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To Professor Henryk Zorski on his 60-th Birthday

Thermodynamics of a heat conducting Maxwellian fluid

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IT IS SHOWN that compatibility of the model with rate type equations for deviatoric stresses and heat flux with thermodynamics yields only the positivity of heat conductivity K, shear viscosity μ and both relaxation times ζ_t and ζ_q, appearing in rate equations. It does not limit the non-objective terms in heat flux equation. Hyperbolicity implies additionally an upper bound for the magnitude of heat flux. Some aspects of transition by regular perturbation to second order fluids are briefly discussed.

W pracy pokazano, że zgodność modelu z równaniami ewolucji dla dewiatora naprężeń i strumienia ciepła z termodynamiką wymaga jedynie dodatności przewodnictwa ciepła K, lepkości μ i obu czasów relaksacji ζ_t i ζ_q, występujących w równaniach ewolucji. Termodynamika nie ogranicza członów nieobiektywnych w równaniu strumienia ciepła. Warunek hiperboliczności wprowadza dodatkowo kres górny dla modułu strumienia ciepła. Przedyskutowano również pewne aspekty przejścia przy pomocy regularnej perturbacji do modelu cieczy drugiego rzędu.

В работе показано, что совпадение модели с уравнениями эволюции для девиатора напряжений и потока тепла с термодинамикой требует только положительности теплопроводности K, вязкости μ и обоих времен релаксации ζ_t и ζ_q, выступающих в уравнениях эволюции. Термодинамика не ограничивает необъективных членов в уравнении потока тепла. Условие гиперболичности вводит дополнительно верхнюю границу для модуля потока тепла. Обсуждены тоже некоторые аспекты перехода, при помощи регулярной пертурбации, к модели жидкости второго порядка.

1. Introduction

IN THE RECENT paper [1], I. MÜLLER and myself have investigated an extended thermodynamics model of a non-newtonian fluid. It was shown that deviatoric stresses $t_{\langle kl \rangle}$ fulfil in such a model the equation of the following form

$$(1.1)\qquad 2\varrho\dot{\varepsilon}_{\langle ij\rangle}-t_{\langle ki\rangle}v_{j,k}-t_{\langle kj\rangle}v_{i,k}+\frac{2}{3}t_{\langle kl\rangle}v_{k,l}\delta_{ij}$$
$$=-2pv_{\langle i,j\rangle}-\beta_1 t_{\langle ij\rangle}-\beta_2\overset{\triangle}{t}_{\langle ij\rangle}+\beta_3\left(t_{\langle ik\rangle}t_{\langle kj\rangle}-\frac{1}{3}t_{\langle kl\rangle}t_{\langle kl\rangle}\delta_{ij}\right),$$

where the inertial Cartesian frame of reference has been used, angular brackets $\langle\cdot\rangle$ denote trace-free symmetric part of tensors,

$$(1.2)\qquad \overset{\triangle}{t}_{\langle ij\rangle}:=\dot{t}_{\langle ij\rangle}+t_{\langle ik\rangle}v_{[k,j]}+t_{\langle jk\rangle}v_{[k,i]}$$

is the objective, so-called corotational, time derivative of $t_{\langle ij\rangle}$, $\varepsilon_{\langle ij\rangle}$ is a constitutive quantity, p — pressure $\left(p := -\frac{1}{3}t_{kk}\right)$, $\beta_1, \beta_2, \beta_3$ — material parameters.

The rate-type equation (1.1) indicates that extended thermodynamics leads to a Maxwell-type model of non-newtonian fluids.

We have shown in the paper that the presence of a non-zero left-hand side of equation (1.1) yields non-objectivity and, simultaneously, it makes impossible the transition to incompressible fluids. We have argued, however, that both these effects are very small in normal circumstances and can be neglected if one considers a non-newtonian fluid as it is understood in rheology.

It should be stressed that the structure of left-hand side of (1.1) is motivated by kinetic theory and it has, for this reason, a very solid physical background.

For technical reasons, we have considered in [1] only adiabatic processes.

In this paper, I intend to show how ordinary rational thermodynamics of such rate-type models can be constructed. However, in contrast to standard thermodynamic approaches, I shall use some of the features of extended thermodynamics, which prove to be particularly convenient in the present approach. Namely, the heat flux vector $\mathbf{q}$ and the stress tensor $\mathbf{t}$ will be considered to be constitutive variables; for this reason no spatial gradients of basic fields are going to appear as constitutive variables, in contrast, for example, to papers [2, 3]. The entropy inequality will be exploited by use of Lagrange multipliers. To preserve the hyperbolicity of field equations, the evolution equation of Cattaneo type for heat flux is postulated.

It is worthwhile to mention that the evolution equation of heat flux yields the existence of the second sound. This phenomenon is of no importance in the case of real non-newtonian fluids due to a very short relaxation time of thermal disturbances. Hence, its presence in the model should be considered as technical means of preserving hyperbolicity rather than an effect of real physical importance. It proves to be particularly useful in constructing wave solutions of field equations. We shall not consider, however, this problem in the present paper.

2. Governing set of equations

We consider a class of processes, which can be described by the following set of fields on the classical space-time $\mathsf{R}\times\mathsf{R}^3$

$$\begin{aligned} &v_k \text{ — velocity,}\\ (2.1)\qquad &T \text{ — temperature,}\\ &q_k \text{ — heat flux,}\\ &t_{kl} \text{ — Cauchy stress tensor.}\end{aligned}$$

Throughout the paper we use a Cartesian rectangular frame of reference.

Field equations for those fields are assumed to have the following form

$$\dot{\varrho}+\varrho v_{k,k} = 0, \tag{2.2}$$

$$\varrho\dot{v}_k = t_{kl,l}, \tag{2.3}$$

$$\varrho\dot{\varepsilon}+q_{k,k} = t_{kl}v_{k,l}, \tag{2.4}$$

$$\zeta_q \overset{\triangle}{q}_k+q_k+KT_{,k} = 0, \tag{2.5}$$

$$\zeta_t \overset{\triangle}{t}_{\langle kl\rangle}+t_{\langle kl\rangle}-2\mu v_{\langle k,l\rangle} = 0, \tag{2.6}$$

where

$$\begin{aligned} &\varrho = \varrho(T, q_k, t_{kl}), \quad \zeta_q = \zeta_q(T,p), \quad K = K(T,p),\\ &\varepsilon = \varepsilon(T, q_k, t_{kl}), \quad \zeta_t = \zeta_t(T,p), \quad p:= -\frac{1}{3}t_{kk}, \end{aligned} \tag{2.7}$$

and

$$\overset{\triangle}{q}_k:= \dot{q}_k+\xi q_i v_{[i,k]}+\gamma_q q_i v_{(i,k)}, \tag{2.8}$$

$$\overset{\triangle}{t}_{\langle kl\rangle}:= \dot{t}_{\langle kl\rangle}+t_{\langle kl\rangle}v_{[i,l]}+t_{\langle li\rangle}v_{[i,k]}+\gamma_t\left(t_{\langle ki\rangle}v_{(i,l)}+t_{\langle il\rangle}v_{(i,k)}-\frac{2}{3}t_{\langle ij\rangle}v_{(i,j)}\delta_{kl}\right). \tag{2.9}$$

Equations (2.2)–(2.4) are, certainly, mass, momentum and specific internal energy conservation equations.

On the other hand, evolution equations (2.5) and (2.6) do not follow in this approach — in contrast to extended thermodynamics — from any balance equations and they should be considered — similarly to (2.7) — as the part of definition of the class of materials under considerations.

The structure of equation (2.5) follows from considerations, concerning the propagation of thermal waves and initiated by the paper of C. Cattaneo [4]. Definition (2.8) of the time derivative of q contains two parameters: ξ and γ_q. In the case: $\xi = 1$, γ_q — arbitrary, this time derivative is objective. Such a choice of ξ is neither motivated by kinetic theories, which would rather yield $\xi = -1$ nor by macroscopic arguments. For instance, Cattaneo rigid heat conductor, a model following from (2.2–9) by assuming $v_k \equiv 0$ and described by the following equation for heat flux

$$\zeta_q \frac{\partial q_k}{\partial t}+q_k+KT_{,k} = 0$$

cannot be made objective. Its transformation to non-inertial frame $(\overset{*}{x}_k)$ yields

$$\zeta_q \frac{\partial \overset{*}{q}_k}{\partial t}+\overset{*}{q}_k+K\frac{\partial T}{\partial \overset{*}{x}_k}+\zeta_q W_{kl}\overset{*}{q}_l = 0, \tag{2.10}$$

where

$\overset{*}{q}_k = O_{kl}q_l$, **O** — orthogonal time-dependent matrix of transformation,

$W_{kl} = \dot{O}_{ki}O_{li}$ — matrix of angular velocities of non-inertial frame with respect to inertial one.

In the steady-state case, we have

$$(\delta_{kl}+\zeta_q W_{kl})q_l = -KT_{,k}$$

and it means that the non-inertial frame would have to rotate with angular velocity $\omega \sim 1/\zeta_q$ for non-objective term to have an influence. It does not seem to be very realistic in normal circumstances. It very well may be that an equation for $\mathbf{q}$, derived as a moment equation from a proper kinetic theory, should contain two different time derivatives — one non-objective, arising from transport terms in kinetic equation and another one — objective, coming from collision term in kinetic equation. Such a situation has been discussed in [1] for deviatoric stresses. A kinetic equation, appropriate for non-newtonian fluids has as yet not been constructed. Therefore, we are satisfied with Eqs. (2.5) and (2.9).

On the other hand, the structure of Eq. (2.6) enables us to classify the material under considerations as a Maxwell-type non-newtonian, belonging to the rate type materials, investigated by J. G. OLDROYD [5]. Definition (2.9) of time derivative of $\mathbf{t}$ has an objective form, which again, does not follow from kinetic theory. However in this case, in contrast to heat conduction, objectivity is supported by macroscopic observations as well as estimations mentioned at the beginning of this work. Simultaneously, the coefficient γ_t may have an arbitrary value — similarly to γ_q. In the case $\gamma_t = 0$, derivative (2.9) is called corotational. Rheologists consider often two cases: $\gamma_t = \pm 1$. Such models are called upper and lower Maxwellian, respectively.

It should be mentioned that lack of direct coupling between Eqs. (2.5) and (2.6), for instance of the type $q_i t_{\langle ik \rangle}$ in (2.5) or $q_{\langle k} q_{l \rangle}$ in Eq. (2.6), leads to a rather artificial separation of waves, which shall be briefly described in the sequel.

3. Entropy inequality

Constitutive quantities (2.7) are still rather arbitrary within the model presented in Sect. 2. They must, certainly, satisfy some smoothness conditions but otherwise they are not specified.

It is customary in thermodynamics to require in addition that all solutions of (2.2)–(2.6) should satisfy the entropy inequality

$$\varrho \dot{\eta} + h_{kk} \geqslant 0, \tag{3.1}$$

where

$$\eta = \eta(T, q_k, t_{kl}) \tag{3.2}$$

is the specific entropy and

$$h_k = h_k(T, q_k, t_{kl}) \tag{3.3}$$

is the entropy flux.

The above requirement imposes some restrictions on Eq. (2.7) which we now investigate.

Due to the fact that inequality (3.1) should hold only for solutions of (2.2)–(2.6) we seek the class of solutions of (3.1) restricted by the field equations. This can be done by the method of Lagrange multipliers (e.g. see [7]). It means that we solve the inequality

$$\begin{aligned} \varrho \dot{\eta} + h_{k,k} - \Lambda^{\varrho}(\dot{\varrho} + \varrho v_{k,k}) - \Lambda^{v_k}(\varrho \dot{v}_k - t_{kl,l}) - \Lambda^{\varepsilon}(\varrho \dot{\varepsilon} + q_{k,k} - t_{kl} v_{k,l}) \\ - \Lambda^{q_k}(\zeta_q \overset{\triangle}{q}_k + q_k + KT_{,k}) - \Lambda^{\langle kl \rangle}(\zeta_t \overset{\triangle}{t}_{\langle kl \rangle} + t_{\langle kl \rangle} - 2\mu v_{\langle k,l \rangle}) \geqslant 0, \end{aligned} \tag{3.4}$$

which should hold for *all* fields v_k, T, q_k, t_{kl}. The multipliers Λ^{ϱ}, Λ^{v_k}, Λ^{ε}, Λ^{q_k}, $\Lambda^{\langle kl\rangle}$ are functions of the above fields themselves and relations (2.7), (3.2) and (3.3) should be satisfied.

The linearity of (3.4) with respect to $\dot{v}_k$ indicates

$$\Lambda^{v_k} = 0. \tag{3.5}$$

The inequality (3.4) is also linear with respect to $v_{k,l}$. The following separation of this gradient

$$v_{k,l} = v_{\langle k,l\rangle} + \frac{1}{3} v_{m,m}\delta_{kl} + v_{[k,l]} \tag{3.6}$$

indicates then

$$\varrho\Lambda^{\varrho} + p\Lambda^{\varepsilon} + \frac{2}{3}\zeta_t\gamma_t\Lambda^{\langle kl\rangle}t_{\langle kl\rangle} + \frac{1}{3}\zeta_q\gamma_q\Lambda^{q_k}q_k = 0, \tag{3.7}$$

$$\Lambda^{\varepsilon}t_{\langle kl\rangle} + 2\mu\Lambda^{\langle kl\rangle} - \zeta_t\gamma_t\left(\Lambda^{\langle ik\rangle}t_{\langle il\rangle} + \Lambda^{\langle il\rangle}t_{\langle ik\rangle} - \frac{2}{3}\Lambda^{\langle ij\rangle}t_{\langle ij\rangle}\delta_{kl}\right) - \frac{1}{2}\zeta_q\gamma_q\left(\Lambda^{q_k}q_l + \Lambda^{q_l}q_k - \frac{2}{3}\Lambda^{q_i}q_i\delta_{kl}\right) = 0, \tag{3.8}$$

$$\zeta_t(\Lambda^{\langle ik\rangle}t_{\langle il\rangle} - \Lambda^{\langle il\rangle}t_{\langle ik\rangle}) + \frac{1}{2}\zeta_q\xi(\Lambda^{q_k}q_l - \Lambda^{q_l}q_k) = 0. \tag{3.9}$$

Making use of the chain rule in evaluation of time derivatives in (3.4), we find it to be linear with respect to $\dot{T}$, $\dot{q}_k$, $\dot{t}_{kl}\delta_{kl}$, $\dot{t}_{\langle kl\rangle}$, which leads to the following identities

$$\varrho\eta_T - \Lambda^{\varrho}\varrho_T - \Lambda^{\varepsilon}\varrho\varepsilon_T = 0, \qquad (\ )_T := \frac{\partial}{\partial T}(\ ), \tag{3.10}$$

$$\varrho\eta_{q_k} - \Lambda^{\varrho}\varrho_{q_k} - \Lambda^{\varepsilon}\varrho\varepsilon_{q_k} - \Lambda^{q_k}\zeta_q = 0, \qquad (\ )_{q_k} := \frac{\partial}{\partial q_k}(\ ), \tag{3.11}$$

$$(\varrho\eta_{t_{ij}} - \Lambda^{\varrho}\varrho_{t_{ij}} - \Lambda^{\varepsilon}\varrho\varepsilon_{t_{ij}})\delta_{ij} = 0, \qquad (\ )_{t_{ij}} := \frac{\partial}{\partial t_{ij}}(\ ), \tag{3.12}$$

$$(\varrho\eta_{t_{ij}} - \Lambda^{\varrho}\varrho_{t_{ij}} - \Lambda^{\varepsilon}\varrho\varepsilon_{t_{ij}})\left(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}\right) - \Lambda^{\langle kl\rangle}\zeta_t = 0. \tag{3.13}$$

In the similar manner, we find the inequality (3.4) to be linear with respect to $T_{,k}$, $q_{l,k}$, $t_{ij,k}$. This yields

$$\begin{aligned} &\frac{\partial h_k}{\partial T} - K\Lambda^{q_k} = 0, \\ &\frac{\partial h_k}{\partial q_l} - \Lambda^{\varepsilon}\delta_{kl} = 0, \\ &\frac{\partial h_k}{\partial t_{ij}} = 0. \end{aligned} \tag{3.14}$$

Bearing in mind the above results, we can write (3.4) in the form of the following residual inequality

$$\Lambda^{q_k} q_k + \Lambda^{\langle kl \rangle} t_{\langle kl \rangle} \leqslant 0. \tag{3.15}$$

Integrability conditions of Eqs. $(3.14)_2$ easily indicate

$$\frac{\partial \Lambda^{\varepsilon}}{\partial q_j} (\delta_{ij}\delta_{kl} - \delta_{lj}\delta_{ik}) = 0 \Rightarrow \Lambda^{\varepsilon} = \Lambda^{\varepsilon}(T, t_{ij}).$$

Substitution of this relation in Eq. $(3.14)_2$ yields

$$h_k = \Lambda^{\varepsilon} q_k + \hat{h}_k(T, t_{ij}). \tag{3.16}$$

For isotropic fluids the second term must vanish — one cannot construct an isotropic vector function from a tensor of the second rank and, siumltaneously, we expect the equilibrium part of h to be zero (compare: (3.21)). Then the substitution in Eq. $(3.14)_3$ yields

$$\Lambda^{\varepsilon} = \Lambda^{\varepsilon}(T). \tag{3.17}$$

Relation $(3.14)_1$, indicates, in turn,

$$\Lambda^{q_k} = \frac{1}{K} \frac{d\Lambda^{\varepsilon}}{dT} q_k. \tag{3.18}$$

On the other hand, relations (3.10)–(3.13) can be written in the following compact form

$$\varrho\, d\eta = \Lambda^{\varrho} d\varrho + \Lambda^{\varepsilon} d\varepsilon + \zeta_q \Lambda^{q_k} dq_k + \zeta_t \Lambda^{\langle kl \rangle} dt_{\langle kl \rangle}, \tag{3.19}$$

where constitutive relations for ε and η have been used. At the same time, formula (3.7) leads to the following relation for multiplier Λ^{ϱ}:

$$\Lambda^{\varrho} = -\frac{p}{\varrho}\Lambda^{\varepsilon} - \frac{2}{3}\frac{\zeta_t \gamma_t}{\varrho}\Lambda^{\langle kl \rangle} t_{\langle kl \rangle} - \frac{1}{3}\frac{\zeta_q \gamma_q}{\varrho K}\frac{d\Lambda^{\varepsilon}}{dT} q_k q_k. \tag{3.20}$$

To proceed further, we need the notion of thermodynamic equilibrium. For processes under considerations, we define a *state of thermodynamic equilibrium* as such for which

$$q_k|_E = 0, \quad t_{\langle kl \rangle}|_E = 0. \tag{3.21}$$

In such a state, relation (3.19) has the form

$$d\eta|_E = \Lambda^{\varepsilon}|_E \left\{ d\varepsilon|_E + p d\left(\frac{1}{\varrho}\right)\Big|_E \right\}. \tag{3.22}$$

Comparison with classical Gibbs relation yields

$$\Lambda^{\varepsilon}|_E = \frac{1}{T}.$$

Bearing in mind relation (3.17), we obtain

$$\Lambda^{\varepsilon} = \frac{1}{T}. \tag{3.23}$$

This result yields quite an explicit form of Eqs. (3.16) and (3.18)

$$h_k = \frac{1}{T} q_k, \quad \Lambda^{q_k} = -\frac{1}{KT^2} q_k. \tag{3.24}$$

Hence, the model, considered in this paper, leads to the classical relation between heat flux and entropy flux.

It remains to eliminate multiplier $\Lambda^{\langle kl \rangle}$. To this aim, let us rewrite relations (3.8) and (3.9). We have

$$t_{\langle kl \rangle} + \frac{1}{2} \frac{\zeta_q \gamma_q}{KT} q_{\langle k} q_{l \rangle} = -L_{\langle kl \rangle \langle ij \rangle} \Lambda^{\langle ij \rangle}, \tag{3.25}$$

$$\Lambda^{\langle ik \rangle} t_{\langle il \rangle} - \Lambda^{\langle il \rangle} t_{\langle ik \rangle} = 0, \tag{3.26}$$

where

$$\begin{aligned} L_{\langle kl \rangle \langle ij \rangle} := \mu T \left(\delta_{ki} \delta_{lj} + \delta_{li} \delta_{kj} - \frac{2}{3} \delta_{kl} \delta_{ij} \right) \\ - \frac{1}{2} \zeta_t \gamma_t T \left(t_{\langle ik \rangle} \delta_{jl} + t_{\langle il \rangle} \delta_{jk} + t_{\langle jk \rangle} \delta_{il} + t_{\langle jl \rangle} \delta_{ik} - \frac{4}{3} t_{\langle kl \rangle} \delta_{ij} - \frac{4}{3} t_{\langle ij \rangle} \delta_{kl} \right). \end{aligned} \tag{3.27}$$

This linear mapping: $\Lambda^{\langle ij \rangle} \mapsto t_{\langle kl \rangle} + \frac{1}{2} \frac{\zeta_q \gamma_q}{KT} q_{\langle k} q_{l \rangle}$ is, as it is easy to see, non-singular and, hence, can be inverted. It is particularly simple for the corotational model ($\gamma_t \equiv 0$):

$$\Lambda^{\langle kl \rangle} = -\frac{1}{2\mu T} \left(t_{\langle kl \rangle} + \frac{1}{2} \frac{\zeta_q \gamma_q}{KT} q_{\langle k} q_{l \rangle} \right). \tag{3.28}$$

Substitution of Eqs. (3.24) and (3.28) in Eq. (3.9) yields

$$\frac{\zeta_t}{2\mu T} \cdot \frac{1}{2} \frac{\zeta_q \gamma_q}{KT} (q_k t_{\langle il \rangle} - q_l t_{\langle ik \rangle}) = 0. \tag{3.29}$$

Hence the non-trivial model is possible only if ($\zeta_t \neq 0$, $\zeta_q \neq 0$)

$$\gamma_q = 0, \tag{3.30}$$

which means that corotational form of derivative (2.9) leads also to the corotational form of Eq. (2.8).

Let us collect the results for this particular case

$$\Lambda^{\varepsilon} = \frac{1}{T}, \quad \Lambda^{\varrho} = -\frac{p}{\varrho T}, \quad \Lambda^{q_k} = -\frac{q_k}{KT^2}, \quad \Lambda^{\langle kl \rangle} = -\frac{t_{\langle kl \rangle}}{2\mu T}. \tag{3.31}$$

Then, according to Eq. (3.19),

$$T d\eta = d\varepsilon + p d\left(\frac{1}{\varrho}\right) - \frac{\zeta_q}{\varrho KT} q_k dq_k - \frac{\zeta_t}{2\varrho\mu} t_{\langle kl \rangle} dt_{\langle kl \rangle}. \tag{3.32}$$

It is convenient to introduce the free enthalpy

$$g := \varepsilon + \frac{p}{\varrho} - T\eta, \quad g = g(T, q_k, t_{kl}). \tag{3.33}$$

Then

$$dg = -\eta dT + \frac{1}{\varrho} dp + \frac{\zeta_q}{\varrho KT} q_k dq_k + \frac{\zeta_t}{2\varrho\mu} t_{\langle kl \rangle} dt_{\langle kl \rangle}. \tag{3.34}$$

The set of integrability conditions for (3.34) is of the form

$$\begin{gathered}
\frac{\partial}{\partial T}\left(\frac{1}{\varrho}\right) = -\frac{\partial \eta}{\partial p}, \quad \frac{\partial}{\partial q_k}\left(\frac{1}{\varrho}\right) = \frac{\partial}{\partial p}\left(\frac{\zeta_q}{\varrho KT}\right) q_k, \\
\frac{\partial}{\partial T}\left(\frac{\zeta_q}{\varrho KT}\right) q_k = -\frac{\partial \eta}{\partial q_k}, \quad \frac{\partial}{\partial T}\left(\frac{\zeta_t}{2\varrho\mu}\right) t_{\langle kl \rangle} = -\frac{\partial \eta}{\partial t_{ij}}\left(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{kl}\delta_{ij}\right), \\
\frac{\partial}{\partial p}\left(\frac{\zeta_t}{2\varrho\mu}\right) t_{\langle kl \rangle} = \frac{\partial}{\partial t_{ij}}\left(\frac{1}{\varrho}\right)\left(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{kl}\delta_{ij}\right).
\end{gathered} \tag{3.35}$$

Then

(3.36)

$$\frac{\partial g}{\partial T} = -\eta, \quad \frac{\partial g}{\partial p} = \frac{1}{\varrho}, \quad \frac{\partial g}{\partial q_k} = \frac{\zeta_q}{\varrho KT} q_k, \quad \frac{\partial g}{\partial t_{ij}}\left(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{kl}\delta_{ij}\right) = \frac{\zeta_t}{2\varrho\mu} t_{\langle kl \rangle}.$$

Particularly simple results follow from Eqs. (3.35) and (3.36) if we assume *incompressibility*, i.e. pressure should enter the constitutive relations only through its explicit presence in Eqs. (3.33). Then, due to Eqs. $(3.35)_{1,2,5}$

$$\varrho = \text{const} \tag{3.37}$$

and relations (3.36) yield

$$g = g_0(T) + \frac{1}{2}\frac{\zeta_q}{\varrho KT} q_k q_k + \frac{1}{4}\frac{\zeta_t}{\varrho\mu} t_{\langle kl \rangle} t_{\langle kl \rangle} \tag{3.38}$$

as well as

$$\begin{gathered}
\eta = -\frac{dg_0}{dT} - \frac{1}{2\varrho}\frac{d}{dT}\left(\frac{\zeta_q}{KT}\right) q_k q_k - \frac{1}{4\varrho}\frac{d}{dT}\left(\frac{\zeta_t}{\varrho\mu}\right) t_{\langle kl \rangle} t_{\langle kl \rangle}, \\
\varepsilon = \varepsilon_0 + \frac{1}{2\varrho}\left[\frac{\zeta_q}{KT} - T\frac{d}{dT}\left(\frac{\zeta_q}{KT}\right)\right] q_k q_k + \frac{1}{4\varrho}\left[\frac{\zeta_t}{\mu} - T\frac{d}{dT}\left(\frac{\zeta_t}{\mu}\right)\right] t_{\langle kl \rangle} t_{\langle kl \rangle}, \\
\varepsilon_0 := g_0 - T\frac{dg_0}{dT} - \frac{p}{\varrho}.
\end{gathered} \tag{3.39}$$

Simultaneously, the residual inequality (3.15) takes the form

$$\frac{1}{KT^2} q_k q_k + \frac{1}{2\mu T} t_{\langle kl \rangle} t_{\langle kl \rangle} \geqslant 0. \tag{3.40}$$

This inequality should hold for arbitrary q_k and $t_{\langle kl \rangle}$, i.e.

$$K > 0, \quad \mu > 0. \tag{3.41}$$

At the same time, *stability of thermodynamic equilibrium* (i.e. convexity of g) indicates

$$\frac{\zeta_q}{\varrho KT} > 0 \Rightarrow \zeta_q > 0, \quad \frac{\zeta_t}{\varrho\mu} > 0 \Rightarrow \zeta_t > 0. \tag{3.42}$$

These inequalities are certainly satisfied if ζ_q and ζ_t are relaxation times for heat flux and deviatoric stresses, respectively.

Complicated form of relations (3.25) and (3.27) does not make it possible to carry through similar considerations in the general case. For this reason, we limit our attention in the remaining part of this work to the corotational model.

It can easily be proved by use of (3.8) and (3.9) that the condition $\gamma_q = 0$ (with $\zeta_q \neq 0$!) implies $\gamma_t = 0$, i.e. again a full corotational form of (2.8) and (2.9).

Let us notice that the above thermodynamic reults do not involve the constant ξ The general result $(3.24)_2$ eliminates the second term in relation (3.9), which is the only place of appearance of constant ξ. It means that both objective: $\xi = 1$ and non-objective form: $\xi = 0$ of definition (2.9) lead to thermodynamically admissible models. From the macroscopic point of view, the choice of ξ should follow from observations. We demonstrate a possible argumentation in the next Section.

4. Simple example of shear flow

To demonstrate some properties of corotational model of incompressible heat conducting Maxwellian fluid, let us consider a shearing flow, shown in Fig. 1.

Let us first select those fields (2.1), which may appear in the description of the above flow under the assumption that *all fields depend only on t and $x_2 = x$.*

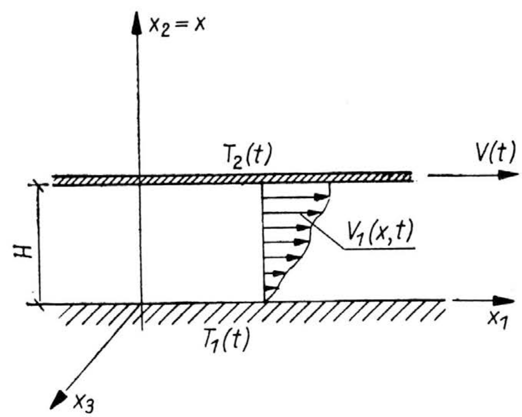

FIG. 1.

Incompressibility condition

$$v_{k,k} = \frac{\partial v_2}{\partial x} = 0 \tag{4.1}$$

implies that, provided the walls are inpenetrable, the second component of velocity must be identically zero

$$v_2 = 0. \tag{4.2}$$

Let us also assume that shear stresses vanish on planes perpendicular to x_3-axis:

(4.3) $$t_{13} = t_{23} = 0.$$

Then the momentum balance implies

(4.4) $$\varrho \dot{v}_3 = t_{32,2} = 0 \Rightarrow v_3 = 0,$$

provided the initial value of v_3 was zero. Hence the flow is one-dimensional, as indicated in Fig. 1. It also means that there is no difference between material and partial time derivatives.

We have

(4.5) $$(v_{\langle k,l\rangle}) = (v_{(k,l)}) = \begin{pmatrix} 0 & \frac{1}{2}\frac{\partial v}{\partial x} & 0 \\ \frac{1}{2}\frac{\partial v}{\partial x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (v_{[k,l]}) = \begin{pmatrix} 0 & \frac{1}{2}\frac{\partial v}{\partial x} & 0 \\ -\frac{1}{2}\frac{\partial v}{\partial x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $v \equiv v_1$. Relations (4.5) indicate

(4.6) $$\overset{\triangle}{q}_3 = \frac{\partial q_3}{\partial t} \Rightarrow \zeta_q \frac{\partial q_3}{\partial t} + q_3 = 0 \Rightarrow q_3 = q_3(t=0)e^{-t/\zeta_q}$$

and, assuming that initial heat flux in x_3-direction was zero, we have

(4.7) $$q_3 = 0.$$

Bearing in mind the above considerations, we see that the flow is described by the following fields:

(4.8) $$v_1 \equiv v, \quad T, \quad t_{\langle 11\rangle} \equiv s_1, \quad t_{\langle 22\rangle} \equiv s_2, \quad t_{12} = \sigma, \quad q_1, q_2$$

with the obvious relation

(4.9) $$t_{\langle 33\rangle} = -(s_1+s_2).$$

Assuming for simplicity that K, μ, ζ_q and ζ_t are constant, we can write Eqs. (2.2)–(2.6) in the form

$$\frac{\partial v}{\partial t} - \frac{1}{\varrho}\frac{\partial \sigma}{\partial x} = 0,$$

$$\frac{\partial T}{\partial t} - \frac{2}{\delta T} q_2 \frac{\partial T}{\partial x} + \frac{1}{\delta}\frac{\partial q_2}{\partial x} = \frac{2}{KT\delta}(q_1^2+q_2^2) + \frac{1}{\mu\delta}[(s_1+s_2)^2+\sigma^2-s_1 s_2],$$

(4.10) $$\frac{\partial q_1}{\partial t} - \frac{1}{2}\xi q_2 \frac{\partial v}{\partial x} = -\frac{1}{\zeta_q} q_1,$$

$$\frac{\partial q_2}{\partial t} + \frac{1}{2}\xi q_1 \frac{\partial v}{\partial x} + \frac{K}{\zeta_q}\frac{\partial T}{\partial x} = -\frac{1}{\zeta_q} q_2,$$

$$\frac{\partial s_1}{\partial t} - \sigma \frac{\partial v}{\partial x} = -\frac{1}{\xi_t} s_1,$$

$$\frac{\partial s_2}{\partial t} + \sigma \frac{\partial v}{\partial x} = -\frac{1}{\zeta_t} s_2,$$

(4.10) [cont.]
$$\frac{\partial \sigma}{\partial t} - \left[\frac{\mu}{\zeta_t} - \frac{1}{2}(s_1 - s_2)\right]\frac{\partial v}{\partial x} = -\frac{1}{\zeta_t}\sigma,$$

where

(4.11)
$$\delta := \varrho c_v - \frac{\zeta_q}{KT^2}(q_1^2 + q_2^2), \qquad c_v := \frac{\partial \varepsilon_0}{\partial T}$$

and Eq. $(3.39)_2$ has been used.

In addition, the momentum balance equation implies the following relation

(4.12)
$$\frac{\partial}{\partial x}(s_2 - p) = 0.$$

This relation, overdetermining the system (4.10), imposes constraints on solutions of (4.10) and may lead to problems of their existence. To avoid those difficulties, we assume the thickness H to be small enough for neglecting (4.12) entirely. In such a case, we consider solutions of Eqs. (4.10) as approximations without going into problems of existence.

Let us first notice a peculiar structure of Eqs. $(4.10)_{3,4}$. In the case of steady-state flow we would have

(4.13)
$$q_1 = -K\frac{\partial T}{\partial x}\,\frac{\frac{1}{2}\xi\zeta_q\frac{\partial v}{\partial x}}{1 + \frac{1}{4}\xi^2\zeta_q^2\left(\frac{\partial v}{\partial x}\right)^2},$$
$$q_2 = -K\frac{\partial T}{\partial x}\,\frac{1}{1 + \frac{1}{4}\xi^2\zeta_q^2\left(\frac{\partial v}{\partial x}\right)^2}.$$

Hence the objective time derivative of $\mathbf{q}$: $\xi = 1$ demands the existence of heat flux component q_1 perpendicular to the temperature gradient $\frac{\partial T}{\partial x}$! This effect would, certainly, not appear in the non-objective case $\xi = 0$:

(4.14)
$$q_1 = 0, \quad q_2 = -K\frac{\partial T}{\partial x}.$$

On the other hand, it can hardly be expected that this effect could be observed in normal circumstances: ratio of transversal to normal component of heat flux for $\xi = 1$

(4.15)
$$\frac{q_1}{q_2} = \frac{1}{2}\zeta_q\frac{\partial v}{\partial x}$$

would be large enough to have any bearing for $\frac{\partial v}{\partial x} \sim \frac{1}{\zeta_q}$ and this, in turn, would require extremally high velocity gradients for usual non-newtonian fluids.

We expect the system (4.10) to be hyperbolic. Let us consider the conditions yielding this type of equations. It is easy to notice that Eq. (4.10) is written in the normal form, which means that characteristic speeds of weak singularities coincide with eigen-values of the matrix of x-derivatives:

(4.16)
$$\begin{vmatrix} -\lambda & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\varrho} \\ 0 & -\frac{2q_2}{\delta T}-\lambda & 0 & \frac{1}{\delta} & 0 & 0 & 0 \\ -\frac{1}{2}\xi q_2 & 0 & -\lambda & 0 & 0 & 0 & 0 \\ \frac{1}{2}\xi q_1 & \frac{K}{\zeta_q} & 0 & -\lambda & 0 & 0 & 0 \\ -\sigma & 0 & 0 & 0 & -\lambda & 0 & 0 \\ \sigma & 0 & 0 & 0 & 0 & -\lambda & 0 \\ -\left[\frac{\mu}{\zeta_t}-\frac{1}{2}(s_1-s_2)\right] & 0 & 0 & 0 & 0 & 0 & -\lambda \end{vmatrix} = 0.$$

The above characteristic equation can be easily solved and we get

(4.17)
$$\lambda_{1,2} = \pm\sqrt{\frac{\mu}{\varrho\zeta_t}-\frac{1}{2}\frac{s_1-s_2}{\varrho}},$$

(4.18)
$$\lambda_{3,4} = -\frac{q_2}{\delta T} \pm\sqrt{\frac{q_2^2}{\delta^2 T^2}+\frac{K}{\delta\zeta_q}},$$

(4.19)
$$\lambda = 0 \quad \text{(triple root)}.$$

In the linear case

(4.20)
$$\lambda_{1,2} = \pm c_{\mathrm{I}}, \quad \lambda_{3,4} = \pm c_{\mathrm{II}}, \quad c_{\mathrm{I}} \cong \sqrt{\frac{\mu}{\varrho\zeta_t}}, \quad c_{\mathrm{II}} \cong \sqrt{\frac{K}{\varrho c_v \zeta_q}},$$

where c_{II} is the so-called speed of second sound.

It coincides with Landau formula for liquid helium if $\frac{K}{\zeta_q} = \varrho T\eta^2 \frac{\varrho_s}{\varrho_n}$, ϱ_s and ϱ_n being the mass densities of supercomponent and normal component of helium, respectively (e.g. see: [7]). As indicated by this relation, we should consider K/ζ_q as temperature-dependent, which would lead to some changes in considerations of this Section. We shall not go, however, into this problem any further.

On the other hand, the eigen-values (4.17), being candidates for speed of shear pulses and eigen-values (4.18) — for speed of thermal pulses are not coupled through the fields. The reason for this discoupling has been explained in Sect. 2.

It is easy to write necessary conditions for hyperbolicity of Eq. (4.10) — to this aim, λ's must be real for arbitrary s_1, s_2 and q_2:

(4.21)
$$\frac{\mu}{\zeta_t} > \frac{1}{2}(s_1-s_2), \quad \frac{K}{\delta\zeta_q} > 0.$$

In rheology of non-newtonian fluids, the normal stress difference s_1-s_2 is related to the so-called first normal stress coefficient

(4.22) $$s_1 - s_2 = 2\left(\frac{\partial v}{\partial x}\right)^2 \alpha_1 < 0,$$

the inequality following from many experimental data. It means that the inequality

(4.23) $$\zeta_t > 0$$

together with Eq. $(3.41)_2$ is sufficient for inequality $(4.21)_1$ to hold.

Simultaneously, the inequality

(4.24) $$\zeta_q > 0$$

together with inequality $(3.41)_1$ leads through inequality $(4.21)_2$ to

(4.25) $$\delta \equiv \varrho c_v - \frac{\zeta_q}{KT^2}(q_1^2 + q_2^2) > 0.$$

This condition imposes an upper bound constraint on the magnitude of the heat flux q:

(4.26) $$|\mathbf{q}| < \sqrt{\frac{\varrho c_v KT^2}{\zeta_q}}$$

which, however, does not seem to be of physical importance due to smallness of ζ_q.

Let us mention that the linear part of formula (4.17): $\sqrt{\frac{\mu}{\varrho\zeta_t}}$ coincides with formula (5.8) of the paper [1] for the speed of propagation of shear pulses.

Easy calculations show that the above conditions are sufficient for linear independence of left eigen-vectors of the matrix, whose determinant appears in Eq. (4.16). It means, however, that system of differential equations (4.10) is indeed hyperbolic.

5. Transition to fluids of the second order

It is customary in rheological models of non-newtonian fluids to approximate relations (2.6) by constitutive relations of the form

(5.1) $$\mathbf{t} = \mathbf{t}(\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \ldots, \mathbf{A}^{(n)})$$

(see: [8]), where $\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)}$ are so-called Rivlin–Ericksen tensors:

(5.2) $$A_{kl}^{(1)} = 2v_{(k,l)}, \quad A_{kl}^{(2)} = A_{kl}^{(1)} + \dot{A}_{ki}^{(1)} v_{i,l} + A_{li}^{(1)} v_{i,k}, \quad \text{etc.}$$

Such models are called fluids of the n-th order. In particular, the fluid of the second order is defined by the relation

(5.3) $$t_{kl} + p\delta_{kl} = \mu A_{kl}^{(1)} + \alpha_1 A_{\langle kl\rangle}^{(2)} + \alpha_2\left[A_{ki}^{(1)} A_{il}^{(1)} - \frac{1}{3} A_{ij}^{(1)} A_{ij}^{(1)} \delta_{kl}\right].$$

It is easy to see that relations of this form follow from Eq. (2.6) by regular perturbation method, which in kinetic theories is called a Maxwell iteration procedure. Namely, the zeroth approximation is defined as

(5.4) $$t_{\langle kl\rangle}^{(0)} = 2\mu v_{\langle k,l\rangle}$$

i.e. Navier–Stokes fluid. Then

(5.5) $$\zeta_t \overset{\triangle}{t}{}_{\langle kl\rangle}^{(0)} + t_{\langle kl\rangle}^{(1)} - 2\mu v_{\langle k,l\rangle} = 0$$

defines the first approximation. It is easy to see that, for incompressible materials: $A^{(1)}_{kk} = 0$, (see Formula (2.9))

$$(5.6)\qquad t^{(1)}_{\langle kl\rangle} = \mu A^{(1)}_{\langle kl\rangle} - \zeta_t \mu \left[A^{(2)}_{\langle kl\rangle} + (\gamma_t - 1)\left(A^{(1)}_{kl} A^{(1)}_{il} - \frac{1}{3} A^{(1)}_{ij} A^{(1)}_{ij} \delta_{kl} \right)\right]$$

if μ is assumed to be constant. Comparison of Eqs. (5.3) and (5.6) yields

$$(5.7)\qquad \alpha_1 = -\mu\zeta_t, \quad \alpha_2 = -\alpha_1(1-\gamma_t).$$

In the case of corotational derivative ($\gamma_t = 0$), the result $(5.7)_2$ coincides with that obtained by DUNN and FOSDICK [9] and, apart from the sign, contradicts experimental data (compare: [1]).

Inequalities for μ and ζ_t yield

$$(5.8)\qquad \alpha_1 < 0.$$

It has been shown, however, by DUNN and FOSDICK [9] that such a model is thermodynamically unstable if considered in its own rights (not as an approximation of a Maxwellian fluid).

Similar problems arise in the case of Maxwell iteration applied to Eq. (2.5)

$$(5.9)\qquad \begin{aligned} q^{(0)}_k &= -KT_{,k}, \\ q^{(1)}_k &= -KT_{,k} - \zeta_q (KT_{,k})^{\triangle}. \end{aligned}$$

In contrast to Fourier relation $(5.9)_1$, the relation $(5.9)_2$ yields the instability of thermodynamic equilibrium for $K > 0$, $\zeta_q > 0$. Moreover, as it is easy to show, Maxwell iteration transforms the hyperbolic system of equations into a parabolic one — the speeds of shear pulses and of the second sound become infinite independently of the degree of approximation.

The reason for those problems is obvious. The expansions of **t** and **q** into power series in ζ_t and ζ_q, respectively, and subsequent use of perturbation method reduces necessarily differential Eqs. (2.5) and (2.6) to algebraic relations. It is due to the fact that ζ_t and ζ_q appear in front of the differential operator. It means, however, that proper approximation of solutions of (2.5) and (2.6) must be sought by the method of *singular* perturbation. It follows that Maxwell iteration method is inappropriate if we want not only to achieve the quantitative agreements but also to preserve such physical features as finite speeds of pulses.

6. Quadratic evolution equations

Maxwell iteration procedure, demonstrated in the previous section, seems to indicate that, if the quadratic model is to be constructed, it is not essential if we replace in Eqs. (2.8) and (2.9) the terms with coefficients γ_t and γ_q by terms quadratic in $t_{\langle kl\rangle}$, q_k, i.e. if the evolution equations are assumed to have the form

$$(6.1)\qquad \begin{aligned} &\zeta_q \overset{\triangle}{q}_k + q_k + KT_{,k} + \beta_1 t_{\langle kl\rangle} q_l = 0. \\ &\zeta_t \overset{\triangle}{t}_{\langle kl\rangle} + t_{\langle kl\rangle} - 2\mu v_{\langle k,l\rangle} + \beta_2 q_{\langle k} q_{l\rangle} + \beta_3 \left(t_{\langle kl\rangle} t_{\langle ll\rangle} - \frac{1}{3} t_{\langle ij\rangle} t_{\langle ij\rangle} \delta_{kl} \right) = 0, \end{aligned}$$

where

$$\overset{\triangle}{q}_k = \dot{q}_k + \xi q_i v_{[i,k]}, \tag{6.2}$$

$$\overset{\triangle}{t}_{\langle kl \rangle} = \dot{t}_{\langle kl \rangle} + t_{\langle ki \rangle} v_{[i,l]} + t_{\langle li \rangle} v_{[i,k]}$$

and coefficients β_1, β_2, β_3 are independent of $t_{\langle kl \rangle}$ and q_k.

Inspection of inequality (3.4), corrected by additional terms arising from Eqs. (6.1), shows that the above statement is false. It is obvious that, in comparison with the case $\gamma_t = 0$, $\gamma_q = 0$, the only change in thermodynamic relations appears in the residual inequality. In the present case, we obtain

$$\Lambda^q(q_k + \beta_1 t_{\langle kl \rangle} q_l) + \Lambda^{\langle kl \rangle}(t_{\langle kl \rangle} + \beta_2 q_{\langle k} q_{l \rangle} + \beta_3 t_{\langle ki \rangle} t_{\langle il \rangle}) \leqslant 0, \tag{6.3}$$

where

$$\Lambda^{q_k} = -\frac{1}{KT^2} q_k, \quad \Lambda^{\langle kl \rangle} = -\frac{1}{2\mu T} t_{\langle kl \rangle}. \tag{6.4}$$

The above inequality can easily be solved:

1) assuming $t_{\langle kl \rangle} = 0$, we get

$$K > 0; \tag{6.5}$$

2) assuming $q_k = 0$, $t_{11} = \sigma$, $t_{22} = \sigma$, $t_{33} = t_{12} = t_{13} = t_{23} = 0$, we obtain

$$\beta_3 = 0, \quad \mu > 0; \tag{6.6}$$

3) assuming $q_1 = q$, $q_2 = q_3 = 0$ and the stress tensor as in 2), we get

$$\beta_2 = -\frac{2\mu}{KT}\beta_1 = :\beta. \tag{6.7}$$

The above relations are necessary and sufficient for the inequality (6.3) to hold for arbitrary $t_{\langle kl \rangle}$ and q_k.

Hence, equations (6.1) must have the form

$$\zeta_q \overset{\triangle}{q}_k + q_k + KT_{,k} - \frac{KT}{2\mu}\beta t_{\langle kl \rangle} q_l = 0, \tag{6.8}$$

$$\zeta_t \overset{\triangle}{t}_{\langle kl \rangle} + t_{\langle kl \rangle} - 2\mu v_{\langle k,l \rangle} + \beta q_{\langle k} q_{l \rangle} = 0.$$

It is obvious that coupling of those two equations is entirely different from coupling of (2.5) and (2.6), whatever the coefficients γ_t, γ_q may be. Particularly striking in such a model is the presence of heat flux in the equation for stresses. It means that, in general, time-dependent deviatoric stresses would appear even in a rigid heat conductor and their evolution would depend on the history of heat flux.

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