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Uniqueness in the elastostatic problem of bending of micropolar plates

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CLASSES of functions are indicated for which Betti and Somigliana relations hold in the exterior of a bounded domain in the problem of bending of thin micropolar plates. A uniqueness theorem is derived, and an example is discussed to illustrate the theory.

1. Introduction

THE EXISTENCE and uniqueness of the solution of the traction boundary value problem for a finite micropolar plate in the theory of bending proposed by ERINGEN [1] were investigated in [2]. The other problems usually covered by the boundary integral equation method were not considered there because of the lack of adequate uniqueness results in exterior domains. This drawback is caused by the fact that the matrix of fundamental solutions corresponding to the equilibrium equations of bending exhibits a growth of $O(|x|^2 \ln |x|)$ as $|x| \to \infty$, which means that the Betti formula cannot be established in exterior domains. A similar obstacle encountered in the theory of classical plates with transverse shear deformation has meanwhile been removed by formulating the problems in special uniqueness classes of finite energy functions characterized by a certain behavioural pattern at infinity (see [3]-[5]). In this paper we construct similar classes for the micropolar case and prove the corresponding uniqueness theorems. The discussion is concluded with a simple illustrative example.

2. Preliminaries

Unless stated otherwise, throughout what follows Latin and Greek suffixes take the values 1, 2, 3 and 1,2, respectively, and the convention of summation over repeated indices is understood.

We consider a homogeneous and isotropic micropolar plate occupying a cylindrical region $\bar{\Omega} \times [-h_0/2, h_0/2]$ in $\mathbf{R}^3$, where Ω is a bounded domain in the (x_1, x_2)-plane and $h_0 =$ const the plate thickness. We assume that the boundary $\partial\Omega$ of Ω is a simple closed C^2-curve. The equilibrium equations of the theory developed in [1], in the absence of body forces and couples and of forces and couples on the faces, can be written in the form [2]

$$L(\partial_x)v(x) = 0, \quad x \in \Omega, \tag{2.1}$$

where $L(\partial_x) = L(\partial/\partial x_\alpha)$, (2.2) $$L(\xi) = L(\xi_\alpha)$$

$$= \begin{pmatrix} \Delta_3+h^2(\lambda+\mu)\xi_1^2 & h^2(\lambda+\mu)\xi_1\xi_2 & -\mu\xi_1 & 0 & \varkappa \\ h^2(\lambda+\mu)\xi_1\xi_2 & \Delta_3+h^2(\lambda+\mu)\xi_2^2 & -\mu\xi_2 & -\varkappa & 0 \\ \mu\xi_1 & \mu\xi_2 & \Delta_1 & -\varkappa\xi_2 & \varkappa\xi_1 \\ 0 & -\varkappa & \varkappa\xi_2 & \Delta_2+(\alpha+\beta)\xi_1^2 & (\alpha+\beta)\xi_1\xi_2 \\ \varkappa & 0 & -\varkappa\xi_1 & (\alpha+\beta)\xi_1\xi_2 & \Delta_2+(\alpha+\beta)\xi_2^2 \end{pmatrix},$$

$v = (v_1, \ldots, v_5)^{\mathrm{T}}$ is a (5×1)-matrix characterizing the displacements (v_1, v_2, v_3) and microrotations (v_4, v_5), λ, μ, $\varkappa$, α, β, γ are the elastic constants of the material, $\Delta = \xi_\alpha\xi_\alpha$, $\Delta_1 = (\mu+\varkappa)\Delta$, $\Delta_2 = \gamma\Delta-2\varkappa$, $\Delta_3 = (\mu+\varkappa)(h^2\Delta-1)$, and $h^2 = h_0^2/12$. For the sake of simplicity, if the $v_1, \ldots, v_5$ are elements of a function space X, then we write $v \in X$.

We also consider the boundary integral operator $T(\partial_x) = T(\partial/\partial x_\alpha)$, where

(2.3) $$T(\xi) = T(\xi_\alpha)$$

$$= \begin{pmatrix} h^2(\mu_1\xi_\alpha n_\alpha+\lambda_1\xi_1 n_1) & h^2(\mu\xi_1 n_2+\lambda\xi_2 n_1) & 0 & 0 & 0 \\ h^2(\lambda\xi_1 n_2+\mu\xi_2 n_1) & h^2(\mu_1\xi_\alpha n_\alpha+\lambda_1\xi_2 n_2) & 0 & 0 & 0 \\ \mu n_1 & \mu n_2 & \mu_1\xi_\alpha n_\alpha & -\varkappa n_2 & \varkappa n_1 \\ 0 & 0 & 0 & \gamma\xi_\alpha n_\alpha+\alpha_1\xi_1 n_1 & \beta\xi_1 n_2+\alpha\xi_2 n_1 \\ 0 & 0 & 0 & \alpha\xi_1 n_2+\beta\xi_2 n_1 & \gamma\xi_\alpha n_\alpha+\alpha_1\xi_2 n_2 \end{pmatrix},$$

$\lambda_1 = \lambda+\mu$, $\mu_1 = \mu+\varkappa$, $\alpha_1 = \alpha+\beta$, and $n = (n_1, n_2)^{\mathrm{T}}$ is the unit vector of the outward normal to $\partial\Omega$. In view of the simplifying assumptions made in [1], Tv is the vector of resultant stress and couple on $\partial\Omega$.

Throughout what follows we assume that

$$2\lambda+2\mu+\varkappa > 0, \quad 2\mu+\varkappa > 0, \quad \varkappa > 0,$$

$$2\alpha+\beta+\gamma > 0, \quad \gamma+\beta > 0, \quad \gamma-\beta > 0,$$

which ensures that the system (2.1) is elliptic and that the internal energy density $E(v, v)$ [2] is a positive quadratic form.

Let $\Omega_{\mathrm{in}} = \Omega$ and $\Omega_{\mathrm{ex}} = \mathbf{R}^2 \setminus \bar{\Omega}_{\mathrm{in}}$. If u, $v \in C^2(\Omega_{\mathrm{in}}) \cap C^1(\bar{\Omega}_{\mathrm{in}})$, then we obtain [2] the reciprocity relation

$$\int_{\Omega_{\mathrm{in}}} (v^{\mathrm{T}}Lu-u^{\mathrm{T}}Lv)\,d\sigma = \int_{\partial\Omega} (v^{\mathrm{T}}Tu-u^{\mathrm{T}}Tv)\,ds$$

and the Betti formula

(2.4) $$\int_{\Omega_{\mathrm{in}}} v^{\mathrm{T}}Lv\,d\sigma + \int_{\Omega_{\mathrm{in}}} 2E(v, v)\,d\sigma = \int_{\partial\Omega} v^{\mathrm{T}}Tv\,ds.$$

Also, $E(v, v) = 0$ if and only if v is a rigid displacement, that is,

(2.5) $$v(x) = (c_2,\ -c_1,\ c_1x_2-c_2x_1+c_3,\ c_1,\ c_2)^{\mathrm{T}},$$

where c_1, c_2 and c_3 are arbitrary constants.

Let $B(\xi)$ be the matrix of cofactors of $L(\xi)$. Then the matrix of fundamental solutions of Eq. (2.1) is [2]

$$D(x, y) = (B(\partial x))^{\mathrm{T}} t(x, y). \tag{2.6}$$

where

$$t(x, y) = \frac{k^2}{8\pi}[(k_5|x-y|^2+4k_4)\ln|x-y|+4k_j K_0(l_j|x-y|)], \tag{2.7}$$

K_0 is the modified Bessel function of order zero, and $k, k_1, \ldots, k_5$, and l_j are well-defined constants expressed in terms of the elastic coefficients. Introducing the matrix of singular solutions

$$P(x, y) = [T(\partial_y)D(y, x)]^T, \tag{2.8}$$

we can now show [2] that every solution $v \in C^2(\Omega_{\mathrm{in}}) \cap C^1(\overline{\Omega}_{\mathrm{in}})$ of Eq. (2.1) admits the Somigliana representation

$$\chi_1(x)v(x) = \int_{\partial\Omega} [D(x, y)(Tv)(y) - P(x, y)v(y)]\,ds_y, \tag{2.9}$$

where

$$\chi_1(x) = \begin{cases} 1, & x \in \Omega_{\mathrm{in}}, \\ 1/2, & x \in \partial\Omega, \\ 0, & x \in \Omega_{\mathrm{ex}}. \end{cases}$$

3. Betti and Somigliana relations in the exterior domain

Since, by Eqs. (2.2) and (2.6)–(2.8) for $y \in \partial\Omega$

$$D(x, y) = O(|x|^2 \ln|x|), \qquad P(x, y) = O(\ln|x|) \qquad \text{as} \qquad |x| \to \infty,$$

the usual technique does not yield the analogues of Eqs. (2.4) and (2.9) in Ω_{ex}. To derive such formulae, we need to restrict the behaviour of the solution of Eq. (2.1) at infinity.

Let $\mathscr{A}$ be the set of (5×1)-matrices v in Ω_{ex} having an asymptotic expansion of the form [6]

$$\begin{aligned}
v_1(r, \theta) &= r^{-1}[a_0\sin\theta + 2a_1\cos\theta - a_0\sin 3\theta + (a_2-a_1)\cos 3\theta] \\
&\quad + r^{-2}[(2b_1+d_1)\sin 2\theta + d_2\cos 2\theta - 2b_1\sin 4\theta + 2b_2\cos 4\theta] \\
&\quad + r^{-3}[2e_1\sin 3\theta + 2f_1\cos 3\theta + 3(e_2-e_1)\sin 5\theta + (f_2-f_1)\cos 5\theta] + O(r^{-4}), \\
v_2(r, \theta) &= r^{-1}[2a_2\sin\theta + a_0\cos\theta + (a_2-a_1)\sin 3\theta + a_0\cos 3\theta]. \\
&\quad + r^{-2}[(2b_2+d_2)\sin 2\theta - d_1\cos 2\theta + 2b_2\sin 4\theta + 2b_1\cos 4\theta] \\
&\quad + r^{-3}[2f_2\sin 3\theta - 2e_2\cos 3\theta + 3(f_2-f_1)\sin 5\theta + 3(e_1-e_2)\cos 5\theta] + O(r^{-4}), \\
v_3(r, \theta) &= -(a_1+a_2)\ln r - [a_1+a_2+a_0\sin 2\theta + (a_1-a_2)\cos 2\theta] \\
&\quad + r^{-1}[(b_1+d_1)\sin\theta + (b_2+d_2)\cos\theta - b_1\sin 3\theta + b_2\cos 3\theta] \\
&\quad + r^{-2}[g_1\sin 2\theta + g_2\cos 2\theta + (e_2-e_1)\sin 4\theta + (f_2-f_1)\cos 4\theta] + O(r^{-3}),
\end{aligned} \tag{3.1}$$

(3.1) [cont.]
$$v_4(r,\theta) = -r^{-1}[2a_{\hat{}}\sin\theta + a_0\cos\theta + (a_2-a_1)\sin 3\theta + a_0\cos 3\theta] - r^{-2}[(2b_2+d_2)\sin 2\theta - d_1\cos 2\theta + 2b_2\sin 4\theta + 2b_1\cos 4\theta] - r^{-3}[(2f_2+a_3)\sin 3\theta - (2e_2-a_4)\cos 3\theta + 3(f_2-f_1)\sin 5\theta + 3(e_1-e_2)\cos 5\theta] + O(r^{-4}),$$

$$v_5(r,\theta) = r^{-1}[a_0\sin\theta + 2a_1\cos\theta - a_0\sin 3\theta + (a_2-a_1)\cos 3\theta] + r^{-2}[(2b_1+d_1)\sin 2\theta + d_2\cos 2\theta - 2b_1\sin 4\theta + 2b_2\cos 4\theta] + r^{-3}[(2e_1+a_4)\sin 3\theta + (2f_1+a_3)\cos 3\theta + 3(e_2-e_1)\sin 5\theta + 3(f_2-f_1)\cos 5\theta] + O(r^{-4}),$$

where (r,θ) are polar coordinates and a_0, a_α, $a_{\alpha+2}$, b_α, d_α, d_α, e_α, f_α, and g_α arbitrary constants. Also, let $\mathscr{A}^* = \{v^* | v^* = v + v_0,\ v \in \mathscr{A},\ v_0$ is of the form (2.5)$\}$.

REMARK. Any solution v of Eq. (2.1) of class $\mathscr{A}$ or $\mathscr{A}^*$ is a finite energy solution [6].

THEOREM 1. *If* $v \in C^2(\Omega_{ex}) \cap \bar{C}^1(\Omega_{ex}) \cap \mathscr{A}$ *is a solution of* Eq. (2.1) *then*

$$\chi_2(x)v(x) = -\int_{\partial\Omega} [D(x,y)(Tv)(y) - P(x,y)v(y)]\,ds_y,$$

where

$$\chi_2(x) = \begin{cases} 0, & x \in \Omega_{in}, \\ 1/2, & x \in \partial\Omega, \\ 1, & x \in \Omega_{ex}. \end{cases}$$

P r o o f. Let K_R be a circle with the centre at $x \in \Omega_{ex}$ and radius R sufficiently large so that $\bar{\Omega}_{in} \subset K_R$. Applying Eq. (2.9) in $\Omega_{ex} \cap K_R$, we find that

$$v(x) = -\int_{\partial\Omega} [D(x,y)(Tv)(y) - P(x,y)v(y)]\,ds_y + \int_{\partial K_R} [D(x,y)(Tv)(y) - P(x,y)v(y)]\,ds_y. \tag{3.2}$$

Choosing the pole at x, from Eqs. (2.2), (2.3), (2.6)–(2.8) and (3.1) it follows [6] that, as $R \to \infty$,

$$T_{3j}v_j = R^{-3}\{(\mu+\varkappa)[(e_1+e_2-2g_1)\sin 2\theta + (f_1+f_2-2g_2)\cos 2\theta] + \varkappa(a_3\cos 2\theta + a_4\sin 4\theta)\} + O(R^{-4}),$$

$$(D_{3\alpha}T_{\alpha j} + D_{3k}T_{kj} - P_{3j})v_j = \frac{k^2}{2\pi}\varkappa^2(2\mu+\varkappa)^2 k_5 R^{-1}[(\beta - h^2\lambda)(2\ln R + 1) - h^2(\lambda+\mu) + \gamma][a_0\sin 2\theta + (a_1-a_2)\cos 2\theta] + O(R^{-2}\ln R),$$

$$(D_{\alpha i}T_{ij} - P_{\alpha j})v_j = O(R^{-2}\ln R),$$

$$(D_{ki}T_{ij} - P_{kj})v_j = O(R^{-2}\ln R), \qquad i,j = 1,\ldots,5, \qquad k = 4,5.$$

Hence, the second term on the right-hand side of Eq. (3.2) is $O(R^{-1}\ln R)$ and the desired relation is obtained by letting $R \to \infty$.

If $x \in \Omega_{\text{in}}$, then, by Eq. (2.9), the right-hand side of Eq. (3.2) is equal to zero, since x is in the exterior of $\Omega_{\text{ex}} \cap K_R$. The proof is similar for $x \in \partial\Omega$.

THEOREM 2. *If* $v \in C^2(\Omega_{\text{ex}}) \cap C^1(\bar{\Omega}_{\text{ex}}) \cap \mathscr{A}^*$ *is a solution of* (2.1), *then*

$$2 \int_{\Omega_{\text{ex}}} E(v, v) d\sigma = - \int_{\partial\Omega} v^{\mathrm{T}} Tv \, ds.$$

Proof. Let $v^* = v + v_0 \in \mathscr{A}^*$, with $v \in \mathscr{A}$ and v_0 of the form (2.5). Then $Tv^* = Tv$, and the desired formula is established by the method used in the proof of Theorem 1 after verifying [6] that, as $R \to \infty$,

$$\begin{aligned} T_{3j} v_j &= O(R^{-3}), \\ T_{\alpha j} v_j &= O(R^{-2}), \\ T_{kj} v_j &= O(R^{-2}), \quad j = 1, \ldots, 5, \quad k = 4, 5. \end{aligned}$$

4. Uniqueness of the solution

Let $A(x)$, $B(x)$, $R(x)$ and $S(x)$ be (5×1)-matrices defined and continuous on $\partial\Omega$. We consider the following interior and exterior Dirichlet and Neumann-type problems:

(D_{in}) Find $v \in C^2(\Omega_{\text{in}}) \cap C^1(\bar{\Omega}_{\text{in}})$ satisfying Eq. (2.1) in Ω_{in} and

$$v(x) = A(x), \quad x \in \partial\Omega.$$

(N_{in}) Find $v \in C^2(\Omega_{\text{in}}) \cap C^1(\bar{\Omega}_{\text{in}})$ satisfying Eq. (2.1) in Ω_{in} and

$$(Tv)(x) = B(x), \quad x \in \partial\Omega.$$

(D_{ex}) Find $v \in C^2(\Omega_{\text{ex}}) \cap C^1(\bar{\Omega}_{\text{ex}}) \cap \mathscr{A}^*$ satisfying Eq. (2.1) in Ω_{ex} and

$$v(x) = R(x), \quad x \in \partial\Omega.$$

(N_{ex}) Find $v \in C^2(\Omega_{\text{ex}}) \cap C^1(\bar{\Omega}_{\text{ex}}) \cap \mathscr{A}$ satisfying Eq. (2.1) in Ω_{ex} and

$$(Tv)(x) = S(x), \quad x \in \partial\Omega.$$

THEOREM 3. (i) (D_{in}), (D_{ex}) *and* (N_{ex}) *have at the most one solution.*

(ii) *Any two solutions of* (N_{in}) *differ by a matrix of the form* (2.5).

Proof. The difference v of two solutions of (D_{in}) satisfies Eq. (2.1) and $v(x) = 0$, $x \in \partial\Omega$. Then from Eq. (2.4) it follows that $\int_{\Omega_{\text{in}}} E(v, v) d\sigma = 0$, therefore, v is of the form (2.5), and the homogeneous boundary condition yields $v(x) = 0$, $x \in \Omega_{\text{in}}$.

The proof for (D_{ex}) and (N_{ex}) is similar, with Eq. (2.4) replaced by Theorem 2. In the case of (N_{in}), the (5×1)-matrix supplied by Eq. (2.4) remains arbitrary.

Since in the application of the boundary integral equation method to the above problems the solution is sought in the form of single or double layer potentials, it is important to check the asymptotic behaviour of such objects. They are defined, respectively, by

$$V(x) = \int_{\partial\Omega} D(x, y) z(y) ds_y,$$

$$W(x) = \int_{\partial\Omega} P(x, y) z(y) ds_y,$$

where z is a density (5×1)-matrix on $\partial\Omega$.

THEOREM 4. *If* $z \in C(\partial\Omega)$, *then*

(i) $W \in \mathscr{A}$;

(ii) $V \in \mathscr{A}$ *if and only if*

$$\begin{aligned} M &= \int_{\partial\Omega} z_3\, ds = 0, \\ M_\alpha &= \int_{\partial\Omega} (x_\alpha z_3 - z_\alpha - \varepsilon_{\alpha\beta}\beta z_{\beta+3}) ds = 0, \end{aligned} \tag{4.1}$$

where $\varepsilon_{\alpha\beta}$ *is the alternating symbol.*

P r o o f. Using series expansions for $|x-y|^{-2}$ and $\ln|x-y|$ with $|x|$ large [4], it is easily seen that W fits the pattern (3.1), while for V we obtain [6]

$$V_1(r, \theta) = \frac{k^2}{8\pi} k_5 A_1 [(2\ln r + 1)(Mr\cos\theta - M_1) - 2M_1\cos^2\theta - M_2\sin 2\theta] + \tilde{V}_1(r, \theta),$$

$$V_2(r, \theta) = \frac{k^2}{8\pi} k_5 A_1 [(2\ln r + 1)(Mr\sin\theta - M_2) - 2M_2\sin^2\theta - M_1\sin 2\theta] + \tilde{V}_2(r, \theta),$$

$$\begin{aligned} V_3(r, \theta) = \frac{k^2}{8\pi} [&k_5 A_1 r(2\ln r + 1)(M_1\cos\theta + M_2\sin\theta) + 4k_5 A_2 M(\ln r + 1) \\ &- A_1 M(4k_4 + k_5 r^2)\ln r] + \tilde{V}_3(r, \theta), \end{aligned}$$

$$V_4(r, \theta) = -\frac{k^2}{8\pi} k_5 A_1 [(2\ln r + 1)(Mr\sin\theta - M_2) - 2M_2\sin^2\theta - M_1\sin 2\theta] + \tilde{V}_4(r, \theta),$$

$$V_5(r, \theta) = \frac{k^2}{8\pi} k_5 A_1 [(2\ln r + 1)(Mr\cos\theta - M_1) - 2M_1\cos^2\theta - M_2\sin 2\theta] + \tilde{V}_5(r, \theta),$$

where the A_α are certain combinations of the elastic coefficients and $\tilde{V} = (\tilde{V}_1, \ldots, \tilde{V}_5)^{\mathrm{T}} \in \mathscr{A}$.

It can be shown [7] that the conditions (4.1) are physically meaningful.

5. Example

We consider an infinite plate with a circular hole of radius q, whose lateral surface is acted upon by a normal force px_3, where $p = \text{const} > 0$, in the absence of body forces and couples and of forces and couples on the faces. Choosing the origin at the centre of the hole, we find that

$$\begin{aligned} (T_{ij}v_j)(x) &= \frac{ph^2}{q} x_\alpha \delta_{\alpha i}, \qquad x \in \partial\Omega, \\ (T_{ij}v_j)(x) &\to 0 \quad \text{as} \quad |x| \to \infty, \qquad i, j = 1, \ldots, 5. \end{aligned} \tag{5.1}$$

where $\delta_{\alpha i}$ is the Kronecker delta. A solution of this problem is [8]

$$\begin{aligned}
v_1(r,\theta) &= [\sigma_1 r^{-1} + \sigma_2\sigma_3 K_1(c_3 r)]\cos\theta,\\
v_2(r,\theta) &= [\sigma_1 r^{-1}\sigma_2\sigma_3 K_1(c_3 r)]\sin\theta,\\
v_3(r,\theta) &= -\sigma_1 - \sigma_2\sigma_4 K_0(c_3 r),\\
v_4(r,\theta) &= -\left[\sigma_1 r^{-1} + \frac{1}{2}k_1\sigma_2 K_1(c_3 r)\right]\sin\theta,\\
v_5(r,\theta) &= \left[\sigma_1 r^{-1} + \frac{1}{2}k_1\sigma_2 K_1(c_3 r)\right]\cos\theta,
\end{aligned}$$

where $\sigma_1, \ldots, \sigma_4$ are uniquely determined combinations of the elastic coefficients and K_1 is the modified Bessel function of first order. The solution of this (N_{ex}) is not unique in general, since $v + v_0$ also satisfies Eqs. (2.1) and (5.1) for any v_0 of the form (2.5). However, we do have uniqueness in the sense of our definition of solution because $v + v_0 \in \mathscr{A}$ if and only if [6] $v_0 = (0, 0, -\sigma_1, 0, 0)^T$.

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