Arch. Mech., **41**, 5, pp. 747-758, Warszawa 1989

The thermodynamic model of a rigid-plastic solid with kinematic hardening, plastic spin and orientation variables

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THE CONCEPT of the m_i-isoclinic motion of a material element is introduced to remove the ambiguity of the multiplicative decomposition of the total deformation gradient. The kinematical constraint relating the plastic spin and the rate of permanent strain is found. Using the Mandel formalism and basic concepts of thermodynamics, the simplest model of the solid with linear kinematic hardening is generalised to illustrate the application of the concept. The resulting constitutive equations do not predict any spurious oscillation of the shear stress curve in simple shear tests.

Wykorzystano pojęcie izoklinalnego ruchu elementu materialnego, z wybraną bazą jednostkowych wektorów, w celu ujednoznacznienia multiplikatywnego rozkładu gradientu deformacji. Znaleziono więzy kinematyczne wiążące spin i prędkość odkształceń trwałych. Stosując formalizm Mandela i podstawowe założenia termodynamiki, uogólniono najprostszy model ciała z liniowym kinematycznym wzmocnieniem. Uogólnienie to nie przewiduje oscylacji naprężeń stycznych w próbach na czyste ścinanie.

Использовано понятие изоклинального движения материального элемента, с избранным базисом единичных векторов, с целью однозначности мультипликативного распределения градиента деформации. Найдены кинематические связи, связывающие спин и скорость остаточных деформаций. Применяя формализм Манделя и основные предположения термодинамики, обобщена самая простая модель тела с линейным кинематическим упрочнением. Это обобщение не предсказывает осцилляций касательных напряжений в образцах на чистой сдвиг.

Notation

$\mathbf{1}$ the unit tensor,
$\mathbf{AB} \rightarrow A_{ij}B_j$ or $A_{ij}B_{jk}$,
$(\mathbf{A}\otimes\mathbf{B})_{ij} = A_iB_j$,
$\mathbf{A}\cdot\mathbf{B} = A_iB_i$ or $A_{ij}B_{ij}$,
$\mathrm{tr}(\mathbf{AB}) = A_iB_i$ or $A_{ij}B_{ij}$,
$\mathbf{A}\times\mathbf{B}$ vector product,
$\dot{\mathbf{A}}$ material derivative,
$\overset{T}{\mathbf{A}}_{ij} = A_{ji}$; $\overset{-1}{\mathbf{A}}\mathbf{A} = \mathbf{1}$.

1. Introduction

THE FORMULATION of models of strain-induced anisotropy for finite strain of rate-independent plastic materials has been devoted much attention to in the recent years. The adequate phenomenological models need to account for the possible mutual rotations of material fibers that accompany, e.g., large torsion of the test specimens. Such rotations are most frequently described by an appropriately defined antisymmetric tensor — the so-called "plastic spin tensor" [1–7]. Its kinematical significance is elucitated by making use of the concept of unloaded configuration and the multiplicative decomposition of the total deformation gradient. The decomposition is not unique since the orientation of a material element in the conceptual momentary unloaded configuration may be arbitrary. To develop the concept, Mandel [8–10] has introduced the notion of the director triad — 3 orthonormal vectors attached in some way to the material element in its unloaded configurations. The director triad constitutes the reference frame that can be used to specify the constitutive properties. The plastic spin, or rather the spin of permanent strains, may then be identified with the mean relative spin of all material fibers measured with respect to the chosen triad. By convention, the conceptual motion of the director triad (associated with the family of unloaded configurations) observed in a background reference frame (laboratory) can be fixed to be translatory. Such a family of unloaded configurations Mandel calls "the isoclinic configurations". Its use simplifies the description of material properties since the usual material derivatives of the constitutive quantities (defined in the unloaded configurations) preserve its constitutive status. In this way Mandel reduced the problem of ambiguous decomposition of the total deformation gradient to the problem of finding the adequate director triad. This, in turn, requires specification of three additional constitutive equations for the plastic spin. Their general form can be deduced from the representation theorems for isotropic tensor functions [4]. However, if the concept of isoclinic unloaded configuration is combined with three kinematical constraints imposed on the elastic [1–2, 7] or plastic part of the deformation gradient, the plastic spin becomes the definite function of the plastic strain rate, and extra constitutive equations for the plastic spin are not required. Two groups of possible kinematical constraints can be distinguished:

The first group is characterized by the fact that the director triad does not appear explicitly in the equations. The assumption that the elastic part of the total deformation gradient $\mathbf{F}^e$ is symmetric [1–2, 7], exemplifies this group of constraints. Unfortunately, when this method is applied the resulting plastic spin cannot be considered to be a measure of the mutual rotations of the material fibers. The equations of the second group of kinematical constraints depend explicitly on the chosen director triad. It seems that this method has not been applied so far. The main objective of this note is to illustrate it on the example of the most simple model of the rigid-plastic solid with linear kinematic hardening. We use here Mandel's suggestion concerning the choice of the director triad:

The director triad is specified by the unit vector representing the single material fiber and the other unit vector representing the material plane in which the chosen material fiber is placed ([1]). In Sect. 2 the notion of the homogeneous isoclinic motion of a material element is introduced, and the equations of kinematical constraints are derived in the context of the usual mechanics of continuum media. In Sect. 3 the relationship between spin and rate of permanent strain is postulated. It follows from the assumption that the fictitious motion of an unloaded material element with the chosen director triad is the isoclinic motion. This relationship is used in Sect. 5 where the model of a solid with linear kinematic hardening is modified using the concepts of classical thermodynamics.

([1]) The trace of a segment of the straight line marked on the definite surface of an unloaded test specimen (undergoing the homogeneous process of deformation) specifies such a triad.

2. The isoclinic motion of a material element

Denote by $\mathbf{e}_i (i = 1, 2, 3)$ the orthonormal base vectors of the background (laboratory). Consider a homogeneous deformation of a material element — the unit cube $\varkappa_0$ in a certain chosen reference configuration. The deformation of the cube into the parallelepiped $\varkappa(t)$ defines the deformation gradient $\mathbf{F}(t)$ ($\mathbf{F}(t_0) = \mathbf{1}$ — Fig. 1). Here the symbol t de-

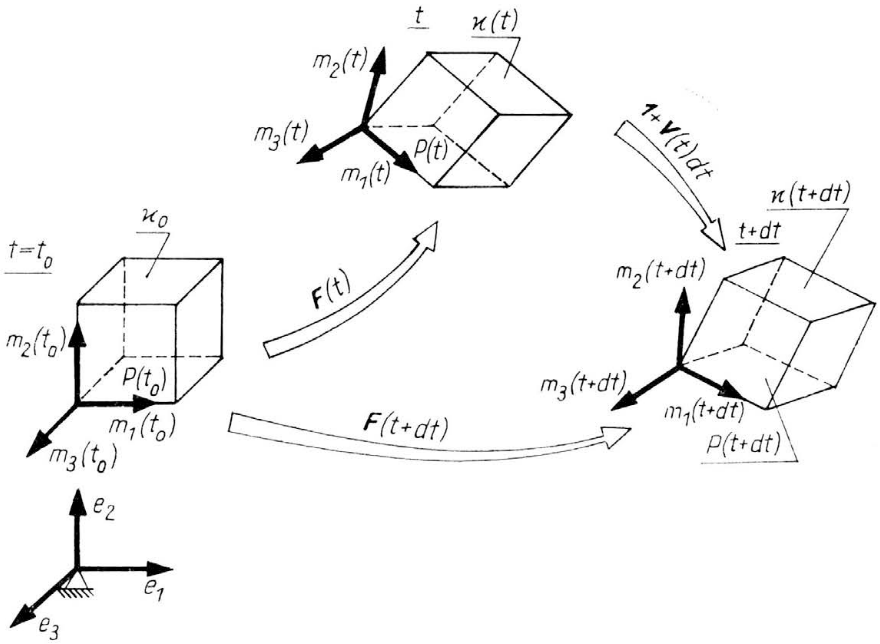

FIG. 1.

notes the time, and $t > t_0$. The infinitesimal deformation of parallelepiped $\varkappa(t)$ into the other parallelepiped $\varkappa(t+dt)$ can be described by an operator $\mathbf{1}+\mathbf{V}(t)dt$, where $\mathbf{V}(t) = \dot{\mathbf{F}}(t)\overset{-1}{\mathbf{F}}(t)$. Consider the other reference frame that rotates with respect to $\mathbf{e}_i$, and it is determined by the following 3 orthonormal vectors: $\mathbf{m}_1$ which represents the chosen material fiber placed on the chosen material plane P defined by the unit vector $\mathbf{m}_2$ normal to P, and $\mathbf{m}_3 = \mathbf{m}_1 \times \mathbf{m}_2$. Three configurations of $\mathbf{m}_i$ at times t_0, t and $t+dt$ are illustrated in Fig. 1. Assume that $\mathbf{V}(t)$ is known. We shall determine the current angular velocity of the triad $\mathbf{m}_i$.

As a measure of angular velocity, one may adopt either the vector $\mathbf{W}^{(m)}(t)$ defined by

$$\mathbf{W}^{(m)} = \sum_{i=1}^{3} \dot{\mathbf{m}}_i \times \mathbf{m}_i \equiv \dot{\mathbf{m}}_i \times \mathbf{m}_i$$

or the antisymmetric tensor $\boldsymbol{\omega}^{(m)}$ corresponding to $\mathbf{W}^{(m)}$:

$$2\boldsymbol{\omega}^{(m)} = \dot{\mathbf{m}}_i \otimes \mathbf{m}_i - \mathbf{m}_i \otimes \dot{\mathbf{m}}_i. \tag{2.1}$$

It can be shown that the material derivatives of $\mathbf{m}_1$ and $\mathbf{m}_2$ are

(2.2) $$\dot{\mathbf{m}}_1 = \mathbf{V}\mathbf{m}_1 - (\mathbf{m}_1 \cdot \mathbf{V}\mathbf{m}_1)\mathbf{m}_1, \qquad \dot{\mathbf{m}}_2 = (\mathbf{m}_2 \cdot \mathbf{V}\mathbf{m}_2)\mathbf{m}_2 - \overset{T}{\mathbf{V}}\mathbf{m}_2.$$

Since $\mathbf{m}_3 = \mathbf{m}_1 \times \mathbf{m}_2$, it is $\dot{\mathbf{m}}_3 \times \mathbf{m}_3 = (\mathbf{m}_2 \otimes \dot{\mathbf{m}}_1 - \mathbf{m}_1 \otimes \dot{\mathbf{m}}_2)\mathbf{m}_3$ or

(2.3) $$\dot{\mathbf{m}}_3 \otimes \mathbf{m}_3 - \mathbf{m}_3 \otimes \dot{\mathbf{m}}_3 = [(\mathbf{m}_2 \cdot \dot{\mathbf{m}}_1) - (\mathbf{m}_1 \cdot \dot{\mathbf{m}}_2)](\mathbf{m}_1 \otimes \mathbf{m}_2 - \mathbf{m}_2 \otimes \mathbf{m}_1) + \dot{\mathbf{m}}_1 \otimes \mathbf{m}_1 - \mathbf{m}_1 \otimes \dot{\mathbf{m}}_1 + \dot{\mathbf{m}}_2 \otimes \mathbf{m}_2 - \mathbf{m}_2 \otimes \dot{\mathbf{m}}_2.$$

Substituting Eqs. (2.3) and (2.2) into Eq. (2.1), we get the final result:

(2.4) $$\boldsymbol{\omega}^{(m)} = \boldsymbol{\omega} + D(m_1, m_2)[\mathbf{m}_1 \otimes \mathbf{m}_2 - \mathbf{m}_2 \otimes \mathbf{m}_1] + (\mathbf{D}\mathbf{m}_1) \otimes \mathbf{m}_1 - \mathbf{m}_1 \otimes (\mathbf{D}\mathbf{m}_1) + \mathbf{m}_2 \otimes (\mathbf{D}\mathbf{m}_2) - (\mathbf{D}\mathbf{m}_2) \otimes \mathbf{m}_2.$$

Here $2\mathbf{D} = \mathbf{V} + \overset{T}{\mathbf{V}}$ is the Eulerian strain rate tensor, $2\boldsymbol{\omega} = \mathbf{V} - \overset{T}{\mathbf{V}}$ is the usual material spin, $D(m_1, m_2) = \mathbf{m}_1 \cdot \mathbf{D}\mathbf{m}_2$.

Note that Eq. (2.4) implies the well-known fact that $\boldsymbol{\omega}^{(m)} = \boldsymbol{\omega}$ whenever $\mathbf{m}_i$ coincides with the principal directions of $\mathbf{D}$ ($\mathbf{D}\mathbf{m}_i \sim \mathbf{m}_i$). The antisymmetric tensor $\boldsymbol{\omega} - \boldsymbol{\omega}^{(m)}$ is the definite function of $\mathbf{D}$ and $\mathbf{m}_i$. It represents the mean angular velocity of all material fibers measured by an observer moving with the triad $\mathbf{m}_i$. Thus it can be regarded as a measure of the mean rate of mutual rotations of material fibers — rotations that are associated with straining of a material element.

If for every t and fixed $\mathbf{m}_i$, $\mathbf{V}(t)$ is such that $\boldsymbol{\omega}^{(m)} = \mathbf{0}$ then, following MANDEL [8–9], the motion of a material element (its homogeneous deformation) will be called $\mathbf{m}_i$ — isoclinic motion. Substituting $\boldsymbol{\omega}^{(m)} = \mathbf{0}$ into Eq. (2.4), one finds the constraint equations

(2.5) $$\boldsymbol{\omega} = D(m_1, m_2)[\mathbf{m}_2 \otimes \mathbf{m}_1 - \mathbf{m}_1 \otimes \mathbf{m}_2] + \mathbf{m}_1 \otimes (\mathbf{D}\mathbf{m}_1) - (\mathbf{D}\mathbf{m}_1) \otimes \mathbf{m}_1 - \mathbf{m}_2 \otimes (\mathbf{D}\mathbf{m}_2) + (\mathbf{D}\mathbf{m}_2) \otimes \mathbf{m}_2.$$

The equation can be transformed into the form

(2.6) $$\boldsymbol{\omega} = \bar{\omega}_{ij}(\mathbf{m}_i \otimes \mathbf{m}_j - \mathbf{m}_j \otimes \mathbf{m}_i)/2,$$

where $\bar{\omega}_{ij}$ are the components of $\boldsymbol{\omega}$ on the triad $\mathbf{m}_i$

(2.7) $$\bar{\omega}_{12} = -\bar{\omega}_{21} = D(m_1, m_2) \equiv \mathbf{m}_1 \cdot \mathbf{D}\mathbf{m}_2, \qquad \bar{\omega}_{13} = -\bar{\omega}_{31} = D(m_3, m_1), \qquad \bar{\omega}_{32} = -\bar{\omega}_{32} = D(m_2, m_3).$$

The three equations (2.5) or (2.6) are the constraint equations which must be satisfied by $\mathbf{V}(t)$ if the homogeneous deformation is to be $\mathbf{m}_i$-isoclinic. The constraints do not impose any restrictions upon the type of straining of the material element. They merely fix its orientation in the physical space.

3. Isoclinic fictitious motion of the unloaded material element

Consider a representative macroscopic material sample of the unit mass. Let the sample be a cube $\varkappa_0$ in the reference configuration ($t = t_0$). Denote by θ_0 and ϱ_0 its initial temperature and the initial mass density. After application of thermo-mechanical loading, the sample changes its thermodynamical state (in the course of a homogeneous process), and becomes at time t parallelepiped $\varkappa(t)$ (the actual configuration of a sample) whose mass density is ϱ and the temperature is θ. Denote by $\boldsymbol{\sigma}(t)$ the true stress (Cauchy's stress) in the actual configuration.

The instantaneous natural state (i.n.s.) of a sample is the conceptual thermodynamic state that would have occurred in the sample had all mechanical loadings and the temperature been instantaneously reduced to zero and θ_0, respectively, keeping the internal structure of the sample unchanged (thermoelastic unloading). The fictitious configuration of the material element at the termination of this conceptual process is called instantaneous unloaded configuration. It will be denoted by $\overset{*}{\varkappa}(t)$. In i.n.s. we have $\boldsymbol{\sigma} = O, \theta = \theta_0$ and mass density $\overset{*}{\varrho}(t)$.

Denoting by $\mathbf{F}$, $\mathbf{F}^e$ and $\mathbf{F}^p$ the deformation gradients that map $\varkappa_0 \to \varkappa(t)$, $\overset{*}{\varkappa}(t) \to \varkappa(t)$ and $\varkappa_0 \to \overset{*}{\varkappa}(t)$, respectively, we arrive at the well-known multiplicative decomposition of the total deformation gradient

$$\mathbf{F}(t) = \mathbf{F}^e(t)\mathbf{F}^p(t). \tag{3.1}$$

The decomposition (3.1) is not unique since the orientation of the sample at the termination of the conceptual process of unloading can be arbitrary. To reduce this ambiguity, we adopt the convention that the conceptual motion of the unloaded element is $\mathbf{m}_i$-isoclinic motion. Introducing the notation

$$\mathbf{V}^* = \dot{\mathbf{F}}^p\overset{-1}{\mathbf{F}^p}, \quad 2\mathbf{D}^* = \mathbf{V}^*+\overset{T}{\mathbf{V}^*}, \quad 2\boldsymbol{\omega}^* = \mathbf{V}^*-\overset{T}{\mathbf{V}^*}, \quad \mathbf{V}^* = \mathbf{D}^*+\boldsymbol{\omega}^* \tag{3.2}$$

and replacing D by D^* and $\boldsymbol{\omega}$ by $\boldsymbol{\omega}^*$ in Eqs. (2.6)–(2.7), one gets

$$\boldsymbol{\omega}^* = \bar{\omega}^*_{ij}(\mathbf{m}_i\otimes\mathbf{m}_j-\mathbf{m}_j\otimes\mathbf{m}_i)/2, \tag{3.3}$$
$$\bar{\omega}^*_{12} = D^*(m_1, m_2), \quad \bar{\omega}^*_{13} = D^*(m_1, m_3), \quad \bar{\omega}^*_{32} = D^*(m_2, m_3).$$

The orthonormal vectors $\mathbf{m}_i$ occurring in Eqs. (3.3) do not change in the course of straining of the material element, i.e., the triad $\mathbf{m}_i$ does not change its orientation in the physical space in which the conceptual $\mathbf{m}_i$-isoclinic motion takes place ($\mathbf{m}_i \cdot \mathbf{e}_j$ = const). The tensor D^* is called "the rate of permanent strain" tensor, whereas $\boldsymbol{\omega}^*$ is the spin of permanent strain. The tensor $\boldsymbol{\omega}^*$ is a well-defined physical quantity that vanishes when the motion of the unloaded configuration is the rigid-body motion. Its physical significance is the same as that of the tensor $\boldsymbol{\omega}-\boldsymbol{\omega}^{(m)}$ discussed in Sect. 2.

4. The transformation rules

When use is made of the multiplicative decomposition of $\mathbf{F}$, the instantaneous unloaded configuration can be regarded as a basic reference configuration for the formulation of the constitutive equations of an elastic-plastic solid. Within this approach, the plastic flow rules represent the response of a material upon the prescribed fictitious $\mathbf{m}_i$-isoclinic motion of $\overset{*}{\varkappa}$, whereas the current $\overset{*}{\varkappa}(t)$ itself is identified with the reference configuration for employed measures of the elastic strain, incremental permanent strain and work-conjugate stress. Since the triad $\mathbf{m}_i$ does not rotate, the material derivatives of all physical quantities, including structural parameters, define their physically plausible rates. Such an approach will be called the Mandel formulation of constitutive equations [8–10]. The constitutive equations, once established according to the Mandel idea, can subsequently be transformed to the actual configuration $\varkappa(t)$. Let us recall some basic transformation rules [8, 11–12]. They consist of kinematical relations: Eqs. (3.2),

$$
\begin{gathered}
\mathbf{V} = \mathbf{V}^e+\mathbf{V}^p, \quad \mathbf{V}^e = \mathbf{F}^e\overset{-1}{\mathbf{F}}{}^e, \quad \mathbf{V}^p = \mathbf{F}^e\mathbf{V}^*\overset{-1}{\mathbf{F}}{}^e,\\
2\mathbf{D} = \mathbf{V}+\overset{T}{\mathbf{V}} = 2\mathbf{D}^e+2\mathbf{D}^p; \quad 2\mathbf{D}^e = \mathbf{V}^e+\overset{T}{\mathbf{V}}{}^e, \quad 2\mathbf{D}^p = \mathbf{V}^p+\overset{T}{\mathbf{V}}{}^p,\\
2\boldsymbol{\omega} = \mathbf{V}-\overset{T}{\mathbf{V}} = 2\boldsymbol{\omega}^e+2\boldsymbol{\omega}^p, \quad 2\boldsymbol{\omega}^e = \mathbf{V}^e-\overset{T}{\mathbf{V}}{}^e, \quad 2\boldsymbol{\omega}^p = \mathbf{V}^p-\overset{T}{\mathbf{V}}{}^p,\\
2\mathbf{E}^* = \overset{T}{\mathbf{F}}{}^e\mathbf{F}^e-\mathbf{1}
\end{gathered}
\tag{4.1}
$$

and the relationships between different measures of stresses

$$
\mathbf{T}^* = \overset{-1}{\mathbf{F}}{}^e\boldsymbol{\sigma}\overset{-T}{\mathbf{F}}{}^e/\varrho, \qquad \mathbf{P}^* = \overset{-1}{\mathbf{F}}{}^e\boldsymbol{\sigma}\mathbf{F}^e/\varrho
\tag{4.2}
$$

that follow from the additive decomposition of the rate of total work $\dot{w}$ into elastic $\dot{w}^e$ and plastic $\dot{w}^p$ parts:

$$
\begin{gathered}
\dot{w} = \boldsymbol{\sigma}\cdot\mathbf{D}/\varrho = \dot{w}^e+\dot{w}^p,\\
\dot{w}^e = \boldsymbol{\sigma}\cdot\mathbf{D}^e/\varrho = \mathbf{T}^*\cdot\dot{\mathbf{E}}^*, \qquad \dot{w}^p = \frac{1}{\rho}\,\boldsymbol{\sigma}\cdot\mathbf{D}^p = \operatorname{tr}(\mathbf{P}^*\mathbf{V}^*).
\end{gathered}
\tag{4.3}
$$

Here $\boldsymbol{\omega}^p$ is the so-called plastic spin tensor. Within the framework of the Mandel formulation, the constitutive equations are expressed in terms of starred quantities. For example, the plastic flow rules may relate $\mathbf{V}^*$ with $\mathbf{P}^*$. According to the idea introduced in this paper, this relationship is supposed to satisfy three constraint equations (3.3).

5. Thermodynamic model of rigid-plastic solid with linear kinematic hardening

5.1. Rate constitutive equations in the Mandel description

i) To illustrate the application of the constraints (3.3), we shall consider here the simplest model of the solid with linear kinematic hardening, preserving all its properties of the infinitesimal theory. The only modification will concern the thermal effects that will be accounted for within the framework of classical thermodynamics. One can proceed along the same line to generalise more complex models of the infinitesimal theory of plasticity.

ii) For rigid-plastic solids $\mathbf{F}^e$ is an orthogonal tensor

$$
\mathbf{F}^e = \mathbf{R}^e, \qquad \mathbf{R}^e\overset{T}{\mathbf{R}}{}^e = \mathbf{1}
\tag{5.1}
$$

such that (cf. Eq. 4.1))

$$
\begin{gathered}
\mathbf{V}^e = \boldsymbol{\omega}^e = \dot{\mathbf{R}}^e\overset{T}{\mathbf{R}}{}^e, \quad \mathbf{D}^e = \mathbf{0}, \quad \mathbf{E}^* = \mathbf{0}, \quad \mathbf{V}^p = \mathbf{R}^e\mathbf{V}^*\overset{T}{\mathbf{R}}{}^e,\\
\mathbf{D} = \mathbf{D}^p = \mathbf{R}^e\mathbf{D}^*\overset{T}{\mathbf{R}}{}^e, \quad \boldsymbol{\omega} = \boldsymbol{\omega}^e+\boldsymbol{\omega}^p, \quad \boldsymbol{\omega}^p = \mathbf{R}^e\boldsymbol{\omega}^*\overset{T}{\mathbf{R}}{}^e.
\end{gathered}
\tag{5.2}
$$

In what follows we shall use the generalised Huber–Mises yield criterion and associated plastic flow rules. Since these assumptions entail plastic incompressibility

(5.3) $$\varrho = \overset{*}{\varrho} = \varrho_0 = \text{const}, \qquad \mathrm{tr}\mathbf{D} = \mathrm{tr}\mathbf{D}^* = 0,$$

it is convenient to employ the tensor $\boldsymbol{\sigma}^* = \overset{*}{\varrho}\mathbf{P}^*$ (instead of P^*) which, for a rigid-plastic solid, becomes the rotated true (Cauchy's) stress tensor

(5.4) $$\boldsymbol{\sigma}^* = \overset{T}{\mathbf{R}^e}\boldsymbol{\sigma}\mathbf{R}^e.$$

Introduce the "cumulative" permanent strain tensor $\mathbf{H}^*$ defined by

(5.5) $$\dot{\mathbf{H}}^* = \mathbf{D}^* = D^*_{ij}\mathbf{m}_i \otimes \mathbf{m}_j, \qquad \mathrm{tr}\mathbf{H}^* = 0.$$

It can be regarded as a macroscopic measure of the change in the geometry of the internal pattern of the material element, associated with its inelastic deformation. $\mathbf{H}^*$ is treated here as the only internal state variable of the geometrical type.

iii) Neglect the influence of the change in the internal structure of the material element on its thermal properties, and assume that the free energy $\varrho_0\phi$ (per unit of the volume) is the quadratic function of the internal parameter $\mathbf{H}^*$. Then the most general form of ϕ for the rigid-plastic solid is

(5.6) $$\varrho_0\phi = \varrho_0 \int_{\theta_0}^{\theta} \left(1 - \frac{\theta}{\theta_1}\right) C_v(\theta_1)d\theta_1 + c(\theta)\mathbf{H}^* \cdot \mathbf{H}^*/2 + \text{const},$$

where

(5.7) $$c(\theta) = c_0 - (\theta - \theta_0)c_T > 0$$

is the modulus of linear kinematic hardening, c_0 and c_T are constants and $C_v(\theta)$ is the specific heat.

The adopted form (5.6) for the free energy function implies the following expressions:

(5.8) $$u^* = (c_0 + c_T\theta_0)\mathbf{H}^* \cdot \mathbf{H}^*/2 \quad \text{and} \quad s^* = c_T\mathbf{H}^* \cdot \mathbf{H}^*/2$$

for the stored internal energy u^* and stored (configurational) entropy s^*, respectively [12].

The partial derivative of ϕ calculated with respect to $\mathbf{H}^*$ defines the thermodynamical force $\boldsymbol{\pi}^*$

(5.9) $$\boldsymbol{\pi}^* = \varrho_0 \frac{\partial\phi}{\partial\mathbf{H}^*} = c(\theta)\mathbf{H}^*.$$

The force $\boldsymbol{\pi}^*$ will be identified with the internal stress (back stress) that determines the position of the symmetry centre of the yield surface. From Eq. (5.9) one obtains the following evolution equation for the internal force:

(5.10) $$\dot{\boldsymbol{\pi}}^* = c\mathbf{D}^* - \frac{c_T}{c}\dot{\theta}\boldsymbol{\pi}^*$$

expressed in terms of $\boldsymbol{\pi}^*$, $\mathbf{D}^*$ and $\dot{\theta}$.

The rigid-plastic solids have the feature that it is always possible to contrive the real process of plastic deformation to be $\mathbf{m}_i$-isoclinic. Then, for a real process $\mathbf{R}^e = \mathbf{1}$ what entails $\mathbf{D} = \mathbf{D}^*$, $\boldsymbol{\omega}^e = \mathbf{0}$, $\boldsymbol{\omega}^p = \boldsymbol{\omega}^* = \boldsymbol{\omega}$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^*$. For tests with controlled constant strain-rate ($\mathbf{D} = \mathbf{D}^* =$ const) $\mathbf{H}^*$ changes proportionally in time. In particular, this concerns the simple shear tests. Therefore Eqs. (5.5) and (5.9) exclude the possibility of oscillations of the shear stress in the course of such tests — the erroneous theoretical effect discussed in [12, 2] and [4].

iv) The dissipation rate for the considered model is [11, 12]

$$\mathscr{D} = \boldsymbol{\sigma}^* \cdot \mathbf{D}^* - \boldsymbol{\pi}^* \cdot \dot{\mathbf{H}}^* = (\overset{*}{\boldsymbol{\sigma}}{}' - \boldsymbol{\pi}^*) \cdot \mathbf{D}^* \geqslant 0, \tag{5.11}$$

where $\overset{*}{\boldsymbol{\sigma}}{}'$ denotes the deviatoric part of $\boldsymbol{\sigma}^*$.

Adopt the generalised Huber–Mises yield criterion

$$f = \sigma^{(i)} - Y(\theta) = 0, \tag{5.12}$$

where $\sigma^{(i)}$ is the equivalent net stress

$$\sigma^{(i)} = [3(\overset{*}{\boldsymbol{\sigma}}{}' - \boldsymbol{\pi}^*) \cdot (\overset{*}{\boldsymbol{\sigma}}{}' - \boldsymbol{\pi}^*)/2]^{1/2}. \tag{5.13}$$

The plastic flow law associated with Eq. (5.12) relates the dissipative force $\overset{*}{\boldsymbol{\sigma}}{}' - \boldsymbol{\pi}^*$ with $\mathbf{D}^*$. Its eventual form is

$$\begin{gathered} \mathbf{D}^* = \frac{\langle L \rangle}{cY} (\overset{*}{\boldsymbol{\sigma}}{}' - \boldsymbol{\pi}^*), \\ L = \frac{3}{2Y} (\overset{*}{\boldsymbol{\sigma}}{}' - \boldsymbol{\pi}^*) \cdot (\dot{\overset{*}{\boldsymbol{\sigma}}}{}' + c_T \dot{\theta} \boldsymbol{\pi}^*/c) - \frac{dY}{d\theta} \dot{\theta}, \end{gathered} \tag{5.14}$$

when $f = 0$, and $\mathbf{D}^* = \mathbf{0}$ when $f < 0$. Here the symbol $\langle \ \rangle$ denotes the so-called Macauley function: $\langle z \rangle = z$ if $z > 0$, and $\langle z \rangle = 0$ if $z \leqslant 0$.

The complete set of the constitutive equations in the Mandel description consists of Eqs. (5.12)–(5.14), (5.10) and (3.3).

5.2. Constitutive equations in the Euler description

i) To transform the flow law (5.14) into a form adequate in the Euler description (or rather updated Lagrange'a description), introduce the true internal stress $\boldsymbol{\pi}$ defined by

$$\boldsymbol{\pi} = \mathbf{R}^e \boldsymbol{\pi}^* \overset{T}{\mathbf{R}^e} \qquad (\mathrm{tr}\boldsymbol{\pi} = 0). \tag{5.15}$$

The evolution equation for $\boldsymbol{\pi}$ can be obtained from Eqs. (5.10), $(5.2)_1$ and

$$\overset{\nabla}{\boldsymbol{\pi}} = c\mathbf{D} - c_T \dot{\theta} \boldsymbol{\pi}/c, \tag{5.16}$$

where the superimposed symbol ∇ denotes the co-rotational rate defined by the spin $\boldsymbol{\omega}^e$,

$$\overset{\nabla}{\mathbf{K}} = \dot{\mathbf{K}} + \mathbf{K}\boldsymbol{\omega}^e - \boldsymbol{\omega}^e \mathbf{K}. \tag{5.17}$$

The yield criterion expressed in terms of $\boldsymbol{\sigma}$ and $\boldsymbol{\pi}$ preserves its form since $\sigma^{(i)}$ can equivalently be written as

$$\sigma^{(i)} = [3(\boldsymbol{\sigma}'-\boldsymbol{\pi})\cdot(\boldsymbol{\sigma}'-\boldsymbol{\pi})/2]^{1/2},$$

where $\boldsymbol{\sigma}'$ denotes the deviatoric part of $\boldsymbol{\sigma}$.

Likewise, the flow rules remain in their general form ($f = 0$)

$$\mathbf{D} = \langle L\rangle(\boldsymbol{\sigma}'-\boldsymbol{\pi})/cY,$$
(5.18)
$$L = \frac{3}{2Y}(\boldsymbol{\sigma}'-\boldsymbol{\pi})\cdot(\overset{\nabla}{\boldsymbol{\sigma}}+c_T\dot{\theta}\boldsymbol{\pi}/c)-\frac{dY}{d\theta}\dot{\theta}.$$

Introduce the orthonormal triad $\mathbf{n}_i(t)$ being the image of the constant triad $\mathbf{m}_i$ under rotation $\mathbf{R}^e(t)$:

(5.19) $$\mathbf{n}_i(t) = \mathbf{R}^e(t)\mathbf{m}_i, \qquad \dot{\mathbf{n}}_i(t) = \boldsymbol{\omega}^e(t)\mathbf{n}_i(t).$$

The plastic spin $\boldsymbol{\omega}^p$ can be determined from (cf. Eqs. $(5.2)_2$ and (3.3))

(5.20) $$\boldsymbol{\omega}^p = \omega^p_{ij}(\mathbf{n}_i\otimes\mathbf{n}_j-\mathbf{n}_j\otimes\mathbf{n}_i)/2,$$

where

(5.21) $$\omega^p_{12} = D(n_1, n_2), \quad \omega^p_{13} = D(n_1, n_3), \quad \omega^p_{32} = D(n_2, n_3)$$

or, equivalently, from

(5.22)
$$\boldsymbol{\omega}^p = D(n_1, n_2)[\mathbf{n}_2\otimes\mathbf{n}_1-\mathbf{n}_1\otimes\mathbf{n}_2]+\mathbf{n}_1\otimes(\mathbf{D}\mathbf{n}_1)-(\mathbf{D}\mathbf{n}_1)\otimes\mathbf{n}_1-\mathbf{n}_2\otimes(\mathbf{D}\mathbf{n}_2)+(\mathbf{D}\mathbf{n}_2)\otimes\mathbf{n}_2.$$

Finally the spin $\boldsymbol{\omega}^e$ can be calculated from

(5.23) $$\boldsymbol{\omega}^e = \boldsymbol{\omega}-\boldsymbol{\omega}^p.$$

Note that $\boldsymbol{\omega}^e$ is the tensor of current angular velocity of the triad $\mathbf{n}_i(t)$. It is the counterpart of the tensor $\boldsymbol{\omega}^{(m)}$ discussed in Sect. 2.

ii) The use of Eqs. (5.18) is not convenient in practical applications since the spin $\boldsymbol{\omega}^e$ depending on $\mathbf{D}$ occurs on the right-hand side of Eq. (5.18). The flow rules can also be expressed in terms of the usual Zaremba–Jaumann stress rate

(5.24) $$\overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}+\boldsymbol{\sigma}\boldsymbol{\omega}-\boldsymbol{\omega}\boldsymbol{\sigma}.$$

Substituting Eqs. (5.23) into Eq. (5.18) and making use of Eq. (5.20) one can solve the resulting equation with respect to $\mathbf{D}$ to obtain

(5.25) $$D = \frac{1}{(1+r)}\langle L_1\rangle(\boldsymbol{\sigma}'-\boldsymbol{\pi})/cY,$$

where

$$L_1 = \frac{3}{2Y}(\boldsymbol{\sigma}'-\boldsymbol{\pi})\cdot(\overset{\circ}{\boldsymbol{\sigma}}'+c_T\dot{\theta}\boldsymbol{\pi}/c)-\frac{dY}{d\theta}\dot{\theta},$$
(5.26)
$$r = \frac{3}{2cY^2}\operatorname{tr}[(\boldsymbol{\sigma}'\boldsymbol{\pi}-\boldsymbol{\pi}\boldsymbol{\sigma}')\boldsymbol{\Omega}]$$

and

$$(5.27)\qquad \begin{gathered} \mathbf{\Omega} = -\overset{T}{\mathbf{\Omega}} = \Omega_{ij}(\mathbf{n}_i \otimes \mathbf{n}_j - \mathbf{n}_j \otimes \mathbf{n}_i)/2, \\ \Omega_{12} = \mathbf{n}_1 \cdot (\boldsymbol{\sigma}' - \boldsymbol{\pi})\mathbf{n}_2, \quad \Omega_{13} = \mathbf{n}_1 \cdot (\boldsymbol{\sigma}' - \boldsymbol{\pi})\mathbf{n}_3, \quad \Omega_{32} = \mathbf{n}_2 \cdot (\boldsymbol{\sigma}' - \boldsymbol{\pi})\mathbf{n}_3. \end{gathered}$$

The evolution equation for internal stress expressed in terms of its Zaremba–Jaumann rate follows directly from Eq. (5.16):

$$(5.28)\qquad \begin{gathered} \overset{\circ}{\boldsymbol{\pi}} = c\mathbf{D} + \boldsymbol{\pi}\boldsymbol{\omega}^p - \boldsymbol{\omega}^p\boldsymbol{\pi} - c_T\dot{\theta}\boldsymbol{\pi}/c, \\ \overset{\circ}{\boldsymbol{\pi}} \equiv \dot{\boldsymbol{\pi}} + \boldsymbol{\pi}\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\pi}. \end{gathered}$$

The final complete set of equations consists of Eqs. (5.19)–(5.20) and (5.23)–(5.28). The movable triad $\mathbf{n}_i(t)$ occurring in this set represents the orientation of the material element in the physical space, at time t. The associated three parameters are, therefore, examples of the "hidden variables of orientation" [8] that are not the internal state parameters. The plastic flow law depends on them through the dependence of the scalar parameter r on $\mathbf{n}_i$. Note that $r = 0$ whenever the principal directions of $\boldsymbol{\sigma}'$ and $\boldsymbol{\pi}$ coincide, and the plastic spin vanishes when $\mathbf{n}_i(t)$ coincides with principal directions of $\mathbf{D}$.

5.3. The equation for the temperature rate

The changes in the temperature that accompany straining of an element of the rigid-plastic solid are caused by three factors: a) the heat exchange with environment, b) thermostatic effect called in [12] "the heat of internal rearrangement", c) the dissipation of the mechanical work. It can be shown that the latter two effects can together be described by $\boldsymbol{\sigma}' \cdot \mathbf{D} - \dot{u}^*$. Hence the equation for the temperature rate is

$$(5.29)\qquad \varrho_0 C_v \dot{\theta} = \boldsymbol{\sigma}' \cdot \mathbf{D} - \dot{u}^* + \dot{q},$$

where $\dot{q}$ represents the rate of the heat absorbed by an element per unit of the volume. The rate of stored internal energy (cf. the expression (5.8)) $\dot{u}^*$ can be transformed into a more useful form:

$$\dot{u}^* = (c_0 + \theta_0 c_T)\boldsymbol{\pi} \cdot \mathbf{D}/c$$

on account of Eqs. (5.5) $(5.2)_2$, (5.9) and (5.15).

6. Discussion

In this paper the concept of $\mathbf{m}_i$-isoclinic homogeneous deformation has been introduced. This notion has been used to remove the ambiguity of the multiplicative decomposition of the total deformation gradient. The $\mathbf{m}_i$-isoclinic motion of the unloaded configuration implies a definite kinematical relation between the spin and the rate of permanent deformation. In effect, there is no need for experimental determination of three additional constitutive equations for the plastic spin. Instead, the flow rules depend explicitly on three hidden parameters of orientation which have definite evolution equations. The notion of the plastic spin is elucitated on the ground of the usual theory of continuous media. It represents the mean rate of rotations of all material fibers measured with respect to the triad attached to the chosen single material fiber. The concept introduced in this note constitutes, in a certain sense, the development of the idea of LEE *et al.* [2]. The final form of constitutive equations for a rigid-plastic solid with kinematic hardening given in 5.2 exemplifies DAFALIAS [4] general relations. They are derived here using a different approach. The obtained equations could also be derived directly in the Euler description by calculating co-rotational Zaremba–Jaumann rates of the true stress $\boldsymbol{\sigma}$ and the internal stress $\boldsymbol{\pi}$ with the spin $\boldsymbol{\omega}^{(m)} = \boldsymbol{\omega}^e$ (Eq. (2.4)) of the movable triad $\mathbf{n}_i$. However, the more involved Mandel formalism used in this note enables to combine the new concept introduced in this paper with other concepts of thermodynamics, and to generalise the more complex models of the infinitesimal plasticity theory (to account for finite strains, elasticity, damage, etc.) along the uniform line. No aspect of invariance under the change of the triad has been discussed.

Acknowledgment

One of the authors (S. K. S.) gratefully acknowledges the financial support provided by the National Science Foundation, U.S.A. under the NSF Grant No.DMC-8712266.

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Received November 10, 1988.