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Gauss principle of least constraint for the solution of heat conduction problem in an infinite circular solid cylinder

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THE GAUSS principle of least constraint has been extended to solve both linear and nonlinear heat conduction problems in an infinite circular solid cylinder. The approximate solutions obtained have been compared with those of other authors, whenever available. Simplicity of the method and favourable agreement with the results of other authors suggest that this method is effective in dealing with nonlinear heat conduction problems.

Zasadę Gaussa zastosowano do rozwiązania problemu liniowego i nieliniowego przepływu ciepła w nieskończonym walcu kołowym. Otrzymane wyniki przybliżone porównano z osiągalnymi rezultatami otrzymanymi przez innych autorów. Zbieżność tych wyników i prostota zaproponowanej tu metody pozwalają przewidywać, że będzie ona efektywna w zastosowaniu do nieliniowych problemów przewodnictwa cieplnego.

Принцип Гаусса применен к решению линейной и нелинейной задач теплопроводности в бесконечном круговом цилиндре. Полученные приближенные результаты сравнены с достигнутыми результатами других авторов. Сходимость этих результатов и простота предложенного здесь метода позволяют предсказать, что он эффективен в применении к нелинейным задачам теплопроводности.

1. Introduction

THE HEAT conduction equation when expressed in the cylindrical coordinates is given by

$$S\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(Kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(K\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right), \tag{1.1}$$

where T is the temperature at time t of a point of the solid whose position coordinates are (r, θ, z), K is the thermal conductivity which is a function of the temperature and/or coordinates, and $S = \varrho c$, where ϱ is the density and c is the specific heat at constant pressure. The two quantities S and K are termed the "thermal parameters". If a circular solid cylinder, Fig. 1, whose axis coincides with the axis of z, is heated, and the initial and boundary conditions are independent of the coordinates θ and z, the temperature will be a function of r and t only and Eq. (1.1) reduces to

$$S\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(Kr\frac{\partial T}{\partial r}\right). \tag{1.2}$$

Equation (1.2) is linear when the thermal parameters S and K are considered constants as in most of the reported works. However, in this paper we are going to illustrate the solution of Eq. (1.2) when K varies linearly with temperature.

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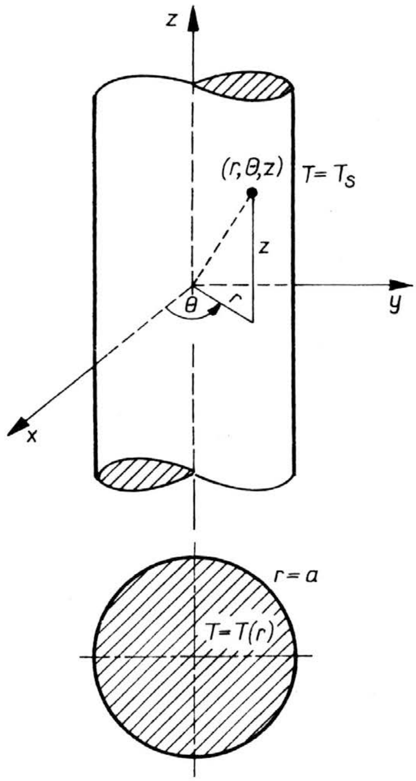

FIG. 1. Flow of heat through an infinite circular solid cylinder.

Section 2 shows how the Gauss principle of least constraint is applied to the solution of Eq. (1.2). In Sect. 3, the problem of linear flow of heat through an infinite solid cylinder is considered, and the results obtained are plotted and compared with those given in [3]. The problem of flow of heat with radiation at the surface is treated in Sect. 4, the results obtained are compared with those given in [4]. Also in this section the black body radiation problem is solved. In Sect. 5, the problem of nonlinear heat conduction along an infinite circular solid cylinder, which is more practical, is considered and the results are plotted. Finally, Sect. 6 contains the general discussion and some remarks.

2. Gauss principle of least constraint

Gauss principle of least constraint is a minimum principle originally applied to problems of dynamics. However, VUJANOVIC and BAČLIC [1] proved that this principle can be applied to treat heat conduction problems. Let us explain how the Gauss principle is applied to the solution of Eq. (1.2).

Let us consider the quantity

(2.1) $$z = \int_v [X-Y]^2 dv,$$

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where V is the volume which is engaged in the process of heat transfer, and

$$X = S\frac{\partial T}{\partial t}, \tag{2.2}$$

$$Y = \frac{1}{r}\frac{\partial}{\partial r}\left(Kr\frac{\partial T}{\partial r}\right), \tag{2.3}$$

are the temporal and the spatial parts, respectively. Then, according to Gauss principle, the quantity z assumes its minimum value in either of the following cases:

(2.4) (i) $\delta X \neq 0$ and $\delta Y = 0$

or

(2.5) (ii) $\delta X = 0$ and $\delta Y \neq 0$,

where δX and δY refer to the variations in X and Y, respectively. This principle differs from the usual variational principles of ordinary mechanics since it applies the technique that all components of the field are held fixed or specified, except one with respect to which the minimization of the integral is carried out. Due to this fact the above variational principle is called a restricted or partial variational principle [2]. The principle may be formulated by means of the generalized coordinates instead of the components of the field itself. In each particular case we must be able to identify the characteristic set of parameters which represents X or Y and then the minimization of z has to be carried out with respect to this set holding all other proper quantities fixed.

3. Linear flow of heat in an infinite circular solid cylinder

In this case $S = S_0$ and $K = K_0$ are considered to be constants, then Eq. (1.2) reduces to

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \tag{3.1}$$

where $\alpha = K_0/S_0$.

The boundary and initial conditions are

$$\begin{aligned} T(a, t) &= T_s, \qquad t > 0;\\ T(r, 0) &= 0, \qquad 0 < r < a; \end{aligned} \tag{3.2}$$

where a is the radius of the cylinder and T_s is a constant. Now if we apply the transformation

$$\tau = \frac{\alpha t}{a^2}, \quad u = \frac{r}{a}, \tag{3.3}$$

then $T(u, \tau)$ should satisfy the equation

$$\frac{\partial T}{\partial \tau} = \frac{1}{u}\frac{\partial}{\partial u}\left(u\frac{\partial T}{\partial u}\right), \tag{3.4}$$

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and the conditions

$$(3.5)\qquad \begin{aligned} T(1,\tau) &= T_s, &\quad \tau > 0,\\ T(u,0) &= 0, &\quad 0 < u < 1. \end{aligned}$$

Now consider the trial solution

$$(3.6)\qquad T(u,\tau) = T_s[1-F(u)G(\tau)],$$

where $F(u)$ is a specified function of the position which has to be chosen in accordance with the shape of the cross-section of the body. The function $F(u)$ has the following properties: $F(1) = 0$, and $F(u) > 0$ for every u inside the region bounded by the surface of the cylinder.

Hence, the problem is reduced to the determination of the unknown function $G(\tau)$ using the Gauss principle. The temporal part of (3.4) is given by

$$(3.7)\qquad X = \frac{\partial T}{\partial \tau} = -T_s F\dot{G} = -FW,$$

where $W = T_s\dot{G}$ is the temporal part which represents X, while the spatial part of (3.4) is given by

$$(3.8)\qquad Y = \frac{1}{u}\frac{\partial}{\partial u}\left(u\frac{\partial T}{\partial u}\right) = -T_s G\left(\frac{\partial^2 F}{\partial u^2}+\frac{1}{u}\frac{\partial F}{\partial u}\right).$$

Let us suppose that

$$(3.9)\qquad F(u) = 1-u^3.$$

Then from (3.7) and (3.8) we have

$$X = -(1-u^3)w,$$
$$Y = 9uT_sG.$$

Now, introducing the quantity

$$z = \int_0^1 [X-Y]^2 du,$$

we get

$$z = \frac{9}{14}W^2+\frac{27}{5}GT_sW,$$

where a term free from W has been omitted. According to Gauss principle, let us minimize z with respect to the temporal part W, i.e.

$$\frac{\partial z}{\partial W} = 0,$$

and using $W = T_s\dot{G}$ we get the following first-order differential equation

$$(3.10)\qquad \dot{G} = -\frac{21}{5}G,$$

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where the initial condition $G(0)$ will be determined by minimizing the initial square residual of the form

$$J = \int_0^1 \{T_s[1-F((u)G_0]\}^2 du, \tag{3.11}$$

where $G_0 = G(0)$.

Integrating (3.11) and using $\partial J/\partial G_0 = 0$ we get

$$G(0) = \frac{7}{6}.$$

Then the solution of the initial value problem of the linear differential equation (3.10) is given by

$$G(\tau) = \frac{7}{6}\exp\left(-\frac{21}{5}\tau\right).$$

Hence the temperature distribution is

$$T(r,t) = T_s\left\{1-\frac{7}{6}\left[1-\left(\frac{r}{a}\right)^3\right]\right\}\exp\left(-\frac{21}{5}\frac{\alpha t}{a^2}\right). \tag{3.12}$$

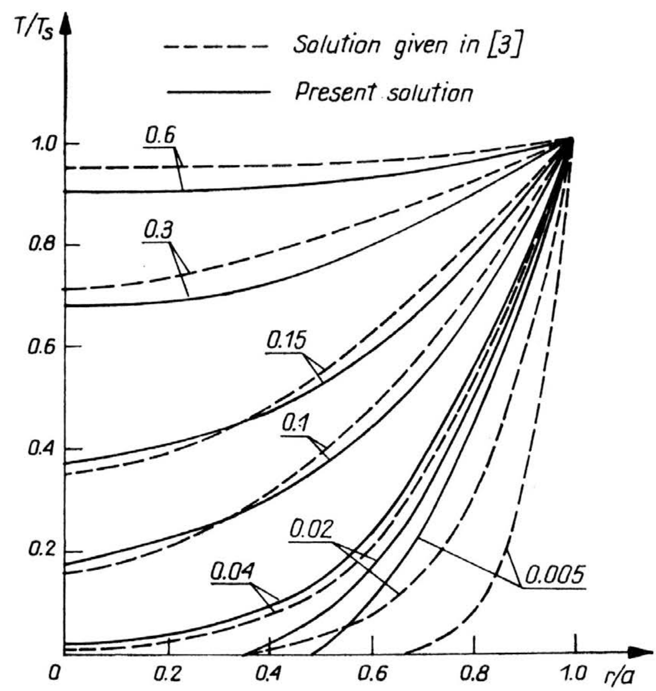

FIG. 2. Temperature distribution at various times in a cylinder of radius a with zero initial temperature and constant surface temperature T_s. The numbers on curves are the values of $\alpha t/a^2$.

The temperature distribution at various times is given in Fig. 2. Comparison with the solution obtained by CARSLAW and JAGER [3] is also illustrated in Fig. 2.

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4. Flow of heat with radiation at surface

In this case we are going to consider the problem

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \tag{4.1}$$

where $\alpha = K_0/S_0$ is a constant, with the initial and boundary conditions

$$\begin{aligned} &T(r, 0) = T_0, \quad 0 < r < a; \\ &\frac{\partial T}{\partial r} = -f(T_s), \quad t > 0, \quad r = a, \end{aligned} \tag{4.2}$$

where T_0 is a constant and $T_s(t)$ is the temperature at the surface of the cylinder, $r = a$. Moreover $f = f(x)$ is a given function. Let us assume a trial solution of the form

$$T(r, t) = T_s(t)[1+F(r)G(t)],$$

where

$$F(r) = 1-\left(\frac{r}{a}\right)^2,$$

Clearly

$$F(a) = 0,$$
$$F(r) > 0 \quad \text{for} \quad 0 < r < a.$$

Then from the given boundary condition we get

$$G(t)\,T_s(t) = \frac{a}{2}f(T_s). \tag{4.3}$$

Differentiating (4.3) with respect to t we have

$$\dot{G} = \frac{a\dot{T}_s}{2T_s^2}(f_sT_s - f), \tag{4.4}$$

where

$$f_s = \frac{df}{dT_s}.$$

Now, consider the quantity

$$z = \int_0^a [X-Y]^2 dr, \tag{4.5}$$

where X, Y are the temporal and spatial parts of Eq. (4.1), respectively. According to Gauss principle, z is to be minimized with respect to either X or Y. Since we have

$$X = \frac{\partial T}{\partial t} = T_sF\dot{G} + \dot{T}_s(1+FG) = W\left\{1+\frac{a}{2}f_s\left[1-\left(\frac{r}{a}\right)^2\right]\right\}, \tag{4.6}$$

where $W = \dot{T}_s$ is the temporal part which represents X, while the spatial part is given by

$$Y = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -\frac{2\alpha}{a}f. \tag{4.7}$$

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Substituting from Eqs. (4.6) and (4.7) into Eq. (4.5), integrating and keeping only terms having W we get

$$z = a\left(1+\frac{2}{3}af_s+\frac{2}{15}a^2f_s^2\right)W^2+4\alpha f\left(1+\frac{a}{3}f_s\right)W. \tag{4.8}$$

From Eq. (4.8) using $\partial z/\partial W = 0$ and $W = \dot{T}_s$ we get

$$\dot{T}_s = -\frac{10\alpha}{a}\left(\frac{3+af_s}{15+10af_s+2a^2f_s^2)}\right)f. \tag{4.9}$$

In the following we are going to consider two possible cases for the shape of the function $f(T_s)$.

CASE 1. $f(T_s) = hT_s$, where h is a constant.

Then Eq. (4.9) gives

$$\dot{T}_s = -\frac{10\alpha h}{a}\left(\frac{3+ah}{15+10ah+2a^2h^2}\right)T_s, \tag{4.10}$$

which is a first-order differential equation with the initial condition

$$T_s(0) = T_{s_0},$$

determined from the minimization of the initial residual error function J given by

$$J = \int_0^a\left[T_{s_0}+\frac{a}{2}f_0F-T_0\right]^2dr, \tag{4.11}$$

where $f_0 = f(T_{s_0})$.

Integrating (4.11) and applying $\partial J/\partial T_{s_0} = 0$ we get

$$T_{s_0} = 5\left(\frac{3+b}{15+10b+2b^2)}\right)T_0, \tag{4.12}$$

where $b = ah$. Solving Eq. (4.10) we get

$$T_s(t) = 5\left(\frac{3+b}{15+10b+2b^2}\right)T_0\exp\left[-\frac{10\alpha t}{a^2}\left(\frac{(3+b)b}{15+10b+2b^2}\right)\right]. \tag{4.13}$$

Hence the temperature distribution is

$$T(r,t) = 5\left(\frac{3+b}{15+10b+2b^2}\right)T_0\left\{1+\left[1-\left(\frac{r}{a}\right)^2\right]\frac{b}{2}\right\}\exp\left[-\frac{10\alpha t}{a^2}\cdot\left(\frac{(3+b)b}{15+10b+2b^2}\right)\right]. \tag{4.14}$$

Results are illustrated in Fig. 3 to Fig. 6 together in comparison with results given by ECKERT and DRAKE [4] at different values of b and r/a.

CASE 2. $f(T_s) = hT_s^4$, where h is a constant.

In this case the cylinder of initial temperature T_0 is suddenly immersed in a medium having zero temperature and black body property. Then from Eq. (4.9) we have

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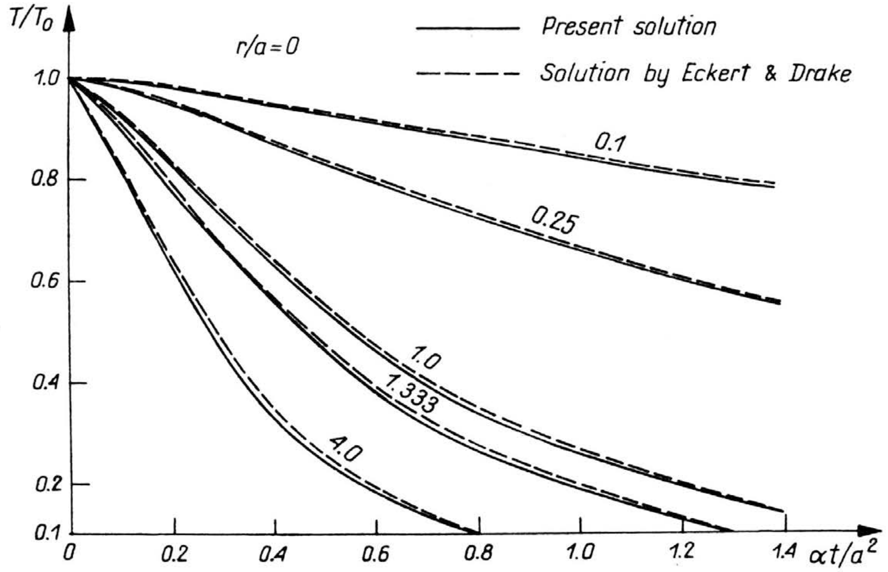

FIG. 3. Transient heat conduction through a cylinder with radiation at its surface. Numbers on curves are the values of *b*.

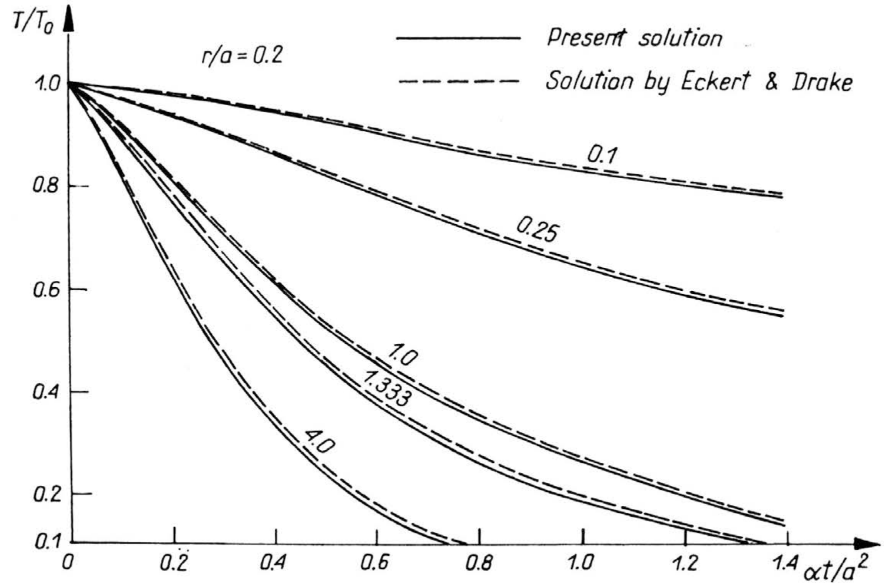

FIG. 4. Transient heat conduction through a cylinder with radiation at its surface. Numbers on curves are the values of *b*.

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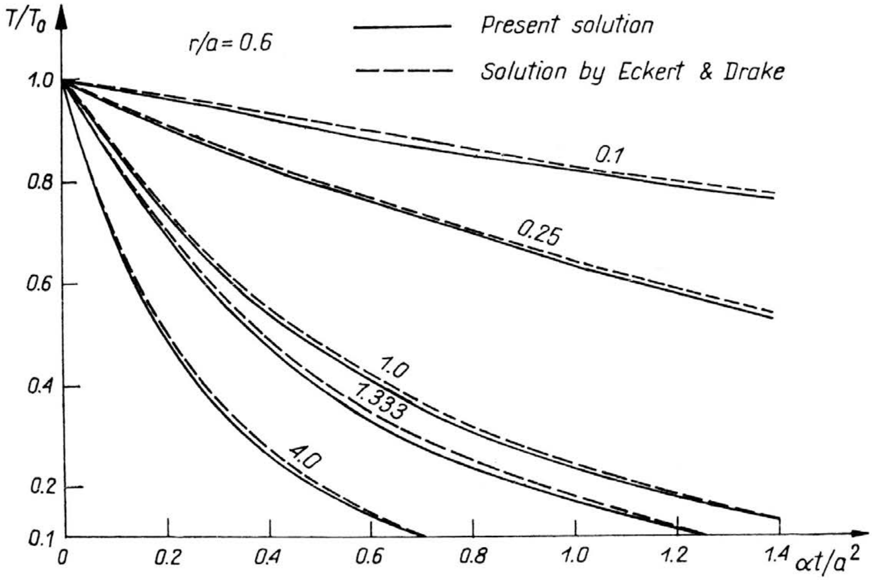

FIG. 5. Transient heat conduction through a cylinder with radiation at its surface. Numbers on curves are the values of *b*.

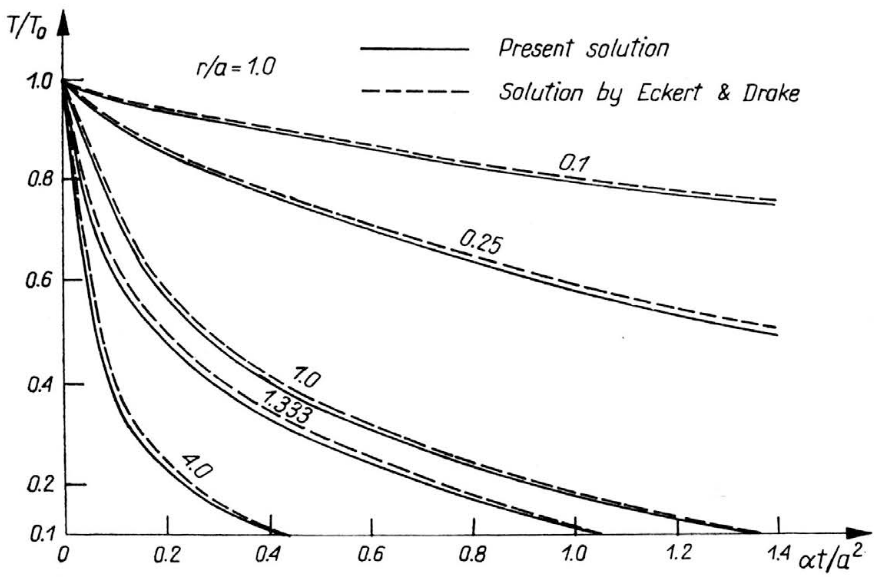

FIG. 6. Transient heat conduction through a cylinder with radiation at its surface. Numbers on curves are the values of *b*.

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(4.15) $$\dot{T}_s = -\frac{10\alpha b}{a^2}\left(\frac{3+4bT_s^3}{15+40bT_s^3+32b^2T_s^6}\right)T_s^4,$$

with

$$T_s(0) = T_0.$$

The solution of the nonlinear initial value differential Eq. (4.15) is obtained using Runge–Kutta fourth-order method for various values of b and the results are illustrated in Fig. 7.

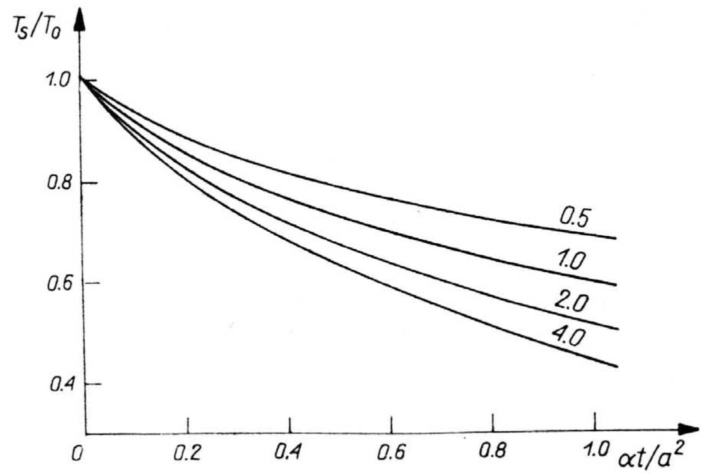

FIG. 7. Transient heat conduction through a cylinder with black-body radiation property at surface. Numbers on curves are the values of b.

5. Flow of heat through an infinite circular solid cylinder whose thermal conductivity varies linearly with temperature

Consider the nonlinear problem

(5.1) $$S_0\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(Kr\frac{\partial T}{\partial r}\right),$$

where the thermal conductivity K is a function of the temperature T and S_0 is a constant. It does not seem possible to solve this equation in general, and empirical expression for K as a function of T must therefore be introduced.

Let us assume

$$K(T) = K_0\left(1+A\frac{T}{T_s}\right),$$

where K_0 and A are constants.

Hence, Eq. (5.1) takes the form

(5.2) $$\frac{\partial T}{\partial t} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left[\left(1+A\frac{T}{T_s}\right)r\frac{\partial T}{\partial r}\right],$$

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where $\alpha = K_0/S_0$ is a constant, with the initial and boundary conditions

$$(5.3)\qquad \begin{aligned} T(r, 0) &= 0, \qquad 0 < r < a, \\ T(a, t) &= T_s, \qquad t > 0, \end{aligned}$$

where T_s is constant.

Now if we apply the transformation

$$(5.4)\qquad \tau = \frac{\alpha t}{a^2}, \quad u = \frac{r}{a}, \quad \theta = \frac{T}{T_s},$$

then $\theta(u, \tau)$ should satisfy the equation

$$(5.5)\qquad \frac{\partial\theta}{\partial\tau} = \frac{1}{u}\frac{\partial}{\partial u}\left[(1+A\theta)u\frac{\partial\theta}{\partial u}\right],$$

and the conditions

$$(5.6)\qquad \begin{aligned} \theta(1, \tau) &= 1, \qquad \tau > 0, \\ \theta(u, 0) &= 0, \qquad 0 < u < 1. \end{aligned}$$

Now, consider the trial solution

$$(5.7)\qquad \theta(u, \tau) = 1 - F(u)G(\tau),$$

and let

$$F(u) = 1 - u^3.$$

Then

and

$$\begin{aligned} &F(1) = 0, \\ &F(u) > 0 \quad \text{for} \quad 0 < u < 1. \end{aligned}$$

Hence, the problem is reduced to the determination of the unknown function $G(\tau)$ using the Gauss principle. The temporal part of Eq. (5.5) is given by

$$(5.8)\qquad X = \frac{\partial\theta}{\partial\tau} = -W(l - u^3),$$

where $W = \dot{G}$ is the temporal set which represents X, while the spatial part of Eq. (5.5) is given by

$$(5.9)\qquad Y = \frac{1}{u}\frac{\partial}{\partial u}\left[(l+A\theta)u\frac{\partial\theta}{\partial u}\right] = 9G[(1+A) - AG(1-2u^3)]u.$$

Introducing the quantity

$$Z = \int_0^1 [X - Y]^2 du,$$

and using Eqs. (5.8) and (5.9), we get

$$(5.10)\qquad z = \frac{9}{14}W^2 + \frac{27}{5}(1+A)GW - \frac{27}{10}AG^2W,$$

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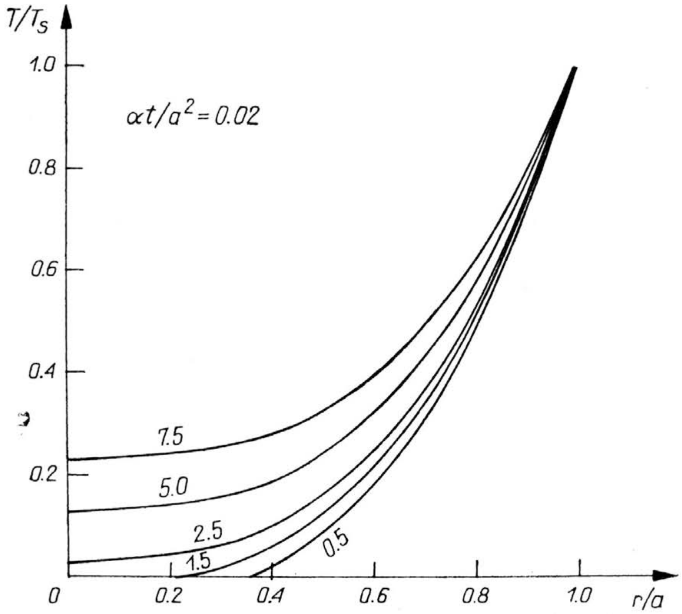

FIG. 8 Temperature distribution curves. Numbers on curves are the values of *A*.

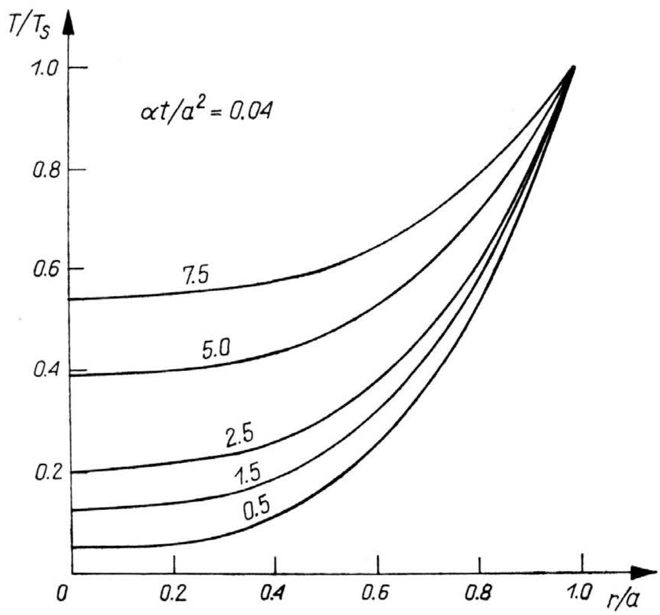

FIG. 9. Temperature distribution curves. Numbers on curves are the values of *A*.

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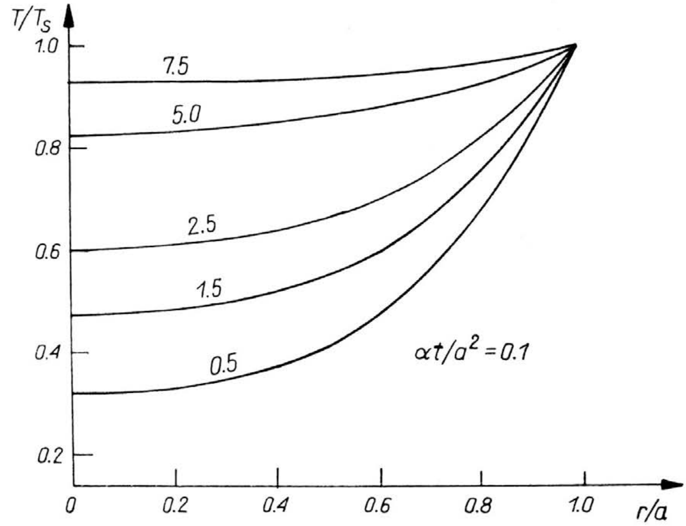

FIG. 10. Temperature distribution curves. Numbers on curves are the values of A.

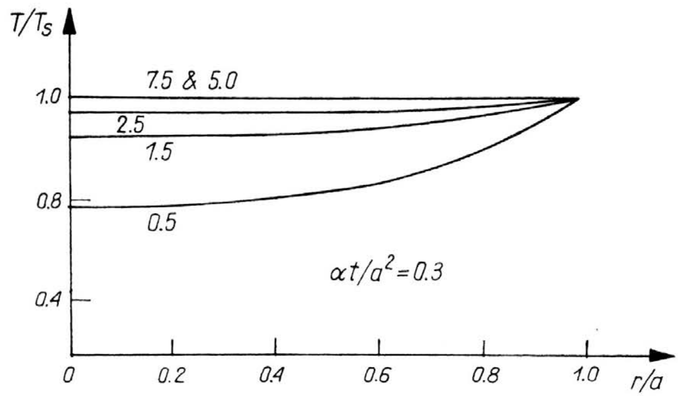

FIG. 11. Temperature distribution curves. Numbers on curves are the values of A.

where terms free from W have been omitted. Minimizing Eq. (5.10) with respect to W, i.e. $\partial z/\partial W = 0$, and using $W = \dot{G}$ we get the following first-order differential equation

$$\dot{G} + \frac{21}{5}(1+A)G = \frac{21}{10}AG^2, \tag{5.11}$$

where the initial condition $G(0)$ will be determined by minimizing the initial square residual of the form

$$J = \int_0^1 [1-F(u)G_0]^2 du, \tag{5.12}$$

where $G_0 = G(0)$.

Integrating Eq. (5.12) and using $\partial J/\partial G_0 = 0$ we get

$$G(0) = \frac{7}{6}.$$

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Then the solution of the initial value problem of the differential equation (5.11) is given by

$$G(t) = \frac{14(1+A)}{2-7A\left[1-\exp\left(-\frac{21}{5}(1+A)\tau\right)\right]}\exp\left[-\frac{21}{5}(1+A)\tau\right]. \tag{5.13}$$

Hence the temperature distribution is

$$T(r,t) = T_s\left\{1-\left[1-\left(\frac{r}{a}\right)^3\right]\frac{14(1+A)}{2-7A\left[1-\exp\left(-\frac{21}{5}(1+A)\frac{\alpha t}{a^2}\right)\right]}\exp\left[-\frac{21}{5}(1+A)\frac{\alpha t}{a^2}\right]\right\}. \tag{5.14}$$

The temperature distribution at various times are given in Fig. 8 to Fig. 11.

6. Conclusion and discussion

In this paper the Gauss principle of least constraint was successfully extended to solve both linear and nonlinear heat conduction problems. Favourable agreement with the results obtained by other methods as well as simplicity of the method suggest that this method is effective in dealing with nonlinear heat conduction problems. It should be noticed that in the case of black body radiation, the solution was found at the surface since $T_s(t)$ is most frequently the quantity of greatest physical interest on account of this extremum character [5]. Also the solution, given here, of the flow of heat when the thermal conductivity depends linearly on temperature, appears to be satisfactory since it is consistent with the linear case when putting $A = 0$.

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