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Instability of the motion of a spherical drop in a vertical temperature gradient

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THE STABILITY of the motion of a spherical drop is analyzed in the presence of Marangoni effect generated by heterogeneity of a temperature field. The solution of the linearised Navier–Stokes equations, describing the motion of a drop is found. Using the classical approach of the linear theory of stability, the equation determining the values of frequences of disturbances is obtained. The given numerical example indicates the existence of disturbances growing in time, what means the instability of the investigated motion.

Badana jest stabilność ruchu sferycznej kropli przy udziale efektu Marangoniego, wywołanego niejednorodnością pola temperatury. Wyprowadzono rozwiązanie zlinearyzowanego układu równań Naviera–Stokesa opisujące ruch sferycznej kropli. Stosując klasyczne podejście liniowej teorii stabilności, wyprowadzono równanie określające częstości zaburzeń. Podano przykład liczbowy świadczący o istnieniu rosnących wraz z czasem zaburzeń i w konsekwencji o niestabilności ruchu kropli.

В работе рассматривается устойчивость движения жидкой сферической капли, движущейся из-за действия сил тяжести и сил возникающих за счет изменения поверхностного напряжения в поле температуры. Найдены решения линеаризованных уравнений Навье–Стокса, описывающих движение капли. Применяя классический аппарат линейной устойчивости, выведены уравнения, определяющие частоты возмущений. Приведен численный пример указывающий на существование возрастающих со временем возмущений и вследствие этого на неустойчивость течения.

1. Introduction

A FLUID DROP immersed in another fluid immiscible with it is driven by buoyancy and by the force resulting from the variation of surface tension. The last effect is called the Marangoni effect. Buoyancy is proportional to the difference of densities of the fluids and to gravitation; it vanishes in the case of equal densities or in the absence of gravity.

In some technological processes migration of droplets is desirable (processing of high quality glasses); in other cases it hinders the achievement of an intentional aim (preparation of composites, foamy materials). In the absence of gravitation in the Spacelab, only the Marangoni effect can be used to eliminate unnecessary bubbles forming in glass processing.

Also, according to current views, the Marangoni effect plays an important role in biological processes but, in that case, surface tension variation is due to chemical reactions on the surfaces of cells. The first attempt at constructing a mathematical model of the

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main aspects of such processes was made by Sorensen in the paper [1]. In that paper the stability of a spherical interface, with chemical reactions on the surface, is investigated. As a result it is shown that for the instability of an interface, the instability of chemical reactions is necessary.

In technical problems, the role of chemical reactions generating the variation of surface tension is played by the temperature gradient. It is interesting to note that for the instability of motion of a drop, in the absence of chemical reactions, the temperature must be unstable. Apart from that, the problem of stability of the motion of a drop is in itself a problem of physical interest.

In the present paper, the stability of the motion of a spherical drop is investigated. As the basic solution, the one describing the motion of a drop and given in [2, 3] has been assumed. Modal expansion of disturbances of the velocity field and pressure is used in the form of spherical functions. A secular equation is obtained. The numerical solution of this equation for the chosen example shows instability. It seems that instability occurs in the majority of cases.

2. Hydrodynamical equations and the solution describing the motion of a drop in unlimited media

Assuming that the velocity of a drop is low, we can use the linearised form of Navier–Stokes equations which, in the spherical coordinate system, with the origin in the centre of a drop, can be written in the form

$$\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial \varphi}+\frac{2v_r}{r}+\frac{v_\theta \operatorname{ctg}\theta}{r}=0, \tag{2.1}$$

$$\varrho\frac{\partial v_r}{\partial t}+\frac{\partial P}{\partial r}=\mu\left[\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \varphi^2}+\frac{2}{r}\frac{\partial v_r}{\partial r}\right.$$
$$\left.+\frac{\operatorname{ctg}\theta}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}-\frac{2}{r^2\sin\theta}\frac{\partial v_\varphi}{\partial \varphi}-\frac{2v_r}{r^2}-\frac{2\operatorname{ctg}\theta}{r^2}v_\theta\right], \tag{2.2}$$

$$\varrho\frac{\partial v_\theta}{\partial t}+\frac{1}{r}\frac{\partial P}{\partial \theta}=\mu\left[\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \varphi^2}+\frac{2}{r}\frac{\partial v_\theta}{\partial r}\right.$$
$$\left.+\frac{\operatorname{ctg}\theta}{r^2}\frac{\partial v_\theta}{\partial \theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\varphi}{\partial \varphi}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta}{r^2\sin^2\theta}\right], \tag{2.3}$$

$$\varrho\frac{\partial v_\varphi}{\partial t}+\frac{1}{r\sin\theta}\frac{\partial P}{\partial \varphi}=\mu\left[\frac{\partial^2 v_\varphi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_\varphi}{\partial \theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\varphi}{\partial \varphi^2}\right.$$
$$\left.+\frac{2}{r}\frac{\partial v_\varphi}{\partial r}+\frac{\operatorname{ctg}\theta}{r^2}\frac{\partial v_\varphi}{\partial \theta}+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \varphi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \varphi}-\frac{v_\varphi}{r^2\sin^2\theta}\right]. \tag{2.4}$$

Buoyancy forces are included in P, $P = p+\varrho gz$, p — static pressure. The polar axis z is vertically directed, v_r, v_θ, v_φ — components of velocity, ϱ — density, the temperature gradient is parallel to the axis z.

We denote all parameters describing flow inside the drop with the subscript i, and outside with the subscript 0.

In accordance with our assumptions concerning the directions of the temperature gradient, we have an axisymmetrical problem. In consequence, the velocity components and pressure $v_r = V_r(r, \theta)$, $v_\theta = V_\theta(r, \theta)$, $P = P(r, \theta)$, are the functions of r and θ, and $v_\varphi = 0$. We have the following boundary conditions:

$$V_{r0} = U\cos\theta \quad \text{for} \quad r \to \infty, \tag{2.5}$$

V_{ri} = finite value, for $r = 0$, U — the axial velocity of the drop.

From Eqs. (2.1), (2.2) and (2.3) follows the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial P}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) = 0, \tag{2.6}$$

the solution of which can be described by the Legendre s polynomial p_k

$$P = \sum_{n=1}^{\infty} A_n r^{-n} p_{n-1} + B_n r^n p_n,$$

and we have to take

$$P_0 = \frac{A}{r^2}\cos\theta, \qquad P_i = Br\cos\theta, \tag{2.7}$$

since otherwise there are no solutions of the system (2.1), (2.2), (2.3) satisfying the conditions (2.5). Continuity equation at $r = R$ (R is the radius of the drop) gives

$$B = \frac{A}{R^3}. \tag{2.8}$$

Substituting Eq. (2.7) in Eqs. (2.1), (2.2), (2.3), we get the solution for the velocity components in the form

$$\begin{aligned} V_{r0} &= \cos\theta\left(U + \frac{C}{r^3} + \frac{A}{\mu_0 r}\right), \\ V_{ri} &= \cos\theta\left(D + \frac{B}{10\mu_i}r^2\right), \\ V_{\theta 0} &= \sin\theta\left(-U + \frac{C}{2r^3} - \frac{A}{2\mu_0 r}\right), \\ V_{\theta i} &= \sin\theta\left(-D + \frac{B}{5\mu_i}r^2\right), \end{aligned} \tag{2.9}$$

where A, B, C, D are constants which are determined by the boundary conditions of the flow. The boundary conditions of velocity at the surface of the drop of the radius R is given as follows:

$$\begin{aligned} V_{r0} &= V_{ri} = 0 \quad \text{at} \quad r = R, \\ V_{\theta 0} &= V_{\theta i} = 0 \quad \text{at} \quad r = R. \end{aligned} \tag{2.10}$$

Also on the surface of the drop the normal and the shearing stresses should be equal:

$$\sigma_{rr0} = -P_0+2\mu_0\frac{\partial V_{r0}}{\partial r} = -P_i+2\mu_i\frac{\partial V_{ri}}{\partial r}+\frac{2\gamma}{R} = \sigma_{rri} \quad \text{at} \quad r = R,$$

(2.11)

$$\sigma_{r\theta 0} = \mu_0\left(\frac{1}{r}\frac{\partial V_{r0}}{\partial \theta}+\frac{\partial V_{\theta 0}}{\partial r}-\frac{V_{\theta 0}}{r}\right) = \mu_i\left(\frac{1}{r}\frac{\partial v_{ri}}{\partial \theta}+\frac{\partial V_{\theta i}}{\partial r}-\frac{V_{\theta i}}{r}\right)+f\gamma' = \sigma_{r\theta i},$$

γ — surface tension,

$$f\gamma' = \gamma'\frac{dT}{dl} = \gamma' T'\sin\theta \tag{2.12}$$

the force due to the thermal variation of the surface tension.

For the calculation of $f\gamma'$, the temperature field is needed. The energy equation has the form

$$\frac{\partial T}{\partial t}+V(\operatorname{grad} T) = \nabla^2 T\varkappa,$$

$\varkappa$ — the coefficient of the thermal diffusivity of the fluid.

If $V \ll 1$ and $\varkappa \gg 1$, the energy equation can be approximated by the Laplacian equation

$$\nabla^2 T = 0,$$

which gives the solution

$$T_i = T_c+T'\left(r+\frac{E}{r^2}\right)\cos\theta,$$

$$T_0 = T_c+Fr\cos\theta,$$

satisfying the boundary conditions suitable to our problem,

$$T = T_c(t)+T'r\cos\theta \quad \text{for} \quad r\to\infty,$$

$$T<\infty \quad \text{for} \quad r = 0,$$

T_c — function of t connected with the motion of the centre of the drop.

Substituting Eqs. (2.9) in Eqs. (2.10) and (2.11) and using (2.8), the constants A, B, C, D and U can be determined. The following form of U,

$$U = \frac{2g(\varrho_i-\varrho_0)(\mu_i-\mu_0)R^2+4\gamma}{3\mu_i(2\mu_i+3\mu_0)}-\frac{2(2\mu_0+\mu_i)\gamma T'R}{3\mu_i(2\mu_i+3\mu_0)} \tag{2.13}$$

describes the velocity of the drop in a laboratory system. The first term is due to gravity and the second to the termocapillary convection.

3. The formulation and the discussion of the stability problem

According to the linear stability approach, for the discussion of the stability of the solution of the system (2.1), (2.2), (2.3), (2.4) obtained in Sect. 2, we introduce the disturbances of the velocity components $\bar{u}_r$, $\bar{u}_\theta$, $\bar{u}_\varphi$ and a disturbance of the static pressure in the gravity field $\bar{p}$. The disturbed velocity components $V_r+\bar{u}_r$, $V_\theta+\bar{u}_\theta$, $\bar{u}_\varphi$ and the disturbed pressure $P+\bar{p}$ must satisfy the equation (2.1)–(2.4). Since the system $V_r, V_\theta, V_\varphi = 0$, P satisfies these equations, the same must apply to the disturbances $\bar{u}_r, \bar{u}_\theta, \bar{u}_\varphi, \bar{p}$. Using the continuity equation (2.1), the terms with $\bar{u}_\theta$ and $\bar{u}_\varphi$ can be eliminated from the momentum equation for the radial component of velocity $\bar{u}_r$. As a result the momentum equation takes the form

$$\frac{\partial \bar{u}_r}{\partial t} = -\frac{1}{\varrho}\frac{\partial \bar{p}}{\partial r} + \frac{\nu}{r}\nabla^2(ru_r), \quad \nu = \frac{\mu}{\varrho}. \tag{3.1}$$

Similarly as P, also $\bar{p}$ satisfies the Laplacian equation:

$$\nabla^2\bar{p} = \frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial \bar{p}}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\bar{p}}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\bar{p}}{\partial\varphi^2} = 0. \tag{3.2}$$

We look for the solution in the form of the modal expansion

$$p = pe^{\omega t}, \quad \bar{u}_r = u_r e^{\omega t}, \quad \bar{u}_\theta = u_\theta e^{\omega t}, \quad \bar{u}_\varphi = u_\varphi e^{\omega t}, \quad \omega = \omega_r + i\omega_i. \tag{3.3}$$

The disturbances with $\omega_r > 0$ grow in time, so if they exist, the basic solution is unstable.

The general solution of Eq. (3.2) has the form

$$\bar{p} = (a_{1l}r^l + a_{2l}r^{-(l+1)})Y_l e^{\omega t}, \tag{3.4}$$

Y_l — spherical functions, a_{1l}, a_{2l} — constants, $l = 0, 1, 2, \ldots$. Since $\bar{p}$ must be finite, in the inner solution $a_{2l} = 0$, and in the outer $a_{1l} = 0$

$$\bar{p}_{li} = a_{1l}r^l Y_l e^{\omega t} = p_{li}Y_l e^{\omega t}, \quad p_{li} = a_{1l}r^l, \tag{3.5}$$

$$\bar{p}_{lo} = a_{2l}r^{-(l+1)}Y_l e^{\omega t} = p_{lo}Y_l e^{\omega t}, \quad p_{lo} = a_{2l}r^{-(l+1)}. \tag{3.6}$$

Substituting instead of $\bar{p}, \bar{p}_{li}$ given by Eq. (3.5) and $\bar{u}_r = u_{ri}Y_l e^{\omega t}$ in Eq. (3.1), the following equation is obtained:

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{r} - \frac{l(1+1)}{r^2} - \frac{\omega}{\nu_i}\right]u_{li} = \frac{la_{1l}r^l}{\varrho_i\nu_i}, \tag{3.7}$$

where $u_{li} = ru_{ri}$, and similarly for the outer region

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{r} - \frac{l(l+1)}{r^2} - \frac{\omega}{\nu_0}\right]u_{lo} = \frac{la_{2l}r^{-(l+1)}}{\varrho_0\nu_0}, \tag{3.8}$$

where $u_{lo} = ru_{ro}$. It is easy to verify that

$$u_{li}^s = -\frac{la_{1l}r^l}{\varrho_i\omega} \tag{3.9}$$

satisfies Eq. (3.7) and

$$u_{lo}^s = \frac{(l+1)a_{2l}r^{-(l+1)}}{\varrho_0\omega} \tag{3.10}$$

is the solution of (3.8).

Now, in order to find the general solution of Eq. (3.7) or Eq. (3.8), we look for the solution of the homogeneous equation corresponding to Eq. (3.7) or Eq. (3.8)

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} - \frac{\omega}{\nu}\right]v_l = 0. \tag{3.11}$$

Introducing the new variable

(3.12) $$z = r\left(\frac{\omega}{\nu}\right)^{1/2}$$

Eq. (3.11) takes the form of the spherical Bessel equation:

(3.13) $$\left[z^2\frac{d^2}{dz^2}+2z\frac{d}{dz}-z^2+l(l+1)\right]v_l = 0,$$

the solution of which are modified Bessel functions having the following series expansions:

(3.14) $$I^*_{l+\frac{1}{2}}(z) = \frac{z^l}{(2l+1)!!}\left[1+\frac{\frac{1}{2}z^2}{1!(2l+3)}+\frac{\left(\frac{1}{2}z^2\right)^2}{2!(2l+3)(2l+5)}+\ldots,\right.$$

(3.15) $$K^*_{l+\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)e^{-z}\sum_{k=0}^{l}\frac{(l+k)!}{k!(l-k)!}(2z)^{-k}.$$

Since for $z = 0$ which corresponds to $r = 0$, $K^*_{l+\frac{1}{2}}(z)$ is infinite, it cannot be used for the construction of the inner solution, and for a similar reason (at infinity $I^*_{l+\frac{1}{2}}(z)$ is not finite) $I^*_{l+\frac{1}{2}}(z)$ cannot be used for the construction of the outer solution. As a result the most general solution of Eq. (3.7) has the form

(3.16) $$u_{li} = A_{li}x^l+B_{li}I^*_{l+\frac{1}{2}}(q_i x),$$

and of Eq. (3.8)

(3.17) $$u_{l0} = A_{l0}x^{-(l+1)}+B_{l0}K^*_{l+\frac{1}{2}}(q_0 x),$$

where $x = r/R$, $q_i = \left(\frac{\omega}{\nu_i}\right)^{1/2}R$, $q_0 = \left(\frac{\omega}{\nu_0}\right)^{1/2}R$, R — the radius of the drop.

The first terms in Eqs. (3.16) and (3.17) represent the special solutions of Eqs. (3.7) and (3.8), so A_{li} and A_{l0} are determined constants:

(3.18) $$A_{li} = -\frac{Rla_{1l}}{\varrho_i\omega},\quad A_{l0} = \frac{(l+1)a_{2l}}{\varrho_0 R^{l+1}}.$$

On the surface of the drop, all components of the velocity vector have to be equal. For the normal components we have

(3.19) $$u_{li} = u_{l0} \quad \text{for} \quad x = 1,$$

or from (3.16) and (3.17)

(3.20) $$A_{li}x^l+B_{li}I^*_{l+\frac{1}{2}}(q_i) = A_{l0}x^{-(l+1)}+B_{l0}K^*_{l+\frac{1}{2}}(q_0).$$

The second condition, following from the equality of tangential components, after using the continuity equation, can be written as

(3.21) $$\frac{du_{li}}{dx} = \frac{du_{l0}}{dx} \quad \text{for} \quad x = 1.$$

It is easy to verify that for the functions $I^*_{l+\frac{1}{2}}(z)$, $K^*_{l+\frac{1}{2}}(z)$, the following identities are true:

$$z\frac{d}{dz}f_l(z) = f_{l-1}(z)-(l+1)f_l(z), \tag{3.22}$$

$$z[f_{l-1}(z)-f_{l+1}(z)] = (2l+1)f_l(z). \tag{3.23}$$

Using Eqs. (3.16), (3.17) and (3.22), the condition (3.21) can be written in the form

$$lA_{li}+[q_iI^*_{l-\frac{1}{2}}(q_i)-(l+1)I^*_{l+\frac{1}{2}}(q_i)]B_{li} = -(l-1)A_{l0}-[q_0K^*_{l-\frac{1}{2}}(q_0)-(l+1)K^*_{l+\frac{1}{2}}(q_0)]B_{l0}.$$

Solving the last equation and Eq. (3.20) with respect to A_{li} and A_{l0}, we obtain

$$A_{li} = -[q_iI^*_{l-\frac{1}{2}}(q_i)B_{li}+q_0K^*_{l-\frac{1}{2}}(q_0)B_{l0}]\frac{1}{2l+1}, \tag{3.24}$$

$$A_{l0} = -[q_iI^*_{l+\frac{1}{2}}(q_i)B_{li}+q_0K^*_{l-\frac{1}{2}}(q_0)B_{l0}]\frac{1}{2l+1}. \tag{3.25}$$

Besides the continuity conditions for the velocity vector, expressed by the equalities (3.24) and (3.25), also the continuity of the components σ_{rr}, $\sigma_{r\theta}$, $\sigma_{r\varphi}$ of the tensor stresses, on the surface of the drop, has to be guaranteed. The condition for normal stresses after using the continuity equation, can be written as follows:

$$\bar{p}_i+2\mu_i\frac{\partial\bar{u}_{ri}}{\partial r}+\frac{\gamma}{r^2}\left[2+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\delta R|_{r=R} = p_0+2\mu_i\frac{\partial u_{r0}}{\partial r}\bigg|_{r=R}. \tag{3.26}$$

δR is a disturbance of the radius of the drop, it can be expressed in the form $\delta R = \left(\frac{\bar{u}_r}{\omega}\right)_{r=R}$ what follows from $u_r|_R = \frac{\partial\delta R}{\partial t}\Big|_{r=R} = \omega\delta R|_{r=R}$.

The continuity conditions for $\sigma_{r\theta}$, $\sigma_{r\varphi}$ can be written in the form

$$\mu_i\left[\frac{1}{R}\frac{\partial u_{ri}}{\partial\theta}+\frac{\partial u_{\theta i}}{\partial r}+\frac{1}{R}u_{\theta i}\right]_{r=R} = \mu_0\left[\frac{1}{R}\frac{\partial u_{r0}}{\partial\theta}+\frac{\partial u_{\theta 0}}{\partial r}+\frac{1}{R}u_{\theta 0}\right]_{r=R}, \tag{3.27}$$

$$\mu_i\left[\frac{1}{R\sin\theta}\frac{\partial u_{ri}}{\partial\varphi}+\frac{\partial u_{\varphi i}}{\partial r}-\frac{1}{R}u_{\varphi i}\right]_{r=R} = \mu_0\left[\frac{1}{R\sin\theta}\frac{\partial u_{r0}}{\partial\varphi}+\frac{\partial u_{\varphi 0}}{\partial r}-\frac{1}{R}u_{\varphi}\right]_{r=R}. \tag{3.28}$$

Applying the operator $\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\sin\theta$ to Eq. (3.27), the operator $\frac{1}{R\sin\theta}\frac{\partial}{\partial\varphi}$ to Eq. (3.28), adding the results and using continuity equation (2.1), the velocity components u_θ and u_φ can be eliminated, and after some calculation the following equation can be obtained:

$$\mu_i\left[\frac{l(l+1)}{R^2}u_{li}+\frac{\partial}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_{li})\right]_{r=R} = \mu_0\left[\frac{l(1+1)}{R^2}u_{l0}+\frac{\partial}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_{l0})\right]_{r=R}. \tag{3.29}$$

Substituting

$$\bar{p}_i = p_{li} Y_l e^{\omega t}, \quad \bar{p}_0 = p_{l0} Y_l e^{\omega t},$$

$$\bar{u}_{ri} = \frac{u_{li}}{r} Y_l \cdot e^{\omega t}, \quad \bar{u}_{r0} = \frac{u_{l0}}{r} Y_l \cdot e^{t\omega}, \quad \delta R = \left(\frac{u_{li}}{\omega R}\right)$$

into Eq. (3.26), we obtain the equation

$$p_{li} + 2\mu_i \frac{d}{dr}\left(\frac{u_{li}}{r}\right) + \frac{\gamma}{r^2}[2 - l(l+1)] \frac{u_{li}}{\omega r}\bigg|_{r=R} = p_{l0} + 2\mu_0 \frac{d}{dr}\frac{u_{l0}}{r}\bigg|_{r=R} \tag{3.30}$$

which, together with Eq. (3.29), gives the system of homogeneous equations joining four functions $u_{li}, u_{l0}, p_{li}, p_{l0}$. The functions p_{li}, p_{l0} depend on two undefined constants a_{1l} and a_{2l}, the functions u_{li}, u_{l0} additionally depend on the constants B_{li}, B_{l0}. Substituting Eqs. (3.5), (3.6), (3.16) and (3.17) in Eqs. (3.29) and (3.30) and eliminating $a_{1l}, a_{2l}, A_{li}, A_{l0}$, using Eqs. (3.18), (3.24) and (3.25), we come to the system of two equations:

$$\begin{aligned} C_{11} B_{li} + C_{12} B_{l0} &= 0, \\ C_{21} B_{li} + C_{22} B_{l0} &= 0, \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} C_{11} &= -\left[\frac{2l(l+2)}{2l+1}(\mu_0 - \mu_i) - \mu_i\right] q_i I^*_{l-\frac{1}{2}}(q_i) - \mu_i q_i^2 I^*_{l+\frac{1}{2}}(q_i), \\ C_{12} &= \left[\frac{2(l+1)(1-l)}{2l+1}(\mu_0 - \mu_i) - \mu_0\right] q_0 K^*_{l-\frac{1}{2}}(q_0) + (q_2^2 - 2l - 1)\mu_0 K^*_{l+\frac{1}{2}}(q_0), \end{aligned} \tag{3.32}$$

$$\begin{aligned} C_{21} &= \frac{1}{2l+1}\left[\left(\frac{\varrho_i}{l} + \frac{\varrho_0}{l+1}\right)\omega + \frac{2(2+l)}{R^2}(\mu_0 - \mu_i) + \frac{l(l+1)-2}{R^3}\gamma\right] q_i I^*_{l+\frac{1}{2}}(q_i) \\ &\quad + \frac{\omega \varrho_i}{l} I^*_{l+\frac{1}{2}}(q_i), \\ C_{22} &= \frac{1}{2l+1}\left[\left(\frac{\varrho_i}{l} + \frac{\varrho_0}{l+1}\right)\omega + \frac{2(1-l)}{R^2}(\mu_0 - \mu_i) + \frac{l(l+1)-2}{R^3 \omega}\gamma\right] q_0 K^*_{l-\frac{1}{2}}(q_0) \\ &\quad + \frac{\omega \varrho_0}{l+1} K^*_{l-\frac{1}{2}}(q_0). \end{aligned} \tag{3.32}$$

The system (3.31) has the nontrivial solution if

$$\det\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = 0. \tag{3.33}$$

The matrix elements C_{ij} are, as we can see from the system (3.31), complicated functions of the frequency parameter (through the parameters q_i, q_0). The solution of the secular equation (3.33) gives values of the frequency ω_{lk}. The existence of ω_{lk} with positive real part means that there are growing disturbances and, in consequence, that flow is unstable. In view of the complexity of the secular equation, the general discussion of the properties of its roots cannot be effective. Only numerical calculation for every interesting case can give the required results.

4. Numerical example

To calculate the roots of the secular equation (3.32), the values of the functions $I^*_{l+\frac{1}{2}}(q)$, $K^*_{l+\frac{1}{2}}(q)$ are necessary. These functions depend on the frequency parameter, through the parameters q_i, q_0, and are defined by Eqs. (3.14) and (3.15). The function $I^*_{l+\frac{1}{2}}(q)$ is given in the form of an infinite power series. For calculation, only finite number of the first terms can be used, what gives approximate values. The accuracy of the approximation depends on the number of terms taken to calculate $I^*_{l+\frac{1}{2}}(q)$. We design the order of approximation by the number of terms used in the calculation of $I^*_{l+\frac{1}{2}}(q)$. Due to the complexity of the equation solved, it is difficult to estimate the condition of calculation of $I^*_{l+\frac{1}{2}}(q)$ ensuring the required accuracy of calculation of ω. But, if for several consecutive orders of approximation the calculated values of ω differ in a sufficiently small manner, we can recognize the result to be satisfactory.

For the construction of a numerical example we selected the problem of air drop, or rather bubble, immersed in water, and moving according to the variation of surface

Table 1. Properties of fluids.

medium	ϱ [kg m^{-3}]	$\mu \cdot 10^4$ [Ns m^{-2}]	$\gamma \cdot 10^3$ [Nm^{-1}]
water	1000	17	20
air	1.3	0.2	20

Table 2. The values of a real part of the frequency parameter ω.

$\left[\frac{1}{s}\right]$

R [mm]	l	First approximation	Second approximation	Third approximation	Fourth approximation
10	1	0.01496	0.01421	0.01432	0.01431
	2	0.00448	0.00451	0.00452	0.00453
	3	0.00783	0.00752	0.00751	0.00745
	5	1.01478	0.01473	0.01473	0.01473
1	1	1.49553	1.42583	1.42763	1.42751
	2	0.44899	0.45114	0.45115	0.45115
	3	0.78595	0.78353	0.78364	0.78359
	5	1.47874	1.47393	1.47411	1.47425
0.1	1	149.553	142.587	137.453	137.211
	2	44.9229	46.2385	45.5643	45.4536
	3	78.5193	76.4372	78.3776	78.4154
	5	145.342	142.215	143.328	143.368

tension in the temperature field and to the gravity forces. Since this example serves only to illustrate the existence of growing modes, the fact that the chosen media do not fully satisfy some previously made assumptions, such as incompressibility, the small difference of densities of fluids is immaterial. After all, the last chosen media are the most common ones.

The properties of water and air necessary for calculation are given in Table 1.

Table 2 illustrates the results of calculation of the real part of frequency. The calculations have been performed up to fourth-order approximation, and for the first five spherical harmonics $l = 1, 2, 3, 4, 5$.

For $l = 4$ and all chosen values of R, there are no ω with a positive real part. The result of lower order approximations are shown in Table 2 to demonstrate the influence of the subsequent term of the series expansion of $I^{*}_{l+\frac{1}{2}}(q)$ on the value of frequency. The small differences suggest that the order of approximation is high enough to make the results correct.

References

1. T. S. SORENSEN, *Chemical and hydrodynamical analysis of stability of a spherical interface*, J. Colloid and Interface Sci., **56**, 2, August 1976.
2. T. NARASAKI, *The effect of temperature gradient on the motion of a bubble in reduced gravity*, Adv. Space Res., **13**, 5, 1983.
4. R. SHANKAR, *Thermocapillary migration of bubbles and droplets*, Adv. Space Res., **3**, 5, 1983.
3. H. F. BAUER, *Combined residual natuaral and Marangoni convection in a liquid...*, Appl. Math. and Mech., **65**, 10, 1985.
5. S. CHANDRASEKHAR, *Hydrodynamic and hydromagnetic stability*, Oxford Univ. Press, 1961.

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