Klasyczna teoria cieczy nie jest dostatecznie ogólna na to, by objąć takie zjawiska jak włoskowatość lub przejścia fazowe; był to jeden z powodów wprowadzenia cieczy drugiego rzędu: ciecze te opisane są równaniami konstytutywnymi w których zmagazynowana energia zależy nie tylko od gęstości, ale i od jej gradientu. W pracy wykorzystuje się zasadę prac wirtualnych dla zbudowania spójnego układu równań pola i warunków brzegowych dla takich cieczy. Teoria pozwala wykazać istnienie napięć powierzchniowych na granicy zewnętrznej cieczy i podać fizycznie uzasadnioną interpretację "hiperprzyciągania".

Классическая теория жидкостей не является достаточно общей, чтобы описать такие явления как капиллярность или фазовые переходы; это одна из причин введения жидкости второго порядка: эти жидкости описаны определяющими уравнениями, в которых накопленная энергия зависит не только от плотности, но и от ее градиента. В работе используется принцип виртуальных работ для построения связной системы уравнений поля и граничных условий для таких жидкостей. Теория позволяет показать существование поверхностных напряжений на внешней границе жидкости и привести физически обоснованную интерпретацию ,,гиперпритяжения".

1. Second-grade fluids. Virtual work

IT IS THE PURPOSE of this note to develop a consistent set field equations and boundary conditions for the equilibrium of second-grade fluids; that is, fluids whose stored energy W, per unit mass, is a function of the density ϱ and its spatial gradient, grad ϱ.

Constitutive equations involving density gradients go back to the work of KORTEWEG [1], and, more recently, to the work of CAHN and HILIARD [2], FIXMAN [3] and FELDERHOF [4] (Cf. ROWLINSON [5], DAVIS and SCRIVEN [6] and DUNN and SERRIN [7] for selected references).

Since W is necessarily isotropic, this functional relationship may be written in the form

$$W(\varrho, M), \qquad M := |\operatorname{grad}\varrho|^2.$$

Following TOUPIN [8, 9], we use the principle of virtual work (PVW) to deduce the underlying equations and boundary conditions. Many, but not all, of our results are simply

(*) This work was supported in part by the Army Research Office (USA) when the author was a Ph. D, student at Carnegie-Mellon University.

specializations of equations of Toupin. However, our derivation, which utilizes a spatial (Eulerian) version of PVW, is, for fluids, simpler and more transparent than the derivation of Toupin, which utilizes the material (Lagrangian) description.

More interestingly, our results lead to the deduction of boundary conditions containing a term which can be naturally interpreted as a surface tension. This should be compared with other attempts of using second grade fluids for the description of capillarity phenomena [10, 11].

We assume that there are no other forms of energy; then, for any configuration $\varkappa$ of the fluid, the total energy $E_\varkappa$ is given by

$$E_\varkappa = \int_\varkappa \varrho W(\varrho, M) \tag{1}$$

with $\varrho = \varrho_\varkappa$. Of course, the density in any two configurations is related through conservation of mass.

Here and in what follows we tacitly assume that all integrals over $\varkappa(\mathscr{B})$ are taken with respect to the ordinary volume measure in the three-dimensional Euclidean space, while all integrals over $\partial\varkappa(\mathscr{B})$ are taken with respec to tthe induced surface measure. Moreover, for simplicity of notation, we shall frequently write $\varkappa$ and $\partial\varkappa$ for $\varkappa(\mathscr{B})$ and $\partial\varkappa(\mathscr{B})$.

To formulate the principle of virtual work (cf., e.g., TRUESDELL and TOUPIN [12]), we make use of the fact that the family of configurations of $\mathscr{B}$ has the structure of a smooth manifold (cf., e.g., [13]) and use the term *virtual displacement* $\mathbf{u}$ from $\varkappa$ for an element of the tangent space at $\varkappa$. For any such $\mathbf{u}$ we define the total variation of E, written $\delta E_\varkappa(\mathbf{u})$, as the derivative of E at $\varkappa$ in the direction $\mathbf{u}$. Here we assume that, for every configuration $\varkappa(\mathscr{B})$, $\partial\varkappa(\mathscr{B})$ is a smooth closed surface. This is crucial to the derivation of Eq. (5) and, for simplicity, we take $\partial\varkappa(\mathscr{B})$ to be of class C^∞. We comment later on this restriction.

For future use we note that, for all practical considerations, we can simply think of $\mathbf{u}$ as a C^∞ "velocity" field on $\varkappa(\mathscr{B})$, and, for any quantity $\psi_\varkappa$, we can compute $\delta\psi_\varkappa$ as the material rate of change of $\psi_\varkappa$ corresponding to $\mathbf{u}$.

The PVW postulates the body to be in equilibrium in the configuration $\varkappa$ if and only if $\delta E_\varkappa(\mathbf{u})$ is equal to the total work $L_\varkappa(\mathbf{u})$ exerted over $\mathscr{B}$ by the external forces. We assume that this work has the form

$$L_\varkappa(\mathbf{u}) = \int_\varkappa \mathbf{b}_\varkappa \cdot \mathbf{u} + \int_{\partial\varkappa} \mathbf{t}_\varkappa \cdot \mathbf{u} + \int_{\partial\varkappa} \mathbf{h}_\varkappa \cdot D\mathbf{u}, \tag{2}$$

where the physical interpretation of the first two terms is classical: $\mathbf{b}_\varkappa$ is the body force per unit mass; $\mathbf{t}_\varkappa$ is the surface traction in the configuration $\varkappa$. The remaining term, with D the normal derivative on $\partial\varkappa(\mathscr{B})$, expresses the possibility of higher-order interactions on the boundary through the introduction of a "hypertraction" $\mathbf{h}_\varkappa$.

Denoting by $\mathscr{V}_\varkappa$ the class of virtual displacements from $\varkappa(\mathscr{B})$ and using the principle of virtual work, together with the definitions (1) and (2), we say that the body is in equilibrium in the configuration $\varkappa$ if and only if

$$\delta E_\varkappa(\mathbf{u}) = L_\varkappa(\mathbf{u}) \qquad \forall \mathbf{u} \in \mathscr{V}_\varkappa. \tag{PVW}$$

REMARK. DUNN and SERRIN [7] have given a complete thermodynamical theory of second grade materials. Their theory masterfully bypasses the well-known obstacle, the incompatibility (cf. COLEMAN and MIZEL [14], GURTIN [15]) between the laws of thermodynamics and the dependence of the stress tensor on higher deformation gradients, by introducing an additional term, the "interstitial working", into the clyssical version of the first law. As one would expect, our results are in some sense related to those of Dunn and Serrin, even if boundary conditions show a meaningful difference.

2. Consequences of the principle of virtual work

We first introduce terminology and establish some preliminary identities (cf., e.g., [16]). We write grad and div for the usual gradient and divergence in R^3. We choose a configuration $\varkappa$ and hold it fixed; for the surface $\partial\varkappa(\mathscr{B})$, $\mathbf{n}$ is the outward unit normal, H the mean curvature, D the normal derivative, ∇_s the surface gradient, and div_s the surface divergence. Then, for any vector field $\mathbf{u}$ on $\varkappa(\mathscr{B})$ and any scalar field λ on $\partial\varkappa(\mathscr{B})$,

$$\mathrm{div}_s\mathbf{u} = \mathrm{div}\,\mathbf{u} - D\mathbf{u}\cdot\mathbf{n}, \tag{3}$$

$$\mathrm{div}_s(\lambda\mathbf{u}) = \lambda\,\mathrm{div}_s\mathbf{u} + \nabla_s\lambda\cdot\mathbf{u} \tag{4}$$

on $\partial\varkappa(\mathscr{B})$, and

$$\int_{\partial\varkappa} \mathrm{div}_s\mathbf{u} = -2\int_{\partial\varkappa} H\mathbf{u}\cdot\mathbf{n}. \tag{5}$$

Moreover, denoting by $\mathbf{I}$ the identity tensor on the tangent space of $\partial\varkappa(\mathscr{B})$ we have

$$\mathrm{div}_s(\lambda\mathbf{I}) = 2H\lambda\mathbf{n} + \nabla_s\lambda. \tag{6}$$

We list three identities, easily verifiable using the divergence theorem in which ϕ, $\mathbf{u}$ and $\mathbf{d}$ are functions defined on $\varkappa(\mathscr{B})$:

$$\begin{aligned}
\int_{\varkappa} \phi\,\mathrm{div}\,\mathbf{u} &= \int_{\partial\varkappa} \phi\mathbf{u}\cdot\mathbf{n} - \int_{\varkappa} \mathrm{grad}\,\phi\cdot\mathbf{u},\\
\int_{\varkappa} \phi(\mathrm{grad}\,\mathbf{u})\mathbf{d}\cdot\mathbf{d} &= \int_{\partial\varkappa} \phi(\mathbf{d}\otimes\mathbf{d})\mathbf{n}\cdot\mathbf{u} - \int_{\varkappa} \mathrm{div}(\phi\mathbf{d}\otimes\mathbf{d})\cdot\mathbf{u},\\
\int_{\varkappa} \phi\mathbf{d}\cdot\mathrm{grad}\,\mathrm{div}\,\mathbf{u} &= \int_{\partial\varkappa} \phi(\mathrm{div}\,\mathbf{u})\mathbf{d}\cdot\mathbf{n} - \int_{\varkappa} (\mathrm{div}\,\mathbf{u})\,\mathrm{div}(\phi\mathbf{d}).
\end{aligned} \tag{7}$$

We may use conservation of mass to derive the relations

$$\begin{aligned}
\delta\varrho_\varkappa(\mathbf{u}) &= -\varrho_\varkappa\,\mathrm{div}\,\mathbf{u},\\
\delta(\mathrm{grad}\,\varrho_\varkappa)(\mathbf{u}) &= -(\mathrm{grad}\,\mathbf{u})^T\mathrm{grad}\,\varrho_\varkappa - (\mathrm{div}\,\mathbf{u})\,\mathrm{grad}\,\varrho_\varkappa - \varrho_\varkappa\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}.
\end{aligned} \tag{8}$$

Analogously, for

$$e_\varkappa := \int_{\partial\varkappa} \psi_\varkappa$$

we have

$$\delta e_\varkappa(\mathbf{u}) = \int_{\partial\varkappa} [\delta\psi_\varkappa(\mathbf{u}) + \psi_\varkappa\,\mathrm{div}_s\mathbf{u}].$$

For the remainder of the paper we shall systematically omit the subscript $\varkappa$ for ϱ, **b**, **t**, **h** and other related quantities; besides, we shall repeatedly use the notation

$$\mathbf{d} := \operatorname{grad}\varrho$$

(so that $M := \mathbf{d}\cdot\mathbf{d}$). Also, partial derivatives with respect to ϱ or M will be denoted by subscripts:

$$\psi_\varrho(\varrho, M) = \frac{\partial\psi(\varrho, M)}{\partial\varrho}, \quad \psi_M(\varrho, M) = \frac{\partial\psi(\varrho, M)}{\partial M}.$$

Our next step is to express the variation of the energy

$$E = \int_\varkappa \varrho W(\varrho, M).$$

To do so we need δM, which is easily derived from Eq. $(8)_2$:

$$\delta M(\mathbf{u}) = -2(\operatorname{grad}\mathbf{u})\mathbf{d}\cdot\mathbf{d} - 2M\operatorname{div}\mathbf{u} - 2\varrho\mathbf{d}\cdot\operatorname{grad}\operatorname{div}\mathbf{u}. \tag{9}$$

Clearly

$$\delta E = \int_\varkappa \varrho\delta W(\varrho, M) = \int_\varkappa (\varrho W_\varrho \delta\varrho + \varrho W_M \delta M).$$

(Here and in what follows we will often omit the argument **u**). Substitution from Eqs. $(8)_1$ and (9) yields

$$\delta E = \int_\varkappa [W_\varrho(-\varrho^2\operatorname{div}\mathbf{u}) - 2\varrho W_M(\operatorname{grad}\mathbf{u})\mathbf{d}\cdot\mathbf{d} - 2\varrho W_M(\operatorname{div}\mathbf{u})\mathbf{d}\cdot\mathbf{d} - 2\varrho^2 W_M\mathbf{d}\cdot\operatorname{grad}\operatorname{div}\mathbf{u}],$$

and, if we apply Eqs. (7) in the obvious way, we arrive at

$$\begin{aligned}\delta E = \int_\varkappa [\operatorname{grad}(\varrho^2 W_\varrho + 2\varrho W_M M) + \operatorname{div}(2\varrho W_M \mathbf{d}\otimes\mathbf{d}]\cdot\mathbf{u} \\ - \int_{\partial\varkappa} [\varrho^2 W_\varrho \mathbf{n} + 2\varrho W_M(\mathbf{d}\otimes\mathbf{d})\mathbf{n} + 2\varrho W_M M\mathbf{n}]\cdot\mathbf{u} \\ - \int_{\partial\varkappa} 2\varrho^2 W_M(\operatorname{div}\mathbf{u})\mathbf{d}\cdot\mathbf{n} + \int_\varkappa (\operatorname{div}\mathbf{u})\operatorname{div}(2\varrho^2 W_M\mathbf{d}).\end{aligned} \tag{10}$$

Use of Eq. $(7)_1$ yields the equality

$$\int_\varkappa (\operatorname{div}\mathbf{u})\operatorname{div}(2\varrho^2 W_M\mathbf{d}) = \int_{\partial\varkappa} \operatorname{div}(2\varrho^2 W_M\mathbf{d})\mathbf{u}\cdot\mathbf{n} - \int_\varkappa \operatorname{grad}\big(\operatorname{div}(2\varrho^2 W_M\mathbf{d})\big)\cdot\mathbf{u}. \tag{11}$$

Defining

$$\lambda := -2\varrho^2 W_M D\varrho \quad \text{on } \partial\varkappa(\mathscr{B}), \tag{12}$$

Equations (3), (4) and (5) yield

$$\int_{\partial\varkappa} 2\varrho^2 W_M(\mathbf{d}\cdot\mathbf{n})\operatorname{div}\mathbf{u} = \int_{\partial\varkappa} 2\lambda H\mathbf{n}\cdot\mathbf{u} + \int_{\partial\varkappa} \nabla_s\lambda\cdot\mathbf{u} - \int_{\partial\varkappa} \lambda D\mathbf{u}\cdot\mathbf{n}. \tag{13}$$

With the definition

$$c := 2\varrho W_M,$$

and by Eqs. (11) and (13), Eqs. (10) becomes

$$(14)\qquad \delta E = \int_{\varkappa} \operatorname{div}[\varrho^2 W_\varrho \mathbf{I} + cM\mathbf{I} - \operatorname{div}(\varrho c\mathbf{d})\mathbf{I} + c\mathbf{d}\otimes\mathbf{d}]\cdot\mathbf{u}$$
$$- \int_{\partial\varkappa} [\varrho^2 W_\varrho \mathbf{I} + cM\mathbf{I} - \operatorname{div}(\varrho c\mathbf{d})\mathbf{I} + c\mathbf{d}\otimes\mathbf{d}]\mathbf{n}\cdot\mathbf{u} + (2\lambda H\mathbf{n} + \nabla_s\lambda)\cdot\mathbf{u} + \int_{\partial\varkappa} \lambda\mathbf{n}\cdot D\mathbf{u},$$

where $\mathbf{I}$ is the identity tensor in the Euclidean space R^3. The expression (14) suggests the introduction of a tensor $\mathbf{T}$ defined as

$$(15)\qquad \mathbf{T} := -[\varrho^2 W_\varrho + cM - \operatorname{div}(\varrho c\mathbf{d})]\mathbf{I} - c\mathbf{d}\otimes\mathbf{d},$$

so that, finally,

$$(16)\qquad \delta E(\mathbf{u}) = \int_{\varkappa} (-\operatorname{div}\mathbf{T})\cdot\mathbf{u} + \int_{\partial\varkappa} (\mathbf{T}\mathbf{n} - 2\lambda H\mathbf{n} - \nabla_s\lambda)\cdot\mathbf{u} + \int_{\partial\varkappa} \lambda\mathbf{n}\cdot D\mathbf{u}.$$

We now state a useful result:

Lemma. Let $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{c}$ be smooth vector fields defined, respectively, on $\varkappa(\mathscr{B})$ and on $\partial\varkappa(\mathscr{B})$.

Assume that

$$(\mathrm{L})\qquad \int_{\varkappa} \mathbf{a}\cdot\mathbf{u} + \int_{\partial\varkappa} (\mathbf{b}\cdot\mathbf{u} + \mathbf{c}\cdot D\mathbf{u}) = 0 \qquad \forall\mathbf{u}\in\mathscr{V}_\varkappa;$$

then

$$\mathbf{a} = \mathbf{b} = \mathbf{c} = 0.$$

This Lemma can be derived using classical techniques of calculus of variations: for the sake of completeness we give a formal proof in the Appendix.

It is important to point out that this result provides a rationale for choosing the form of the external work as given by Eq. (2): indeed, the Lemma is sometimes loosely stated saying that "the fields $\mathbf{u}$ and $D\mathbf{u}$ are independent over $\partial\varkappa(\mathscr{B})$" (cf., e.g., [8, 17, 18]).

The physical idea underlying this requirement is in fact that the virtual work should be *independently* exerted by $\mathbf{b}$, $\mathbf{t}$ and $\mathbf{h}$.

Combining Eq. (16) with PVW and using the Lemma, we obtain that:

The body is in equilibrium in the configuration $\varkappa$ if and only if the following set of equations is satisfied:

$$(17)\qquad \begin{cases} \text{in } \varkappa(\mathscr{B}) & \operatorname{div}\mathbf{T} + \mathbf{b} = 0, \\ \text{on } \partial\varkappa(\mathscr{B}) & \begin{cases} \mathbf{T}\mathbf{n} - 2H\lambda\mathbf{n} - \nabla_s\lambda = \mathbf{t}, \\ \lambda\mathbf{n} = \mathbf{h}. \end{cases} \end{cases}$$

It is natural to think of $\mathbf{T}$, defined in Eq. (14), as the stress tensor: we note that $\mathbf{T}$ is *symmetric* and that its role in Eq. $(17)_1$ is classical. In view of this result we feel that some considerations contained in Dunn and Serrin [7, p. 106] may require appropriate modifications.

The boundary condition $(17)_2$, which contains an interesting dependence on the mean curvature H, by use of Eq. (6) may be rewritten as

$$\operatorname{div}_s(\lambda\mathbf{I}) + \mathbf{t} - \mathbf{T}\mathbf{n} = 0.$$

Comparison with boundary conditions for bodies with surface stresses [16, pp. 304–306] suggests at once the possibility of interpreting λ as a *surface tension*. This observation, together with Eq. $(17)_3$ gives an interesting "physical" meaning to the hypertraction vector $\mathbf{h}$. Of course, at this stage, this interpretation is rather formal but, nevertheless, we believe it to be of some interest.

We may consider the scalar appearing in the first part of Eq. (15) as a "generalized" pressure:

$$\mathbf{T} = -p\mathbf{I} - c\mathbf{d}\otimes\mathbf{d}.$$

Then

$$p = \varrho^2 W_\varrho - \varrho c_\varrho M - \varrho c\Delta\varrho - 2\varrho c_M \operatorname{grad}^2\varrho \cdot \mathbf{d}\otimes\mathbf{d}$$

(where Δ is the Laplacian and grad^2 is the second gradient), so that $\mathbf{T}$ has exactly the form found by DUNN and SERRIN [7, p. 104] provided we identify W with their Helmholtz free energy. On the other hand, they have only classical boundary conditions (i.e., $\mathbf{Tn} = \mathbf{t}$), which, in our scheme, represent the special case obtained assuming

$$\mathbf{h} = 0, \tag{18}$$

on $\partial\varkappa(\mathscr{B})$. Also, it is important to point out that Eq. (18) together with Eqs. $(17)_3$ and (12) imply

$$D\varrho = 0 \quad \text{on } \partial\varkappa(\mathscr{B}),$$

a boundary condition *not* apparent from the work of DUNN and SERRIN [7].

The removal of the restriction placed by the assumption that $\partial\varkappa(\mathscr{B})$ be smooth is of some interest: indeed, a weakening of this hypothesis leads to the introduction of edge forces (i.e., line distribution of forces over the edges of the body) and other mechanical quantities (cf. [8]). In the case of fluids these results would seem to strengthen the role of λ as a surface tension.

Appendix

Proof of Lemma: Suppose that $\mathbf{a}(\mathbf{x}_0) \neq 0$ at some $\mathbf{x}_0$ in the interior of $\partial\varkappa(\mathscr{B})$: because of continuity and with no loss of generality we may assume, in a neighborhood of $\mathbf{x}_0$, the positiveness of a_1, the Cartesian component corresponding to the unit vector $\mathbf{e}_1$.

Choose $\alpha > 0$ such that $\Sigma_\alpha(\mathbf{x}_0)$, the open ball of radius α centered at $\mathbf{x}_0$, is contained within this neighborhood and consider a virtual displacement given by $\mathbf{u} := \varphi\mathbf{e}_1$ where $\varphi: \varkappa(\mathscr{B}) \to R$ is defined as

$$\varphi(\mathbf{x}) := \begin{cases} \exp[1/(|\mathbf{x}-\mathbf{x}_0|^2-\alpha^2)], & \mathbf{x}\in\Sigma_\alpha(\mathbf{x}_0), \\ 0, & \text{elsewhere}. \end{cases}$$

It is not difficult to see that φ and $\mathbf{u}$ belong to $C^\infty(\varkappa(\mathscr{B}))$. Moreover, for α sufficiently small, $\mathbf{u} = D\mathbf{u} = 0$ on $\partial\varkappa(\mathscr{B})$ and, since $\varphi > 0$ within $\Sigma_\alpha(\mathbf{x}_0)$,

$$\int_\varkappa \mathbf{a}\cdot\mathbf{u} + \int_{\partial\varkappa} (\mathbf{b}\cdot\mathbf{u} + \mathbf{c}\cdot D\mathbf{u}) = \int_{\Sigma_\alpha(\mathbf{x}_0)} \mathbf{a}\cdot\mathbf{u} = \int_{\Sigma_\alpha(\mathbf{x}_0)} a_1\varphi > 0.$$

By contradiction, we are forced to conclude that $\mathbf{a} = 0$ over $\varkappa(\mathscr{B})$. Thus (L) is reduced to the assuption that

$$\text{(L}'\text{)}\qquad \int_{\partial\varkappa} (\mathbf{b}\cdot\mathbf{u}+\mathbf{c}\cdot D\mathbf{u}) = 0 \qquad \forall \mathbf{u}\in\mathscr{V}_{\varkappa}.$$

Suppose that $\mathbf{b}(\mathbf{x}_0) \neq 0$ at some $\mathbf{x}_0 \in \partial\varkappa(\mathscr{B})$: as before, we may assume $b_1 > 0$ on $\mathscr{N}$, a neighborhood of $\mathbf{x}_0$ within $\partial\varkappa(\mathscr{B})$.

Let $\chi:\mathscr{W} \to R^3$ be a local system of curvilinear coordinates defined on an appropriate space neighborhood $\mathscr{W}$ of $\mathbf{x}_0$, such that at all points in the intersection $\mathscr{W}\cap\partial\varkappa(\mathscr{B})$ the local basis vector $\mathbf{g}_3$ coincides with the outward unit normal $\mathbf{n}$. Moreover, we take $\mathscr{W}$ such that this intersection is contained in $\mathscr{N}$ and is mapped by χ onto an open region of the xy plane, with $\chi(\mathbf{x}_0) = (0, 0, 0)$. The existence of such a coordinate system for a C^∞ boundary $\partial\varkappa(\mathscr{B})$ can be derived using standard results of differential geometry (cf., e.g., [19]).

Let $\mathscr{W}^- := \mathscr{W}\cap\varkappa(\mathscr{B})$ and choose $\alpha > 0$ such that the solid hemisphere

$$S_\alpha^- := \{(x, y, z)\in R^3 | x^2+y^2+z^2 < \alpha^2, z \leqslant 0\}$$

is contained in $\chi(\mathscr{W}^-)$. Moreover, we denote by D_α the intersection of S_α^- with the xy plane and we define

$$\mathscr{N}_\alpha := \chi^{-1}(D_\alpha).$$

Obviously, $\chi(D_\alpha) = \mathscr{N}_\alpha$ and $\mathscr{N}_\alpha \subset \mathscr{N} \subset \partial\varkappa(\mathscr{B})$.

Let $R^3_- := \{(x, y, z)\in R^3 | z \leqslant 0\}$ and define $\tilde\varphi: R^3_- \to R$ as

$$\tilde\varphi(x, y, z) := \begin{cases} \exp[1/(x^2+y^2+z^2-\alpha^2)], & (x, y, z)\in S_\alpha^-, \\ 0, & \text{elsewhere.}\end{cases}$$

We can easily prove that:

i) $\qquad \tilde\varphi \in C^\infty(R^3_-);$

ii) $\qquad \tilde\varphi > 0 \quad \text{on } S_\alpha^-,$

$\qquad \tilde\varphi = 0 \quad \text{on } R^3_-\setminus S_\alpha^-;$

iii) $\qquad \partial_z\varphi|_{z=0} = 0.$

Now let $\mathbf{u} := \varphi\mathbf{e}_1$, where

$$\text{(A.1)}\qquad \varphi(x) := \begin{cases} \tilde\varphi(\chi(\mathbf{x})), & \mathbf{x}\in\mathscr{W}^- \\ 0, & \mathbf{x}\in\varkappa(\mathscr{B})\setminus\mathscr{W}^-.\end{cases}$$

The listed properties of $\tilde\varphi$ imply:

j) $\qquad \mathbf{u}\in\mathscr{V}_\varkappa;$

jj) $\qquad D\mathbf{u} = 0 \quad \text{on } \partial\varkappa(\mathscr{B});$

jjj) $\qquad \mathbf{u} = 0 \quad \text{on } \partial\varkappa(\mathscr{B})\setminus\mathscr{N}_\alpha.$

Observing that $\mathbf{b}\cdot\mathbf{u} = b_1\varphi > 0$ on $\mathscr{N}_\alpha$, we deduce

$$\int_{\partial\varkappa} (\mathbf{b}\cdot\mathbf{u}+\mathbf{c}\cdot D\mathbf{u}) = \int_{\mathscr{N}_\alpha} \mathbf{b}\cdot\mathbf{u} = \int_{\mathscr{N}_\alpha} b_1\varphi > 0.$$

Thus, by contradiction, we are forced to conclude that $\mathbf{b} = 0$ on $\partial\varkappa(\mathscr{B})$, and (L) is further reduced to

$$\text{(L}''\text{)} \qquad \int_{\partial\varkappa} \mathbf{c}\cdot D\mathbf{u} = 0 \qquad \forall \mathbf{u}\in\mathscr{V}_{\varkappa}.$$

Again we suppose that $\mathbf{c}(\mathbf{x}_0) \neq 0$ at some $\mathbf{x}_0 \in \partial\varkappa(\mathscr{B})$: with no loss of generality we may assume that $c_1 > 0$ in a neighborhood $\mathscr{N}$ of $\mathbf{x}_0$ in $\partial\varkappa(\mathscr{B})$.

Let α, χ, $\mathscr{W}$, $\mathscr{W}^-$, S_α^-, D_α and $\mathscr{N}_\alpha$ be defined as before. For some strictly positive $\varepsilon < \alpha$, let $\tilde{\varphi}: R_-^3 \to R$ be given by

$$\text{(A.2)} \qquad \tilde{\varphi}(x,y,z) := \begin{cases} \exp[1/(x^2+y^2+(z-\varepsilon)^2-\alpha^2)], & x^2+y^2+(z-\varepsilon)^2 \in S_\alpha^-, \\ 0, & \text{elsewhere.} \end{cases}$$

Let $\mathbf{u} = \varphi\mathbf{e}_1$, where φ is defined through (A.1) and (A.2) and deduce that:

I) $$\mathbf{u}\in\mathscr{V}_{\varkappa};$$

II) $$D\mathbf{u} = 0 \quad \text{on } \partial\varkappa(\mathscr{B})\setminus\mathscr{N}_\alpha;$$

III) $$D\mathbf{u}(\mathbf{x}) = (\partial_z\varphi, 0, 0)|_{\chi(\mathbf{x})} \quad \text{on } \mathscr{N}_\alpha.$$

Since $\chi(\mathscr{N}_\alpha) = D_\alpha$ and $\partial_z\varphi|_{z=0} > 0$ we conclude that

$$\int_{\partial\varkappa} \mathbf{c}\cdot D\mathbf{u} = \int_{\mathscr{N}_\alpha} \mathbf{c}\cdot D\mathbf{u} > 0,$$

contradicting (L''). This implies that $\mathbf{c} = 0$ on $\partial\varkappa(\mathscr{B})$. ■

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Received May 6, 1988.