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Dynamic properties of two elastic layers

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THERE EXIST numerous papers and monographs dealing with the problem of a layered elastic medium. The dynamic properties of such a medium depend essentially on the order of the layers. This allows, in particular, the experimental detection of the structure by measurements of the reflected waves. In the present paper we shall show that for two layers embedded in one medium the transmitted wave does not depend on the order of the layers. It seems that this invariance was never noticed.

Płaska fala sinusoidalna pada prostopadle na układ dwu warstw sprężystych. Pokazuje się, że transmitancja jest niezależna od kolejności warstw. Falę nieciągłości modeluje się jako skończoną sumę fal sinusoidalnych. Fala odbita zależy istotnie od kolejności warstw. Fala przechodząca jest niezmiennicza względem zmiany kolejności warstw.

Плоская синусоидальная волна падает перпендикулярно на систему двух упругих слоев. Показывается, что передаточная функция не зависит от очередности слоев. Волна разрыва моделируется как конечная сумма синусоидальных волн. Отраженная волна зависит существенным образом от очередности слоев. Проходящая волна инвариантна по отношению к изменению очередности слоев.

1. One layer. Sinusoidal wave

CONSIDER an elastic layer immersed in an elastic medium, Fig. 1. The elastic properties of the layer and the medium are assumed to be different. In the further calculations ϱ and c denote the density and speed of the longitudinal wave. It is assumed

$$\varrho_1 = \varrho_3 = \varrho, \quad c_1 = c_3 = c, \quad \varrho_2 c_2^2 \neq \varrho c^2. \tag{1.1}$$

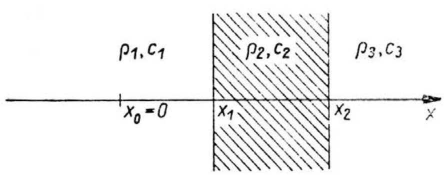


FIG. 1.

The incident wave running in the direction of the x-axis produces the reflected wave and the transmitted wave. In the case of a monochromatic sinusoidal wave the solution is well known, cf. eg. [1]. Here we intend to discuss some properties of other profiles.

We shall concentrate on profiles approximating the discontinuity wave. Such profiles will be produced by finite sums of sinusoidal waves.

First we quote the known results for monochromatic waves, cf. [1, 2]. In the regions 1, 2 and 3, Fig. 1, the displacements are

$$(1.2)\qquad \begin{aligned} u_1 &= A_1 \exp i\omega\left(t-\frac{x-x_0}{c_1}\right)+B_1\exp i\omega\left(t+\frac{x-x_0}{c_1}\right),\\ u_2 &= A_2 \exp i\omega\left(t-\frac{x-x_1}{c_2}\right)+B_2\exp i\omega\left(t+\frac{x-x_1}{c_2}\right),\\ u_3 &= A_3 \exp i\omega\left(t-\frac{x-x_2}{c_3}\right)+B_3\exp i\omega\left(t+\frac{x-x_2}{c_3}\right),\end{aligned}$$

where ω is the constant frequency. The terms proportional to A_K represent waves running to the right, the terms proportional to B_K the waves running to the left. Both the real, or the imaginary part of u satisfy the equations of motion.

The displacement and stress are continuous at $x = x_1$ and $x = x_2$. This condition gives two algebraic relations between A_1, B_1, A_2, B_2 and two relations between A_2, B_2, A_3, B_3, namely,

$$(1.3)\qquad \begin{bmatrix} A_2\\ B_2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}(1+\varkappa)G_1 & (1-\varkappa)F_1\\ (1-\varkappa)G_1 & (1+\varkappa)F_1\end{bmatrix}\begin{bmatrix} A_1\\ B_1\end{bmatrix},$$

$$(1.4)\qquad \begin{bmatrix} A_3\\ B_3\end{bmatrix} = \frac{1}{2}\begin{bmatrix}(1+1/\varkappa)G_2 & (1-1/\varkappa)F_2\\ (1-1/\varkappa)G_2 & (1+1/\varkappa)F_2\end{bmatrix}\begin{bmatrix} A_2\\ B_2\end{bmatrix},$$

where

$$\varkappa = \frac{\varrho_1 c_1}{\varrho_2 c_2},$$

$$(1.5)\qquad \begin{aligned} G_1 &= \exp(-i\alpha_1), & F_1 &= \exp i\alpha_1, & \alpha_1 &= \omega\frac{x_1-x_0}{c_1},\\ G_2 &= \exp(-i\alpha_2), & F_2 &= \exp i\alpha_2, & \alpha_2 &= \omega\frac{x_2-x_1}{c_2}.\end{aligned}$$

From the above follow the relations

$$(1.6)\qquad \varrho_1 c_1(A_1\bar{A}_1-B_1\bar{B}_1) = \varrho_2 c_2(A_2\bar{A}_2-B_2\bar{B}_2) = \varrho_3 c_3(A_3\bar{A}_3-B_3\bar{B}_3).$$

We now give special interpretation to the waves (1.2), The term $A_1\dots$ is the incident wave. Its phase

$$t-\frac{x-x_0}{c_1}$$

is constant at points x moving with speed c_1 to the right, toward the layer. The term $B_1\dots$ is the reflected wave moving from the layer to the left. The term $A_3\dots$ is the transmitted wave. The term $B_3\dots$ represents the wave running from the right to the left. Because it is assumed that $A_1\dots$ is the only incident wave, we take $B_3 = 0$. The terms $A_2\dots$, $B_2\dots$ represent waves in the layer; the first running to the right, the second running to the left.

The system of Eqs. (1.3) and (1.4) for $B_3 = 0$ allows to express the intensities B_1 and A_3 of the reflected and transmitted waves in terms of A_1:

$$B_1 = \frac{(1-\varkappa^2)\left(\exp(-i\alpha_2)-\exp(i\alpha_2)\right)}{-(1-\varkappa)^2\exp(-i\alpha_2)+(1+\varkappa)^2\exp(i\alpha_2)} A_1\exp(-2i\alpha_1), \tag{1.7}$$

$$A_3 = \frac{1}{4\varkappa}\{[(1+\varkappa)^2\exp(-i\alpha_2)-(1-\varkappa)^2\exp(i\alpha_2)]A_1\exp(-i\alpha_1) + (1-\varkappa^2)[\exp(-i\alpha_2)-\exp(i\alpha_2)]B_1\exp(i\alpha_1)\}. \tag{1.8}$$

Because of the further numerical calculations, remove the imaginary part from the denominators in the above formulae. Finally we have

$$B_1 = P_r(\omega)A_1\exp(-2i\alpha_1), \qquad A_3 = P_t(\omega)A_1\exp(-i\alpha_1), \tag{1.9}$$

where

$$P_r(\omega) = \frac{2(1-\varkappa^2)}{M}[(1+\varkappa^2)(\cos 2\alpha_2-1)-2i\varkappa\sin 2\alpha_2], \tag{1.10}$$

$$P_t(\omega) = \frac{8\varkappa}{M}[2\varkappa\cos\alpha_2+i(1+\varkappa^2)\sin\alpha_2],$$

$$M = (1+\varkappa)^4+(1-\varkappa)^4-2(1+\varkappa)^2(1-\varkappa)^2\cos 2\alpha_2. \tag{1.11}$$

From Eqs. (1.9) and (1.10) it follows that the product $P_r\bar{P}_r$ is invariant under replacement of $\varkappa$ by $1/\varkappa$. The same identity holds for the product $P_t\bar{P}_t$.

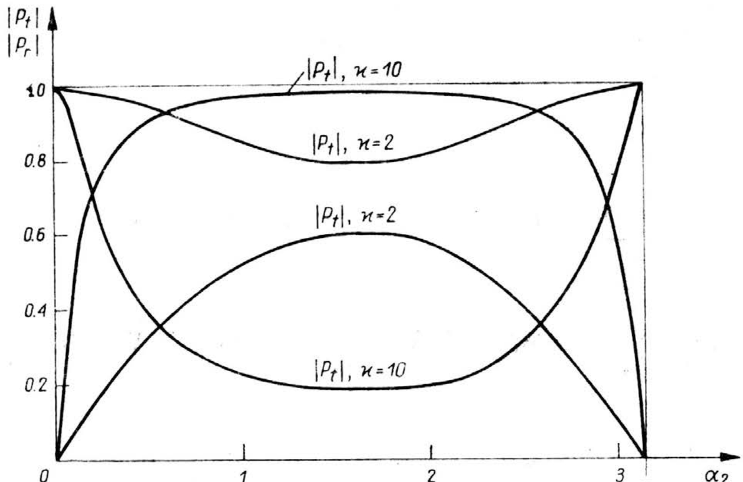


FIG. 2.

Note that the coefficients of A_1 in Eqs. (1.9) and (1.10) depend on the frequency ω. The transmittance of short waves is different from the transmittance of long waves. The same holds for the reflection. Therefore the wave profile does not change travelling in the medium or in the layer, but it does change when passing through the boundaries.

Figure 2 gives the values $|P_r|$ and $|P_t|$ for $\varkappa = 2$ and $\varkappa = 10$ as a function of α_2.

Further calculations will be based on the dimensionless quantities

$$H = \frac{x_2}{x_1}, \quad T = \frac{ct}{Wx_1}, \quad X = \frac{x}{Wx_1}, \quad N = W\frac{\omega x_1}{c}, \tag{1.12}$$

where W is a fixed dimensionless scaling parameter. We have

$$\begin{aligned} \omega\left(t \pm \frac{x}{c}\right) &= N(T \pm x), \\ \omega\left(t - \frac{x-x_2}{c}\right) &= N(T-x+H/W), \\ \alpha_1 &= N/W. \end{aligned} \tag{1.13}$$

Taking into account the above considerations we have

$$\begin{aligned} u_{(i)} &= A_1 \exp iN(T-X), \\ u_{(r)} &= A_1 P_r \exp iN(T+X-2/W), \\ u_{(t)} &= A_1 P_t \exp iN(T-X+H/W-1/W). \end{aligned} \tag{1.14}$$

Here (i), (r), (t) stands for "incident", "reflected" and "transmitted". There is $u_1 = u_{(i)} + u_{(r)}$, $u_3 = u_{(t)}$.

2. One layer. Step function

We shall assume that at $T = 0$ the speed equals zero but the deformation, at least in some regions, is different from zero. In particular, we shall consider the situation when some part of the structure is at $t = 0$ in the state of homogeneous strain and the other part is stress free.

Take

$$\begin{aligned} u_{(i)} &= \operatorname{Im} \sum_{N=1}^{K} [\exp iN(T-x) + \exp iN(-T-x)] \frac{1}{N^2} \sin N\varphi, \\ u_{(r)} &= \operatorname{Im} \sum_{N=1}^{K} P_r [\exp iN(T+X-2/W) + \exp iN(-T+X-2/W)] \frac{1}{N^2} \sin N\varphi, \\ u_{(t)} &= \operatorname{Im} \sum_{N=1}^{K} P_t [\exp iN(T-X+H/W-1/W) + \exp iN(-T-X+H/W-1/W)] \frac{1}{N^2} \sin N\varphi. \end{aligned} \tag{2.1}$$

The expressions satisfy the equations of motion and continuity conditions because each term separately satisfies them. It is easy to check that

$$\dot{u}_{(i)} = \dot{u}_{(r)} = \dot{u}_{(t)} = 0 \quad \text{for} \quad T = 0. \tag{2.2}$$

The infinite Fourier series for $\varphi = \text{const}$

$$S(x) = \frac{4}{\pi}\frac{1}{\varphi}\left(\frac{1}{1^2}\sin\varphi \sin x + \frac{1}{3^2}\sin 2\varphi \sin 3x + \frac{1}{S^2}\sin 5\varphi \sin 5x + \ldots\right) \tag{2.3}$$

has the period 2π and equals (Fig. 3)

$$(2.4)\qquad S(x) = \begin{cases} x/\varphi & \text{for} \quad 0 < x \leqslant \varphi, \\ 1 & \text{for} \quad \varphi \leqslant x < \pi-\varphi, \\ (\pi-x)/\varphi & \text{for} \quad \pi-\varphi < x < \pi+\varphi, \\ -1 & \text{for} \quad \pi+\varphi < x < 2\pi-\varphi, \\ -(2\pi-x)/\varphi & \text{for} \quad 2\pi-\varphi < x < 2\pi. \end{cases}$$

The finite sum in Eq. (2.1) approximates the function (2.4). We give now the wave profiles fixing the scaling parameters

$$(2.5)\qquad W = 4, \quad \varphi = 0.05.$$

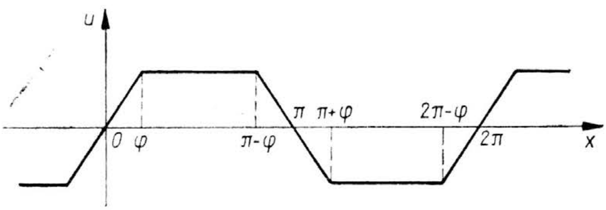


FIG. 3.

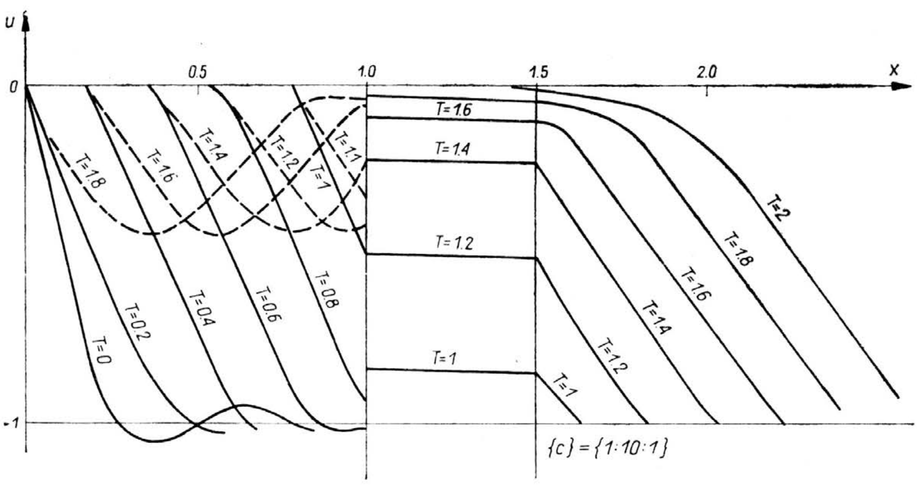


FIG. 4.

Figure 4 gives the wave profile for the case $x_2/x_1 = 1.5$, $\varkappa = 0.1$. The heavy line $T = 0$ is the initial profile. It is seen that the wave front moves to the right. At approximately $T = 8$ it reaches the layer. At $T = 1$ already appears the wave at the other side of the layer. The dotted line gives the reflected wave.

Figure 5 gives the wave fronts for $\varkappa = 2$. In this case the wave in the layer is slower than in the medium. Note the negative sign of the reflected wave. The intensity of the transmitted wave is smaller than that of the incident wave.

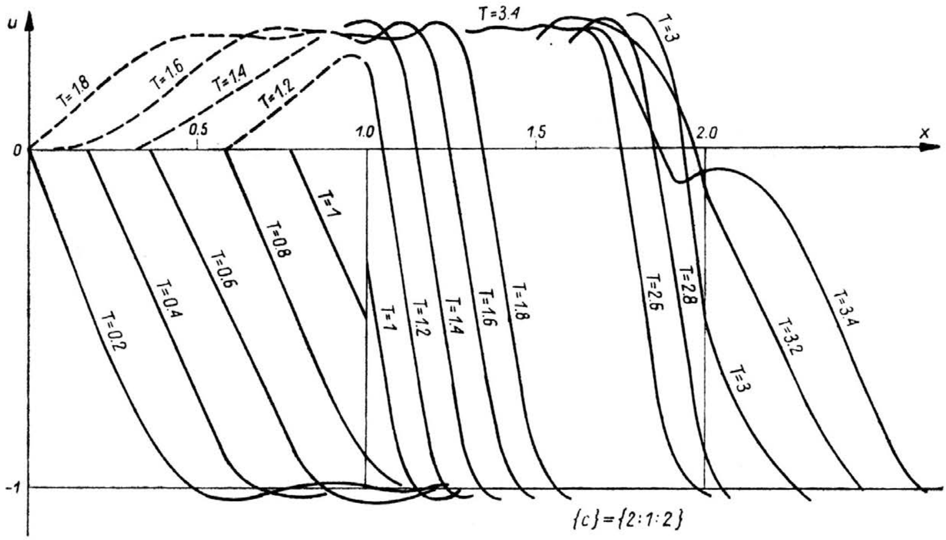


FIG. 5.

3. Two layers. Sinusoidal wave

The set of two different elastic layers is immersed in an elastic medium, Fig. 6. The speeds and densities are denoted by c_k, ϱ_k, $k = 1, 2, 3, 4$. The regions 2 and 3 correspond to the layers and the regions 1 and 4 to the medium. It is assumed

$$\varrho_1 = \varrho_4 = \varrho, \quad c_1 = c_4 = c. \tag{3.1}$$

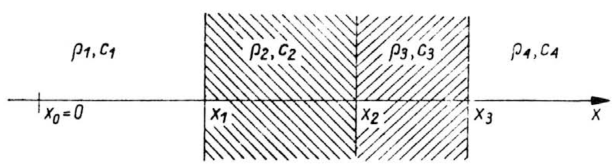


FIG. 6.

The boundaries between regions are situated at x_1, x_2 and x_3. The following notation is introduced, cf. Eq. (1.5)

$$\varkappa_1 = \frac{\varrho_1 c_1}{\varrho_2 c_2}, \quad \varkappa_2 = \frac{\varrho_2 c_2}{\varrho_3 c_3}, \quad \varkappa_3 = \frac{\varrho_3 c_3}{\varrho_4 c_4}, \tag{3.2}$$

$$\alpha_1 = \frac{x_1 - x_0}{c_1}, \quad \alpha_2 = \frac{x_2 - x_1}{c_2}, \quad \alpha_3 = \frac{x_3 - x_2}{c_3}, \tag{3.3}$$

$$\begin{gathered} P_k = 1 + \varkappa_k, \quad Q_k = 1 - \varkappa_k, \\ F_k = \exp(i\omega\alpha_k), \quad G_k = \exp(-i\omega\alpha_k), \quad K = 1, 2, 3. \end{gathered} \tag{3.4}$$

In each region the displacement cosists of one sinusoidal wave of frequency ω running to the right and one wave of the same frequency running to the left.

(3.5)
$$\begin{aligned}
u_1 &= A_1 \exp i\omega\left(t - \frac{x-x_0}{c_1}\right) + B_1 \exp i\omega\left(t + \frac{x-x_0}{c_1}\right),\\
u_2 &= A_2 \exp i\omega\left(t - \frac{x-x_1}{c_2}\right) + B_2 \exp i\omega\left(t + \frac{x-x_1}{c_2}\right),\\
u_3 &= A_3 \exp i\omega\left(t - \frac{x-x_2}{c_3}\right) + B_3 \exp i\omega\left(t + \frac{x-x_2}{c_3}\right),\\
u_4 &= A_4 \exp i\omega\left(t - \frac{x-x_3}{c_4}\right) + B_4 \exp i\omega\left(t + \frac{x-x_3}{c_4}\right).
\end{aligned}$$

It is assumed that at the boundaries both the displacement u_k and stress $\varrho_k c_k^2 u_k'$ are continuous. This condition leads to the following relations between the amplitudes A_k, B_k, cf. Eqs. (1.3) and (1.4)

(3.6)
$$\begin{bmatrix} A_{k+1} \\ B_{k+1} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} P_k G_k & Q_k F_k \\ Q_k G_k & P_k F_k \end{bmatrix}\begin{bmatrix} A_k \\ B_k \end{bmatrix}, \quad K = 1, 2, 3.$$

Similarly as in Sect. 1 the term proportional to A_1 is the incident wave running to the right and the term proportional to B_1 is the reflected wave. The term proportional to A_4 represents the transmitted wave. The term proportional to B_4 represents the incident wave running to the left. We assume that no such wave arrives from ∞, therefore $B_4 = 0$. The remaining terms represent the superposition of the transmitted and reflected waves in the layers.

Chaining the formulae (1.6), we obtain

(3.7)
$$\begin{aligned}
8A_4 = {} & A_1 G_1[(P_2P_3G_3 + Q_2Q_3F_3)P_1G_2 + (Q_2P_3G_3 + P_2Q_3F_3)Q_1F_2]\\
& + B_1F_1[(P_2P_3G_3 + Q_2Q_3F_3)Q_1G_2 + (Q_2P_3G_3 + P_2Q_3F_3)P_1F_2,
\end{aligned}$$

(3.8)
$$\begin{aligned}
8B_4 = {} & A_1G_1[(P_2Q_3G_3 + Q_2P_3F_3)P_1G_2 + (Q_2Q_3G_3 + P_2P_3F_3)Q_1F_2]\\
& + B_1F_1[(P_2Q_3G_3 + Q_2P_3F_3)Q_1G_2 + (Q_2Q_3G_3 + P_2P_3F_3)P_1F_2].
\end{aligned}$$

Putting $B_4 = 0$, we get from the last relation

(3.9)
$$B_1 = -A_1 \frac{G_1}{F_1} \frac{(P_1P_2G_2 + Q_1Q_2F_2)Q_3G_3 + (P_1Q_2G_2 + Q_1P_2F_2)P_3F_3}{(Q_1P_2G_2 + P_1Q_2F_2)Q_3G_3 + (Q_1Q_2G_2 + P_1P_2F_2)P_3F_3}.$$

The denominator here is a complex number. In more useful form the formula reads

(3.10)
$$\begin{aligned}
B_1 = -\frac{1}{M} A_1 G_1^2\{ & [P_1Q_1(P_2^2 + Q_2^2) + P_2Q_2(P_1^2G_2^2 + Q_1^2F_2^2)](P_3^2 + Q_3^2)\\
& + [P_1^2P_2^2G_2^2 + Q_1^2Q_2^2F_2^2 + 2P_1Q_1P_2Q_2]P_3Q_3G_3^2\\
& + [P_1^2Q_2^2G_2^2 + Q_1^2P_2^2F_2^2 + 2P_1Q_1P_2Q_2]P_3Q_3F_3^2\},
\end{aligned}$$

where the real denominator M is

$$\begin{aligned}
M = {} & (Q_1P_2G_2 + P_1Q_2F_2)(Q_1P_2F_2 + P_1Q_2G_2)Q_3^2 + (Q_1Q_2G_2 + P_1P_2F_2)(Q_1P_2F_2\\
& + P_1Q_2G_2)P_3Q_3F_3^2 + (Q_1P_2G_2 + P_1Q_2F_2)(Q_1Q_2F_2 + P_1P_2G_2)P_3Q_3G_3^2\\
& + (Q_1Q_2G_2 + P_1P_2F_2)(Q_1Q_2F_2 + P_1P_2G_2)P_3^2.
\end{aligned}$$

Taking into account Eq. (3.4), we obtain the simple formula

$$M = (P_1^2P_2^2+Q_1^2Q_2^2)P_3^2+(P_1^2Q_2^2+Q_1^2P_2^2)Q_3^2 \tag{3.11}$$
$$+2P_1Q_1P_2Q_2(P_3^2+Q_3^2)\cos 2\omega\alpha_2+2(P_1^2+Q_1^2)P_2Q_2P_3Q_3\cos 2\omega\alpha_3.$$
$$+2P_1Q_1Q_2^2P_3Q_3\cos 2\omega(\alpha_3-\alpha_2)+2P_1Q_1P_2^2P_3Q_3\cos 2\omega(\alpha_3+\alpha_2).$$

Pass to the expression for the transmitted wave. Substituting Eq. (3.10) into Eq.(3.7), the expression for the amplitude is obtained:

$$A_4 = A_1\frac{K}{M}[Q_1P_2Q_3F_2F_3+P_1Q_2Q_3G_2F_3+Q_1Q_2P_3F_2G_3+P_1P_2P_3G_2G_3], \tag{3.12}$$

where

$$K = \frac{1}{8}(P_1^2-Q_1^2)(P_2^2-Q_2^2)(P_3^2-Q_3^2). \tag{3.13}$$

The formula (3.10) and (3.12) give the intensity coefficient and phase shift for the reflected and transmitted waves.

The terms F_k, G_k result in the phase shift only. Therefore we include those terms in the phase. The formulae (3.10) and (3.12) result in the following expressions for the reflected and transmitted waves:

$$u_{(r)} = -\frac{A_1}{M}\Big\{P_1Q_1(P_2^2+Q_2^2)(P_3^2+Q_3^2)\exp i\omega\left(t+\frac{x}{c}-2\alpha_1\right) \tag{3.14}$$
$$+P_1^2P_2Q_2(P_3^2+Q_3^2)\exp i\omega\left(t+\frac{x}{c}-2\alpha_1-2\alpha_2\right)$$
$$+Q_1^2P_2Q_2(P_3^2+Q_3^2)\exp i\omega\left(t+\frac{x}{c}-2\alpha_1+2\alpha_2\right)$$
$$+2P_1Q_1P_2Q_2P_3Q_3\exp i\omega\left(t+\frac{x}{c}-2\alpha_1+2\alpha_3\right)$$
$$+2P_1Q_1P_2Q_2P_3Q_3\exp i\omega\left(t+\frac{x}{c}-2\alpha_1-2\alpha_3\right)$$
$$+P_1^2P_2^2P_3Q_3\exp i\omega\left(t+\frac{x}{c}-2\alpha_1-2\alpha_2-2\alpha_3\right)$$
$$+Q_1^2Q_2^2P_3Q_3\exp i\omega\left(t+\frac{x}{c}-2\alpha_1+2\alpha_2-2\alpha_3\right)$$
$$+P_1^2Q_2^2P_3Q_3\exp i\omega\left(t+\frac{x}{c}-2\alpha_1-2\alpha_2+2\alpha_3\right)$$
$$+Q_1^2P_2^2P_3Q_3\exp i\omega\left(t+\frac{x}{c}-2\alpha_1+2\alpha_2+2\alpha_3\right)\Big\},$$

$$(3.15)\qquad u_{(t)} = A_1 \frac{K}{M}\Big\{Q_1 P_2 Q_3 \exp i\omega\Big(t - \frac{x-x_3}{c} - \alpha_1 + \alpha_2 + \alpha_3\Big)$$
$$+ P_1 Q_2 Q_3 \exp i\omega\Big(t - \frac{x-x_3}{c} - \alpha_1 - \alpha_2 + \alpha_3\Big)$$
$$+ Q_1 Q_2 P_3 \exp i\omega\Big(t - \frac{x-x_3}{c} - \alpha_1 + \alpha_2 - \alpha_3\Big)$$
$$+ P_1 P_2 P_3 \exp i\omega\Big(t - \frac{x-x_3}{c} - \alpha_1 - \alpha_2 - \alpha_3\Big)\Big\}.$$

The given above expressions for $u_{(i)}$, $u_{(r)}$, $u_{(t)}$ are the complex-valued functions of x, t. Both the real and the imaginary parts of Eqs. (3.14) and (3.15) satisfy the equations of motion. Both parts describe some dynamically possible motion. Here we consider the imaginary part. The displacements corresponding to the incident, reflected and transmitted waves are the imaginary parts of $u_{(i)}$, $u_{(r)}$, $u_{(t)}$. Note that the replacement of (t) by $(-t)$ in Eqs. (3.5) results in the same replacement in all further formulae.

Introduce the dimensionless quantities

$$(3.16)\qquad \Omega = \frac{\omega a}{c}, \quad T = \frac{ct}{a}, \quad X = \frac{x}{a}, \quad a = x_1.$$

The formula for the reflected and transmitted waves may be written in the shorthand notation

$$(3.17)\qquad u_r = -A_1 \sum_{L=1}^{8} C_L \exp i\Omega(T+X+D_L),$$

$$(3.18)\qquad u_{(t)} = A_1 \sum_{L=1}^{4} C_L^* \exp i\Omega(T-X+D_L^*).$$

Here C_L, D_L, C_L^*, D_L^* denote the constants depending on the geometry of the system. They may be expressed by P_k, Q_k, α_1, α_2, α_3. The obvious algebraic relations between both systems of constants can be read from Eqs. (3.14) and (3.15).

Figure 7 gives two subsequent positions of the incident reflected and transmitted waves for $A_1 = 1$. The reflected wave is defined by Eq. (3.17) and the transmitted wave is defined by Eq. (3.18). The layers are situated at $1 < x < 1.5$ and $1.5 < x < 2$. Heavy lines correspond to $c_1:c_2:c_3:c_4 = 4:2:1:4$. It is seen that the intensity of the reflected wave is about 0.73 and the intensity of the transmitted wave about 0.69. Note that $0.73^2 + 0.69^2 \approx 1$.

The dotted lines correspond to $c_1:c_2:c_3:c_4 = 4:1:2:4$. for the reflected wave. The incident wave and transmitted waves remain unchanged. The change of the order of layers influences therefore the reflected monochromatic wave, but does not change the transmitted wave. This result may be confirmed by algebraic transformations of Eqs. (3.14) and (3.15). Such simple transformations demand much space. Because of this we do not quote them.

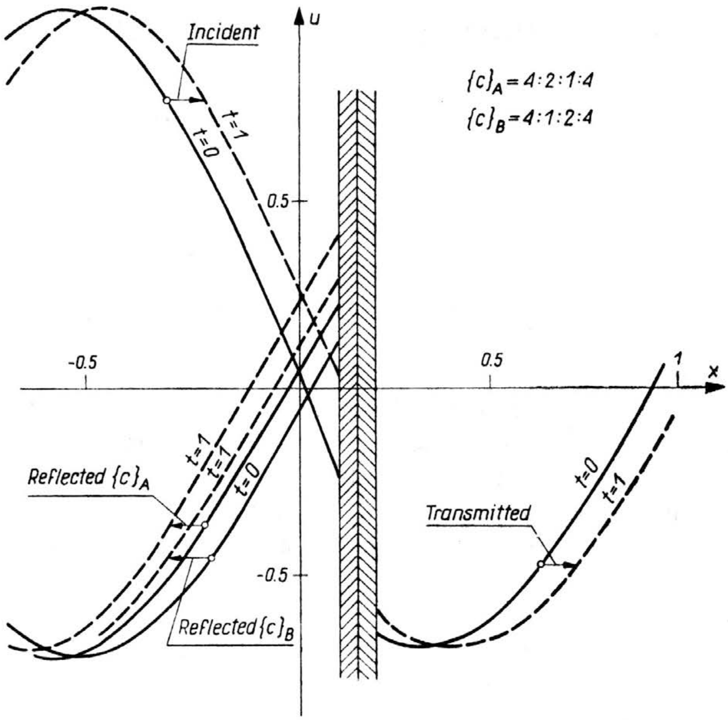


FIG. 7.

4. Two layers. Step function

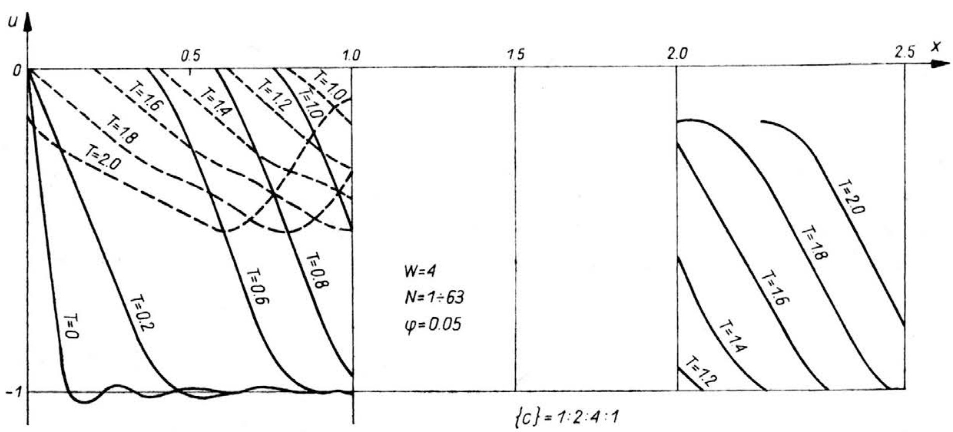


FIG. 8.

Consider the function

$$\bar{u}_{(i)} = \sum_{\Omega} A(\Omega)[\exp i\Omega(T-X) + \exp i\Omega(-T-X)]. \tag{4.1}$$

In accord with the results of the previous section this motion is dynamically possible if it is accompanied by the motion

$$\bar{u}_{(r)} = \sum_{\Omega} A(\Omega) \sum_{L=1}^{8} C_L[\exp i\Omega(T+X+D_L)+\exp i\Omega(-T+X+D_L)], \tag{4.2}$$

$$\bar{u}_{(t)} = \sum_{\Omega} A(\Omega) \sum_{L=1}^{4} C_L^*[\exp i\Omega(T-X+D_L^*)+\exp i\Omega(-T+X+D_L^*)]. \tag{4.3}$$

Differentiating with respect to time, we obtain

$$\bar{u}^{\boldsymbol{\cdot}}_{(i)} = \bar{u}^{\boldsymbol{\cdot}}_{(t)} = \bar{u}^{\boldsymbol{\cdot}}_{(r)} = 0 \quad \text{for} \quad t = 0. \tag{4.4}$$

The motion (2.1)–(2.3) is therefore the motion following static (= zero speed) deforma tion.

Take the discrete spectrum

$$\Omega = N/W, \quad N = 1, 3, 5, \ldots, N^* \tag{4.5}$$

and the amplitudes

$$A(\Omega) = \frac{2}{\pi\varphi}\frac{\sin N\varphi}{N^2}, \quad \varphi = \text{const}. \tag{4.6}$$

For $t = 0$ we have

$$\bar{u}_{(i)} = -\frac{4}{\pi\varphi}\left(\frac{1}{1^2}\sin X/W\sin\varphi + \frac{1}{3^2}\sin 3X/W\sin 3\varphi + \frac{1}{5^2}\sin 5X/W\sin 5\varphi + \ldots + \frac{1}{N^{*2}}\sin N^*X/W\sin N^*\varphi\right). \tag{4.7}$$

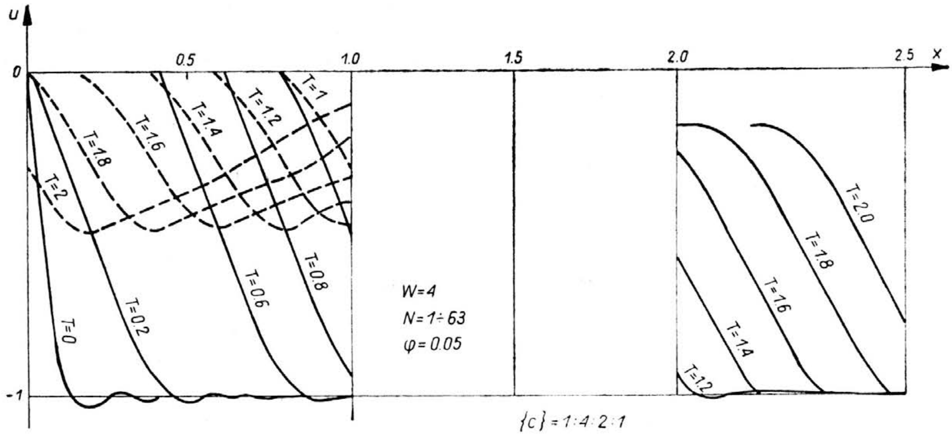
This coincides with Eq. (2.3) representing the profile for one layer. The curve for infinite number of terms is given in Fig. 3.

Each term of the expression for the reflected wave and the incident wave has now the same form as Eq. (2.7) for shifted X, namely $X+D_L$ instead of X. The period for each expression is $2\pi W$.

In the subsequent numerical calculations two layers are assumed. Their boundaries are situated at $X = 1$, 1.5 and 2. We take into account the first 32 terms, $N = 1$ to $N = 63$.

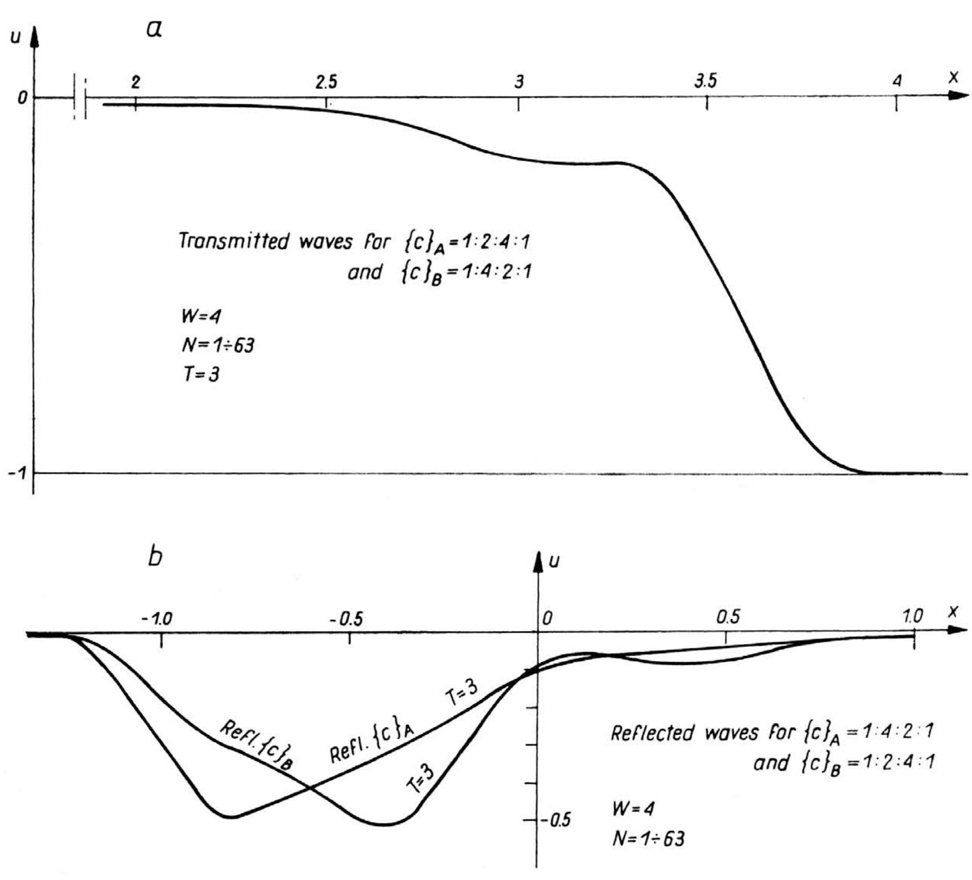
Consider first the case when the speed ratio $\{c\} = 1{:}2{:}4{:}1$. Taking $W = 4$, we obtain the wave fronts given in Fig. 8. The heavy line marked as $t = 0$ is the initial position. At $t = 2$ the incident wave already moved to the right. The displacement in the layers and in the region $X > 2$ behind the layers equals zero. At $T = 1$ there appears the reflected wave. The corresponding profile is given by the broken line. Finally, at $T = 1.2$ there appears the transmitted wave. The profile is given by the broken line. The profiles in the layers were not given. The corresponding curves are similar to that given in Figs. 5 and 6.

After large time there develops a steady wave profile of the reflected and the transmitted waves. This developed profile moves without distortion. We will show this later.



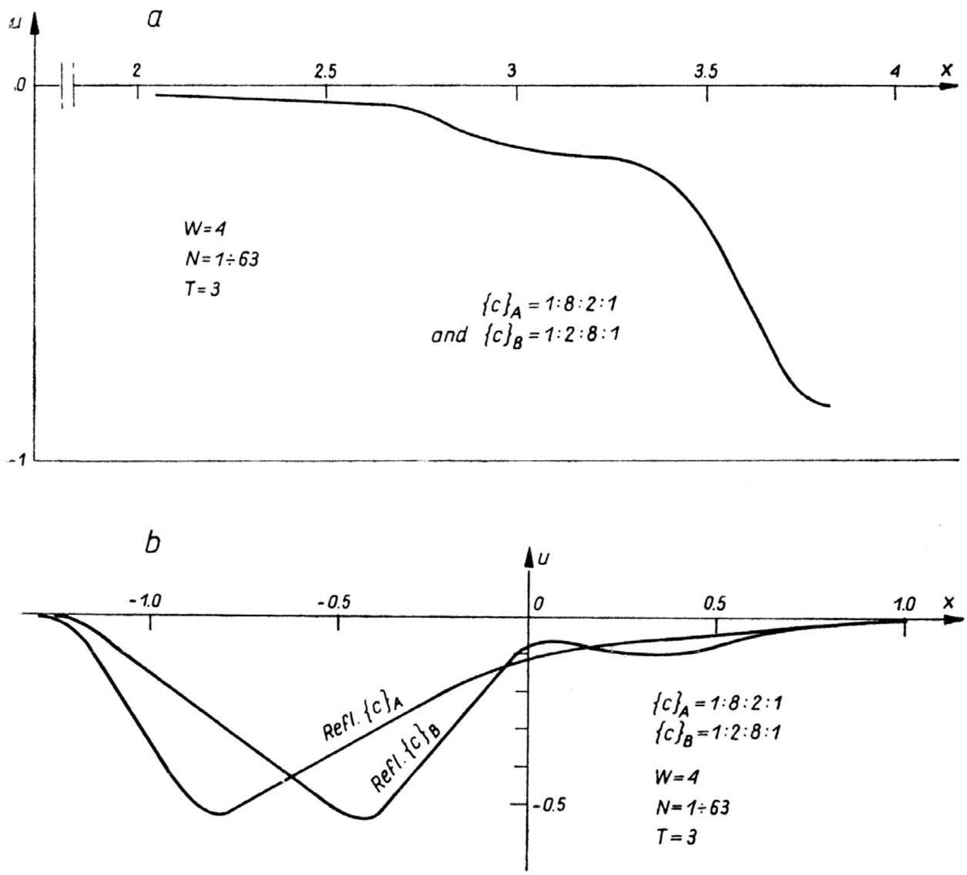
u
x
0
-1
0.5
1.0
1.5
2.0
2.5
T=0
T=0.2
T=0.6
T=0.8
T=1
T=1.2
T=1.4
T=1.6
T=1.8
T=2
T=2.0
W=4
N=1÷63
φ=0.05
{c}=1:4:2:1

FIG. 9.



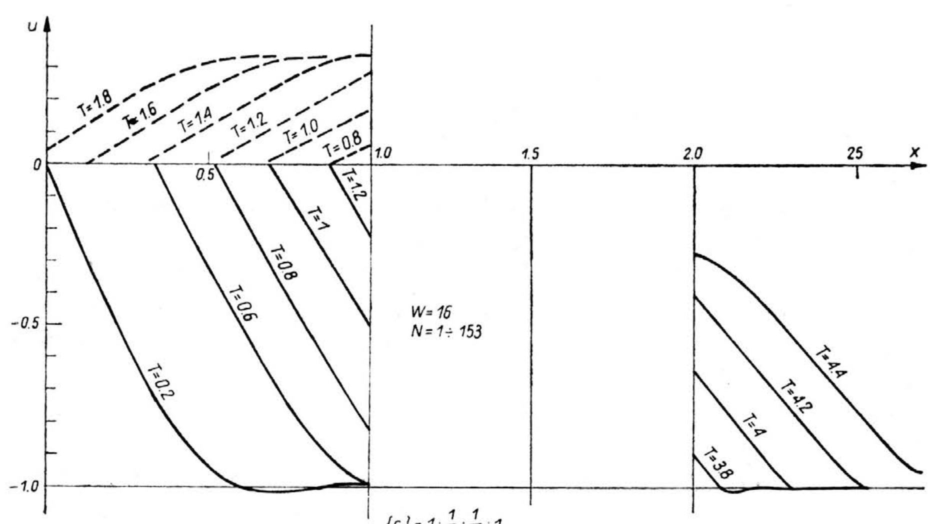
a
u
x
0
-1
2
2.5
3
3.5
4
Transmitted waves for {c}A=1:2:4:1
and {c}B=1:4:2:1
W=4
N=1÷63
T=3
b
u
x
-1.0
-0.5
0
0.5
1.0
-0.5
Refl. {c}B
Refl. {c}A
T=3
T=3
Reflected waves for {c}A=1:4:2:1
and {c}B=1:2:4:1
W=4
N=1÷63

FIG. 10.



a
b
W=4
N=1÷63
T=3
{c}_A = 1:8:2:1
and {c}_B = 1:2:8:1
Refl. {c}_A
Refl. {c}_B

FIG. 11.



W=16
N=1÷153
{c} = 1 : 1/2 : 1/4 : 1

FIG. 12.

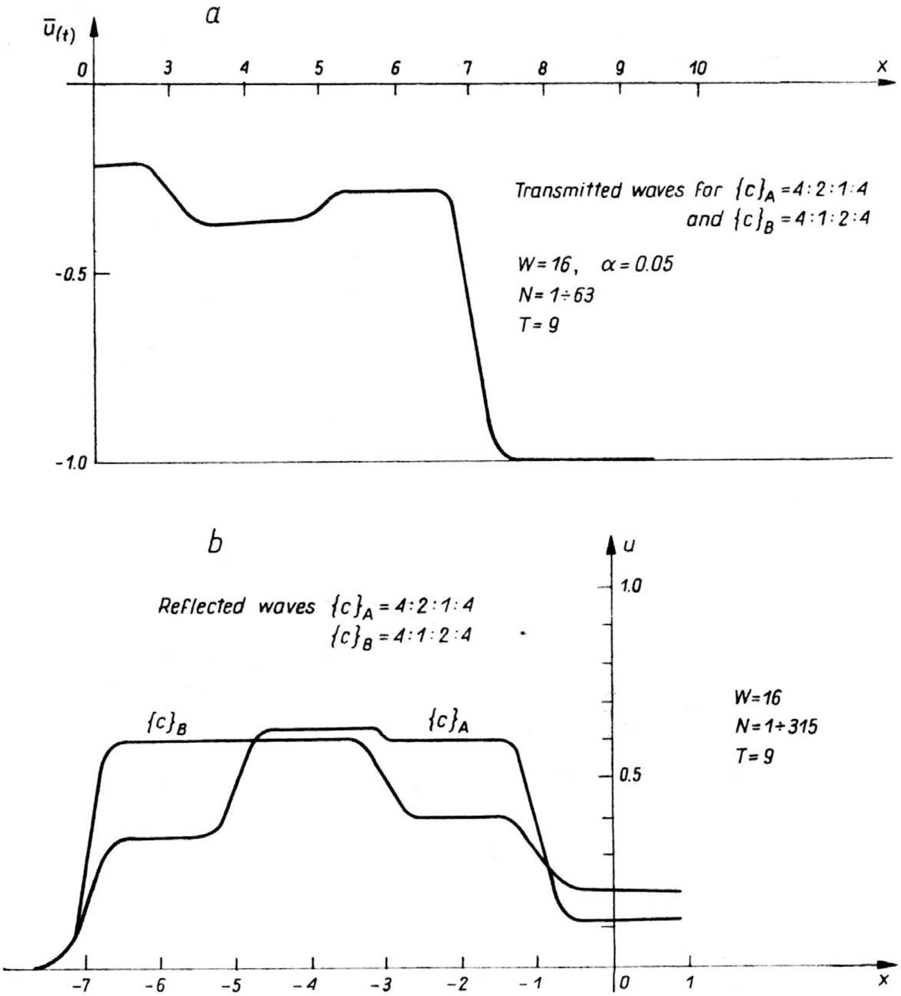


FIG. 13.

Figure 9 shows the wave profiles, for the case when the speed ratio $\{c\} = 1:4:2:1$. The wave profiles have similar character as before.

In order to show the differences between Fig. 4 and Fig. 5 the wave profiles at $T = 3$ (large time) are sketched in Fig. 10. The transmitted wave $u_{(t)}$ for both speed ratios is exactly the same, Fig. 10a. The reflected wave is essentially different, Fig. 10b. Figure 11 gives the curves for $\{c\} = 1:8:2:1$ and $\{c\} = 1:2:8:1$.

Figure 12 gives the wave profile for $\{c\} = 4:2:21:4$. The wave profiles for $\{c\} = 4:1:2:4$ are similar, but qualitatively different. To save space we do not give the corresponding curves. To check the calculations the summation was performed for $N^* = 153$. Figure 13 shows the profiles for $T = 9$. Figure 13a gives the transmitted waves for both cases. Figure 13b gives the reflected waves for $\{c\} = 4:2:1:4$ and $\{c\} = 4:1:2:4$. Again the transmitted profiles do not differ.

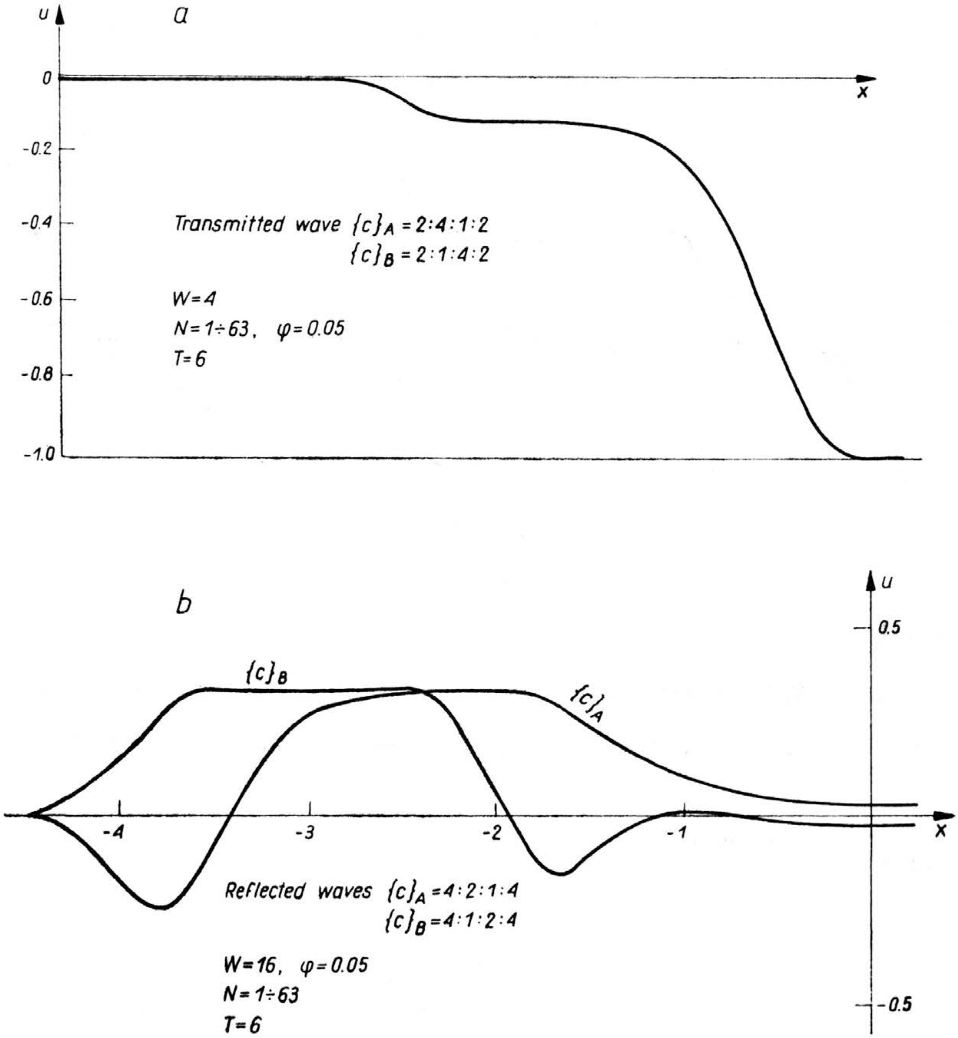


FIG. 14.

Figure 14 shows the wave profiles for the cases $\{c\} = 4{:}2{:}1{:}4$ and $\{c\} = 4{:}1{:}2{:}4$ for $T = 6$. Again there is no difference in the profiles of the transmitted waves, but the reflected waves differ essentially.

This qualitative result trivially holds in the case when the last region is very soft or very rigid. In both cases the transmitted waves do not exist.

Finally we return to the development in time of the reflected wave profile. Figure 15 gives the wave profiles for (large) times for $\{c\} = 8{:}4{:}2{:}1$. Here $c_1 \neq c_2$. The reflections on the inner boundaries may be traced.

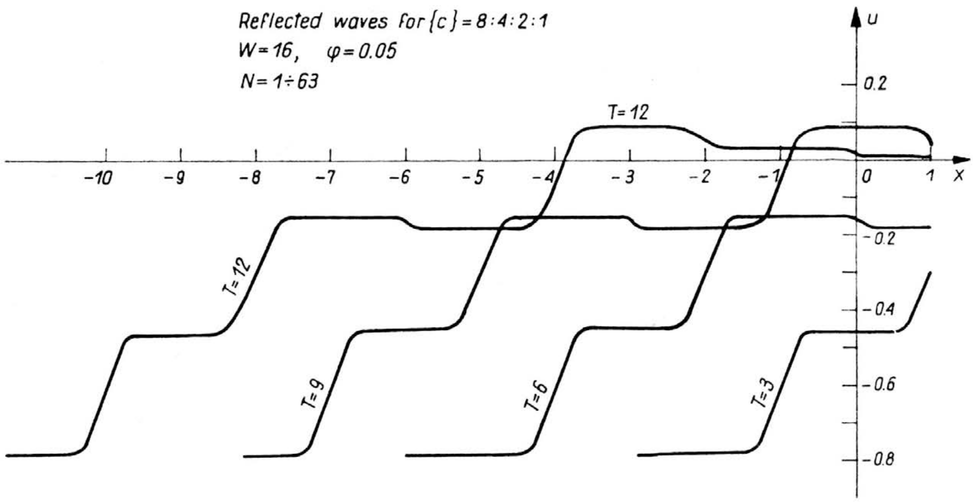


FIG. 15.

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