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Wave propagation in strongly heterogeneous media

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THE EFFECTIVE wave equation describing the propagation of the mean field in a stochastic medium is considered. The medium is specified as a random arrangement of homogeneous regions (phase mixture). A self-consistent approach is used to calculate the velocity and the damping of the mean wave as functions of the wave number. The method can be applied to the case of strong heterogeneity. Detailed calculations are carried out for scalar waves, approximately spherical grains and long wavelengths. Some numerical results are given for a two-phase mixture.

Rozpatruje się konkretne równanie falowe opisujące propagację pola średniego w ośrodku stochastycznym. Ośrodek jest zdefiniowany jako przypadkowa rozmaitość obszarów jednorodnych (mieszanina faz). Dla wyliczenia prędkości i tłumienia fali średniej w funkcji liczby falowej wykorzystuje się metodę zapewniającą zgodność. Rozważana metoda może być stosowana do przypadku silnej niejednorodności. Szczegółowe obliczenia przeprowadzono dla przypadku fal skalarnych, ziaren w przybliżeniu kulistych i fal długich. Podano wyniki numeryczne dla mieszaniny dwufazowej.

Рассматривается эффективное волновое уравнение описывающее распространение средполя в стохастической среде. Среда определена как случайное многообразие однородных областей (смесь фаз). Для расчета скорости и затухания средней волны в функции волнового числа используется самосогласованный подход. Рассматриваемый метод может быть применен к случаю сильной неоднородности. Детальные расчеты проведены для случая скалярных волн, сферических зерен и длинных волн. Приведены численные результаты для двухфазной смеси.

1. Introduction

THIS PAPER deals with the wave propagation in a heteregeneous medium consisting of a random arrangement of homogeneous regions (or grains), for example a phase mixture. The regions differ from each other by their material properties. Our aim is to predict the behaviour of the mean wave from information about the randomly varying material parameters. We are mainly interested in phenomena which are related to the heterogeneity of the material as dispersion and damping of the mean wave due to scattering processes at the inhomogeneities.

Most of former work which has been done in this field is based on perturbation methods (see e.g. [1–3]). With the aid of the multiple-scattering formalism, several authors investigated a material containing a random distribution of scatterers (e.g. [4]). In order to treat the case of strong heterogeneity, we use a self-consistent method developed in a former paper [5] hereafter referred to as I. This method is presumed to give useful results if all components of the mixture have comparable morphology. On the contrary, the proposed treatment seems to be less appropriate for a material consisting, for example, of a dense arrangement of isolated grains which are separated from each other by thin layers of a matrix material. Furthermore, the material is assumed to show negligible

correlations between the material properties of adjacent grains. In the present paper, practical calculations are restricted to the simplest case of geometry, where all homogeneous regions can be approximated by spheres.

We consider a stochastic wave equation of the type

$$L_{st}u(\mathbf{r}, t) \equiv \varrho_{st}(\mathbf{r}) \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial \mathbf{r}} \left(\varepsilon_{st}(\mathbf{r}) \frac{\partial u}{\partial \mathbf{r}} \right) = q(\mathbf{r}, t). \tag{1.1}$$

(In comparison with I we slightly change the notation indicating stochastic quantities by the subscript st, whereas effective parameters are written without any subscript). Analogously to Eq. (2.1) of I, the stochastic material parameters $\varrho_{st}(\mathbf{r})$ and $\varepsilon_{st}(\mathbf{r})$ can be represented as

$$\varrho_{st}(\mathbf{r}) = \sum_i \varrho_i \Theta_i(\mathbf{r}), \qquad \varepsilon_{st}(\mathbf{r}) = \sum_i \varepsilon_i \Theta_i(\mathbf{r}), \tag{1.2}$$

where ϱ_i and ε_i denote their values inside the i-th grain. The Θ_i are step functions([1]) defined by

$$\Theta_i(\mathbf{r}) = \begin{cases} 1 & \text{inside} \\ 0 & \text{outside} \end{cases} \quad \text{the } i\text{-th grain}, \qquad \sum_i \Theta_i(\mathbf{r}) = 1. \tag{1.3}$$

The concrete form of the source term $q(\mathbf{r}, t)$ is of no consequence in the following. Thus, we assume it to be a deterministic function. In order to get rid of boundary effects, we consider an infinite medium. The sources q are located in a finite region, and the field u vanishes at great distances.

For the sake of simplicity we restrict ourselves to the scalar fields $u(\mathbf{r}, t)$. Waves of practical interest are almost vector fields (elastic waves, electromagnetic waves). The vector character would involve additional computational difficulties without, however, changing the essence of the reasoning.

The mean wave $\langle u(\mathbf{r}, t) \rangle$ is governed by an effective wave equation (see [1–4])

$$L\langle u(\mathbf{r}, t) \rangle = q(\mathbf{r}, t) \tag{1.4}$$

containing a linear effective wave operator L. The brackets denote an ensemble average over many samples showing identical macroscopic properties, but differing from each other by the random positions (and shapes) of the grains [9].

In Sect. 2 the self-consistent procedure to determine the effective operator which was outlined in I is briefly reviewed and extended to the dynamic case. In Sect. 3 calculations of the effective wave operator are carried out for the special case of approximately spherical grains and long wavelengths. From this operator the dispersion relation of the mean wave is readily obtained. It gives the velocity and the decay of the coherent mean wave $\langle u \rangle$ as a function of the wavelength $\lambda = 2\pi/k$. Finally, in Sect. 4 the example of a two-phase mixture is discussed numerically.

([1]) Mathematical difficulties which could arise in Eq. (1.1) from the step functions are physically irrelevant. They can be avoided by imagining the steps as rapid, but continuous transitions. Equation (1.1) remains valid in the limit (1.3).

2. Self-consistent approach to the effective wave equation

Due to the linearity of Eq. (1.1), the field u generated by the given sources $q(\mathbf{r}, t)$ can be expressed as a linear functional of the source term

$$u(\mathbf{r}, t) = \mathbf{g}_{st} q(\mathbf{r}, t) = \int d\mathbf{r}' \int_{-\infty}^{t} dt' g_{st}(\mathbf{r}, \mathbf{r}', t, t') q(\mathbf{r}', t'). \tag{2.1}$$

$\mathbf{g}_{st}$ and g_{st} are called the stochastic Green's operator and the stochastic Green's function, respectively. Ensemble averaging gives, for deterministic sources,

$$\langle u(\mathbf{r}, t)\rangle = \int d\mathbf{r}' \int_{-\infty}^{t} dt' \langle g_{st}(\mathbf{r}, \mathbf{r}', t, t')\rangle q(\mathbf{r}', t'). \tag{2.2}$$

If we consider a material which is time independent (Eq. (1.1)) and statistically homogeneous, all averages containing only material properties have to be invariant under translations of space and time. This also holds for the mean Green's function

$$\langle g_{st}(\mathbf{r}, \mathbf{r}', t, t)\rangle =: g(\mathbf{r}, \mathbf{r}', t, t') = g(\mathbf{r}-\mathbf{r}', t-t'). \tag{2.3}$$

Therefore, Eq. (2.2) becomes a convolution integral, and Fourier-transformation leads to a simple product of numbers

$$\begin{aligned} &\langle u(\mathbf{k}, \omega)\rangle = g(\mathbf{k}, \omega) q(\mathbf{k}, \omega), \\ \text{or} \quad &\big(g(\mathbf{k}, \omega)\big)^{-1} \langle u(\mathbf{k}, \omega)\rangle = q(\mathbf{k}, \omega). \end{aligned} \tag{2.4}$$

A comparison with Eq. (1.4) shows that in the Fourier-space the effective wave operator takes the form of a number, too:

$$L\langle u\rangle|_{\mathbf{k},\omega} = L(\mathbf{k}, \omega)\langle u(\mathbf{k}, \omega)\rangle = \big(g(\mathbf{k}, \omega)\big)^{-1} u(\mathbf{k}, \omega) = q(\mathbf{k}, \omega). \tag{2.5}$$

In accordance with the form of the wave operator of a homogeneous material $L_0 = -\omega^2 \varrho_0 + k^2 \varepsilon_0$, we write the effective wave operator

$$L(\mathbf{k}, \omega) = -\omega^2 \varrho(\mathbf{k}, \omega) + k^2 \varepsilon(\mathbf{k}, \omega). \tag{2.6}$$

But it has to be noticed that the decomposition of L into two terms ϱ and ε is not yet defined precisely. Only the meaning of the whole operator L is fixed by Eqs. (1.4) or (2.5).

Analogously to the treatment of the static case in I, we consider a Fourier component of the mean field

$$\begin{aligned} q(\mathbf{r}, t) &= q_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \\ \langle u(\mathbf{r}, t)\rangle &= u_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \end{aligned} \tag{2.7}$$

and then split up the stochastic operator L_{st} (1.1) into

$$\begin{aligned} L_{st} &= L_0 + \sum_i L_i, \\ L_0 &= \varrho(\mathbf{k}, \omega) \frac{\partial^2}{\partial t^2} - \varepsilon(\mathbf{k}, \omega) \left(\frac{\partial^2}{\partial \mathbf{r}^2}\right), \\ L_i &= \big(\varrho_i - \varrho(\mathbf{k}, \omega)\big) \Theta_i(\mathbf{r}) \frac{\partial^2}{\partial t^2} - \big(\varepsilon_i - \varepsilon(\mathbf{k}, \omega)\big) \frac{\partial}{\partial \mathbf{r}} \Theta_i(\mathbf{r}) \frac{\partial}{\partial \mathbf{r}}. \end{aligned} \tag{2.8}$$

Let us note that the operators L_0 and L_i defined above depend on the frequency ω and on the wave vector $\mathbf{k}$ of the mean wave, whereas the whole stochastic operator L_{st} does not.

The field u is decomposed into the mean field $\langle u\rangle$ given by Eq. (2.7) and the fluctuations u_i

$$u = \langle u\rangle + \sum_i u_i, \tag{2.9}$$

which satisfy the equations

$$\begin{aligned} L_0\langle u\rangle &= L\langle u\rangle = q, \\ L_{st}u_i &= \Big(L_0 + \sum_j L_j\Big)u_i = -L_i\langle u\rangle. \end{aligned} \tag{2.10}$$

In the frame of our self-consistent approximation, we neglect all contributions from grains $j \neq i$ in the last equation of the set (2.10), and obtain

$$(L_0 + L_i)u_i = -L_i\langle u\rangle. \tag{2.11}$$

Thus, in order to calculate the field u_i, we imbed the i-th grain into a homogeneous material characterized by the wave operator L_0 which contains the effective material parameters $\varrho(\mathbf{k}, \omega)$, $\varepsilon(\mathbf{k}, \omega)$. Let us notice that this material differs from the effective material described by L, for L_0 is a local operator which contains the wave vector $\mathbf{k}$ of the mean wave only parametrically, whereas L in general is an integral operator. The solution u_i of Eq. (2.11) depends on the properties of the i-th grain and the effective parameters ϱ and ε.

From Eq. (2.9) it follows that the mean value of the field fluctuations must vanish

$$\Big\langle \sum_i u_i \Big\rangle = 0. \tag{2.12}$$

This self-consistency condition yields one equation for the two unknown effective quantities $\varrho(\mathbf{k}, \omega)$, $\varepsilon(\mathbf{k}, \omega)$. A second equation can be obtained from defining the decomposition of $L(\mathbf{k}, \omega)$ into ϱ and ε (Eq. (2.6)), which is somewhat arbitrary, but has to be chosen in a physically reasonable way. This choice influences the validity of the self-consistent approximation in Eq. (2.11). The role of the self-consistency condition (2.12) may also be characterized in other words. Namely, after defining the decomposition (2.6), Eq. (2.12) provides us an implicit equation for the effective wave operator $L(\mathbf{k}, \omega)$.

In order to fix the meaning of ϱ and ε, let us proceed as follows. Applying the operator L_0 to the self-consistency condition (2.12) and using Eq. (2.11), we get

$$L_0\Big\langle \sum_i u_i \Big\rangle = -\Big\langle \sum_i L_i(\langle u\rangle + u_i)\Big\rangle = 0. \tag{2.13}$$

This equation, in connection with the last line in Eq. (2.5), highly suggets the following decomposition:

$$\begin{aligned} \Big\langle \sum_i (\varrho_i - \varrho(\mathbf{k}, \omega))\Theta_i(\langle u\rangle + u_i)\Big\rangle &= 0, \\ \Big\langle \sum_i (\varepsilon_i - \varepsilon(\mathbf{k}, \omega))\Theta_i \frac{\partial}{\partial \mathbf{r}}(\langle u\rangle + u_i)\Big\rangle &= 0, \end{aligned} \tag{2.14}$$

which may be considered as a definition of one of the parameters ϱ and ε. Thus, instead of Eq. (2.12), two equations (2.14) are now at our disposal to calculate the two effective quantities ϱ and ε.

It may be interesting to point out that Eqs. (2.14) can be written in another form:

$$\langle \varrho_{st} u \rangle = \varrho(\mathbf{k}, \omega) \langle u \rangle,$$

$$\left\langle \varepsilon_{st} \frac{\partial}{\partial \mathbf{r}} u \right\rangle = \varepsilon(\mathbf{k}, \omega) \frac{\partial}{\partial \mathbf{r}} \langle u \rangle$$

apart from terms of the type $\Theta_j u_i$ $(j \neq i)$ which have been omitted in Eqs. (2.11) and (2.13). Finally, let us mention that the approach outlined above coincides completely with a self-consistent imbedding procedure proposed by KORRINGA [10].

3. Evaluation for spherical grains

Further calculations are restricted to the simplest case when all grains may be approximated by spheres. Then, Eq. (2.11) can be solved by expanding the fields $\langle u \rangle$ and u_i into series of spherical harmonics. The case of most interest is that of free waves $q = 0$ in which, in consequence of the first equation in the set (2.10), the frequency and wave vector $\mathbf{k}$ are connected by the dispersion relation

$$L(\mathbf{k}, \omega) = -\varrho(\mathbf{k}, \omega)\omega^2 + \varepsilon(\mathbf{k}, \omega)k^2 = 0. \tag{3.1}$$

After having found the solutions of Eq. (2.11), we put them into Eqs. (2.14), perform the averages, and thus get two equations to determine ϱ and ε.

If we choose the center of the i-th grain as origin of the coordinates, Eq. (2.11) reads

$$\begin{aligned} &(\varrho\omega^2 + \varepsilon\Delta)u_i = 0 \quad (r > R_i), \\ &(\varrho_i\omega^2 + \varepsilon_i\Delta)u_i = -[(\varrho_i - \varrho)\omega^2 - (\varepsilon_i - \varepsilon)k^2]\langle u \rangle \quad (r < R_i), \end{aligned} \tag{3.2}$$

R_i denotes the radius of the considered grain. At the surface of the grain, the following boundary conditions hold:

$$\begin{aligned} u_i|_{ext} &= u_i|_{int}, \\ \varepsilon \frac{\partial}{\partial r}(\langle u \rangle + u_i)|_{ext} &= \varepsilon_i \frac{\partial}{\partial r}(\langle u \rangle + u_i)|_{int}. \end{aligned} \tag{3.3}$$

The labels ext and int indicate the outer and the inner side of the grain surface, respectively.

In order to solve Eqs. (3.2) and (3.3), we first expand the angular dependence of the mean field u into Legendre polynomials P_l:

$$\langle u \rangle = u_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} = u_0 e^{-i\omega t} \sum_{l=0}^{\infty} i^l (2l+1) \sqrt{\frac{\pi}{2}} \frac{J_{l+\frac{1}{2}}(kr)}{\sqrt{kr}} P_l(\zeta),$$

where

$$\zeta = \frac{\mathbf{k}\cdot\mathbf{r}}{kr} = \cos(\sphericalangle \mathbf{k}, \mathbf{r}). \tag{3.4}$$

The Bessel functions $J_{l+\frac{1}{2}}(x)$ satisfy the equations

$$\frac{d^2J}{dx^2}+\frac{1}{x}\frac{dJ_\nu}{dx}+\left(1-\frac{\nu^2}{x^2}\right)J_\nu=0. \tag{3.5}$$

Performing for u_i an analogous expansion into a series of Legendre polynomials $P_l(\varrho)$ and inserting it into Eqs. (3.2), we obtain the following solution:

$$\begin{aligned} u_i &= u_0 e^{-i\omega t}\sum_{l=0}^{\infty} A_l^i \frac{H^{(1)}_{l+\frac{1}{2}}(kr)}{\sqrt{kr}} P_l(\zeta) \qquad (r > R_i), \\ \langle u\rangle + u_i &= u_0 e^{-i\omega t}\sum_{l=0}^{\infty} B_l^i \frac{J^{(1)}_{l+\frac{1}{2}}(k_i r)}{\sqrt{kr}} P_l(\zeta) \qquad (r < R_i), \end{aligned} \tag{3.6}$$

where $k_i \equiv \omega\sqrt{\varrho_i/\varepsilon_i}$ is the wave number inside the i-th grain. The $H^{(1)}_{l+\frac{1}{2}}(x)$ are Hankel functions satisfying the same equations (3.5) as $J_{l+\frac{1}{2}}(x)$. They describe waves spreading out from the center and tending to zero at large distances. Within the grain, the Bessel functions $J_{l+\frac{1}{2}}$ have to be used in order to obtain a field u_i which remains finite at the origin.

The coefficients A_l^i, B_l^i can be determined from the boundary conditions (3.3). They are

$$\begin{aligned} A_l^i &= i^l(2l+1)\sqrt{(\pi/2)}\ (b_l^i-1)J_{l+\frac{1}{2}}(kR_i)/H^{(1)}_{l+\frac{1}{2}}(kR_i), \\ B_l^i &= i^l(2l+1)\sqrt{(\pi/2)}\ b_l^i J_{l+\frac{1}{2}}(kR_i)/J_{l+\frac{1}{2}}(k_i R_i), \end{aligned} \tag{3.7}$$

where the abbreviations

$$b_l^i = \frac{\varepsilon kR_i([J,l,kR_i]-[H,l,kR_i])}{(\varepsilon-\varepsilon_i)l+\varepsilon_i k_i R_i[J,l,k_iR_i]-\varepsilon kR_i[H,l,kR_i]}, \tag{3.7'}$$

$$[J,l,x]\equiv J_{l+\frac{3}{2}}(x)/J_{l+\frac{1}{2}}(x), \qquad [H,l,x]\equiv H^{(1)}_{l+\frac{3}{2}}(x)/H^{(1)}_{l+\frac{1}{2}}(x)$$

are introduced.

Now we have to insert the solutions (3.6) and (3.7) into Eqs. (2.14) and to carry out the ensemble average. Generally, the ensemble can be described by a probability function for the positions $\mathbf{r}_i$ of the grains, their material parameters ϱ_i, ε_i, and their sizes R_i. But, because of statistical homogeneity, all the positions $\mathbf{r}_i$ of the grains have equal probability independently of the values of the other stochastic parameters. Thus, as discussed in I, the ensemble average in Eqs. (2.14) reduces to the form

$$\begin{aligned} &\Big\langle\sum_i(\varrho_i-\varrho)\Theta_i(\langle u\rangle+u_i)\Big\rangle = ne^{i\mathbf{k}\cdot\mathbf{r}}\sum_\alpha p_\alpha(\varrho_\alpha-\varrho)\int\limits_{r'<R_\alpha} d\mathbf{r}' e^{-i\mathbf{k}\cdot\mathbf{r}'}\{\langle u(\mathbf{r}')\rangle+u_\alpha(\mathbf{r}')\}=0, \\ &\Big\langle\sum_i(\varepsilon_i-\varepsilon)\Theta_i\frac{\partial}{\partial\mathbf{r}}(\langle u\rangle+u_i)\Big\rangle \\ &\qquad = ne^{i\mathbf{k}\cdot\mathbf{r}}\sum_\alpha p_\alpha(\varepsilon_\alpha-\varepsilon)\int\limits_{r'<R_\alpha} d\mathbf{r}' e^{-i\mathbf{k}\cdot\mathbf{r}'}\frac{\partial}{\partial\mathbf{r}'}\{\langle u(\mathbf{r}')\rangle+u_\alpha(\mathbf{r}')\}=0, \end{aligned} \tag{3.8}$$

where n is the number of grains per unit volume. The sum goes over the kinds α of the grains characterized by certain values ϱ_α, ε_α, R_α of the stochastic parameters, and p_α denotes the probability for an arbitrary grain to be of kind α.

The integrals over the functions (3.6) which are contained in Eqs. (3.8) are evaluated in the Appendix. Finally, inserting the results there obtained into Eqs. (3.8) gives

$$\sum_\alpha v_\alpha \sum_{l=0}^{\infty} (2l+1) b_l^\alpha \frac{\varrho_\alpha - \varrho}{(kR_\alpha)^2 - (k_\alpha R_\alpha)^2} \left(\frac{J_{l+\frac{1}{2}}(kR_\alpha)}{\sqrt{kR_\alpha}} \right)^2 \times \{kR_\alpha [J, l, kR_\alpha] - k_\alpha R_\alpha [J, l, k_\alpha R_\alpha]\} = 0,$$

(3.9)

$$\sum_\alpha v_\alpha \sum_{l=0}^{\infty} (2l+1) b_l^\alpha \frac{\varepsilon_\alpha - \varepsilon}{(kR_\alpha)^2 - (k_\alpha R_\alpha)^2} \cdot \frac{k_\alpha}{k} \cdot \left(\frac{J_{l+\frac{1}{2}}(kR_\alpha)}{\sqrt{kR_\alpha}} \right)^2 \times \left\{ k_\alpha R_\alpha [J, l, kR_\alpha] - kR_\alpha [J, l, k_\alpha R_\alpha] + \frac{k^2 - k_\alpha^2}{kk_\alpha} l \right\} = 0,$$

where $v_\alpha = np_\alpha 4\pi R_\alpha^3/3$ means the relative volume part of the grains of kind α. We have to keep in mind that Eqs. (3.9) on $\varrho(\omega, \mathbf{k})$ and $\varepsilon(\omega, \mathbf{k})$ are valid only for the frequencies ω and wave vectors $\mathbf{k}$ which are connected through the dispersion relation (3.1). Since with the aid of this relation ω can be expressed as a function of $\mathbf{k}$, the effective parameters ϱ and ε for free waves are considered here as functions of $\mathbf{k}$ only.

A general analytic solution of Eqs. (3.7) being impossible, we restrict ourselves to long wavelengths:

$$kR_\alpha \ll 1, \quad k_\alpha R_\alpha \ll 1. \tag{3.10}$$

Let us notice that shorter waves exhibit a strong damping so that their dispersion relation is of little interest. In the limiting case of vanishing frequency $\omega \to 0$ and infinite wavelength $k \to 0$, Eqs. (3.9) give for $\varrho^0 \equiv \varrho(\mathbf{k} = 0)$, $\varepsilon^0 \equiv \varepsilon(\mathbf{k} = 0)$:

$$\sum_\alpha v_\alpha (\varrho_\alpha - \varrho^0) = 0 \to \varrho^0 = \sum_\alpha v_\alpha \varrho_\alpha = \langle \varrho_{st} \rangle,$$

$$\sum_\alpha v_\alpha \frac{\varepsilon_\alpha - \varepsilon^0}{2\varepsilon^0 + \varepsilon_\alpha} = 0. \tag{3.11}$$

Thus, the effective "density" ϱ^0 turns out to be equal to the average density, whereas ε^0 coincides with the static result given in Eq. (4.16) of I. This result is in accordance with [3, 4, 6].

For long, but finite wavelengths, the effective material properties $\varrho(\mathbf{k})$ and $\varepsilon(\mathbf{k})$ as well as the dispersion relation may be expanded with respect to $\mathbf{k}$:

$$\begin{aligned} \varrho(\mathbf{k}) &= \varrho^0 + \varrho^{(2)} k^2 + \varrho^{(3)} k^3 + \ldots, \\ \varepsilon(\mathbf{k}) &= \varepsilon^0 + \varepsilon^{(2)} k^2 + \varepsilon^{(3)} k^3 + \ldots, \\ \omega(\mathbf{k}) &= \sqrt{\frac{\varepsilon(\mathbf{k})}{\varrho(\mathbf{k})}}\, k = c^0 k (1 - \beta' k^2 - i\varkappa' k^3 + \ldots), \\ c^0 &= \sqrt{\frac{\varepsilon^0}{\varrho^0}}, \quad \beta' = \frac{1}{2}\left(\frac{\varrho^2}{\varrho^0} - \frac{\varepsilon^2}{\varepsilon^0} \right), \quad \varkappa' = \frac{1}{2i}\left(\frac{\varrho^{(3)}}{\varrho^0} - \frac{\varepsilon^{(3)}}{\varepsilon^0} \right). \end{aligned} \tag{3.12}$$

The coefficients of these series are obtained from Eqs. (3.9) as

$$\varrho^{(2)} = -\frac{1}{15}\sum_\alpha v_\alpha R_\alpha^2(\varrho_\alpha-\varrho^0)\left[\frac{9(\varepsilon^0+\varepsilon_\alpha)}{2\varepsilon^0+\varepsilon_\alpha} - \frac{\varrho_\alpha}{\varrho^0}\left(5+\frac{\varepsilon^0}{\varepsilon_\alpha}\right)\right],$$

$$\varrho^{(3)} = \frac{i}{3}\sum_\alpha v_\alpha R_\alpha^3 \frac{(\varrho_\alpha-\varrho^0)^2}{\varrho^0},$$

(3.13)

$$\varepsilon^{(2)} = \frac{1}{15}\sum_\alpha v_\alpha R_\alpha^2(\varepsilon_\alpha-\varepsilon^0)\frac{\dfrac{\varepsilon^0}{\varepsilon_\alpha}+\dfrac{10\varepsilon^0}{3\varepsilon^0+2\varepsilon_\alpha}-\dfrac{27\varepsilon^0\varepsilon_\alpha}{(2\varepsilon^0+\varepsilon_\alpha)^2}+\dfrac{\varrho_\alpha-\varrho^0}{\varrho_0}\left[\dfrac{\varepsilon^0}{\varepsilon_\alpha}+\left(\dfrac{3\varepsilon^0}{2\varepsilon^0+\varepsilon_\alpha}\right)^2\right]}{1-2\displaystyle\sum_\alpha v_\alpha\left(\frac{\varepsilon_\alpha-\varepsilon^0}{2\varepsilon^0+\varepsilon_\alpha}\right)^2},$$

$$\varepsilon^{(3)} = -i\varepsilon^0\frac{\displaystyle\sum_\alpha v_\alpha R_\alpha^3\left(\frac{\varepsilon_\alpha-\varepsilon^0}{2\varepsilon^0+\varepsilon_\alpha}\right)^2}{1-2\displaystyle\sum_\alpha v_\alpha\left(\frac{\varepsilon_\alpha-\varepsilon^0}{2\varepsilon^0+\varepsilon_\alpha}\right)^2}.$$

The coefficients β' and $\varkappa'$ turn out to be real. Thus, β' describes the dependence of the phase and group velocities on the wave vector, i.e. the dispersion

$$\frac{\Delta c_{ph}}{c^0} \equiv \frac{c_{ph}(k)-c^0}{c^0} = -\beta' k^2+\ldots,$$

(3.14)

$$\frac{\Delta c_{gr}}{c^0} \equiv \frac{c_{gr}(k)-c^0}{c^0} = -3\beta' k^2+\ldots,$$

whilst $\varkappa'$ represents the damping of the wave due to scattering processes at the heterogeneities of the material. It consists of two parts $\varkappa'_\varrho$ and $\varkappa'_\varepsilon$, the first of which depends only on ϱ_{st}, whereas the second is a functional of ε_{st} only. This feature of the result is not due to the approximation, but can be proved generally by investigating an arbitrary order of the perturbation series.

From a general reasoning given in the Appendix of I, it follows that the above results (3.12) to (3.14) of the self-consistent approach agree with a perturbational treatment up to third order of the fluctuations of the material parameters.

The damping constant $\varkappa'$ given in the result (3.13) differs from that which would be obtained by simply considering the scattering losses at all grains imbedded in an effective medium [11]. The difference lies in the denominator of $\varepsilon^{(3)}$ in Eqs. (3.13), which roughly estimates the influence of the heterogeneity of the material on the scattering properties of a single grain.

4. Results for a two-component mixture

For a two-component mixture consisting of grains of equal size $R_\alpha = R$, the dimensionless coefficients

(4.1) $$\beta \equiv \beta'/R^2, \quad \varkappa \equiv \varkappa'/R^3$$

have been calculated numerically. The results are drawn in Figs. 1 to 4. Figures 1 and 2 show the two parts of the damping coefficient $\varkappa_\varrho$ and $\varkappa_\varepsilon$ as functions of the ratios ϱ_2/ϱ_1 and $\varepsilon_2/\varepsilon_1$, respectively, where ϱ_α and ε_α now denote the material properties of the α-th component. Different curves belong to different mixture ratios characterized by the volume fraction of the second component $v_2 = 1-v_1$. With increasing heterogeneity $\varrho_2/\varrho_1 \to \infty$, $\varepsilon_2/\varepsilon_1 \to \infty$ the damping tends to finite asymptotic values. The contribution $\varkappa_\varepsilon$ takes maximum values for a mixture ratio corresponding to the percolation threshold [5, 7] $v_2 = v_c = 1/3$. In this case $\varkappa_\varepsilon$ even diverges for heterogeneities going to infinity $\varepsilon_2/\varepsilon_1 \to \infty$. This behaviour is physically reasonable. It results from the fact that near the percolation threshold of a strongly heterogeneous material the fluctuations of the fields

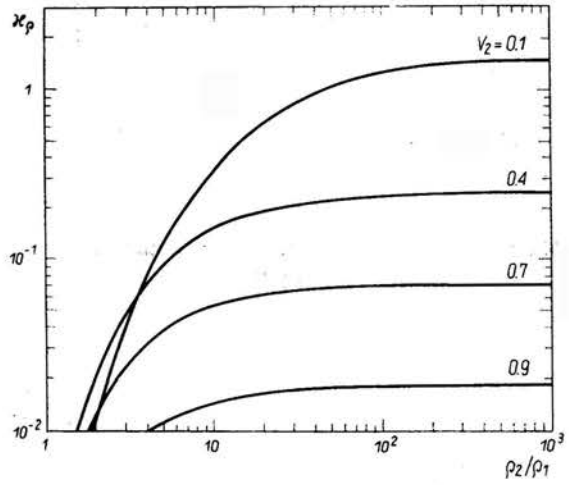

FIG. 1. Damping coefficient $\varkappa_\varrho$ of a two-component mixture.

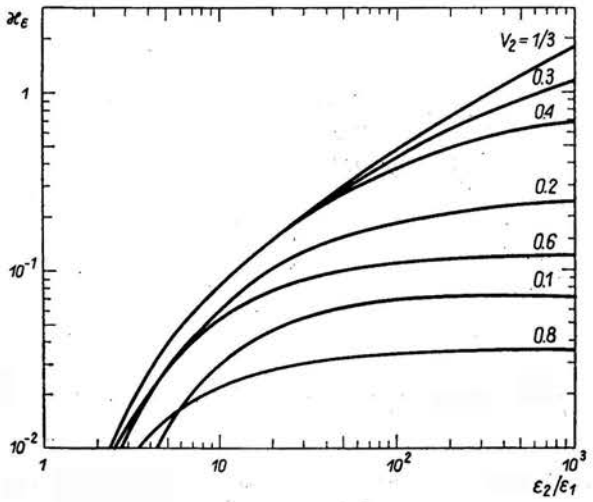

FIG. 2. Damping coefficient $\varkappa_\varepsilon$ of a two-component mixture.

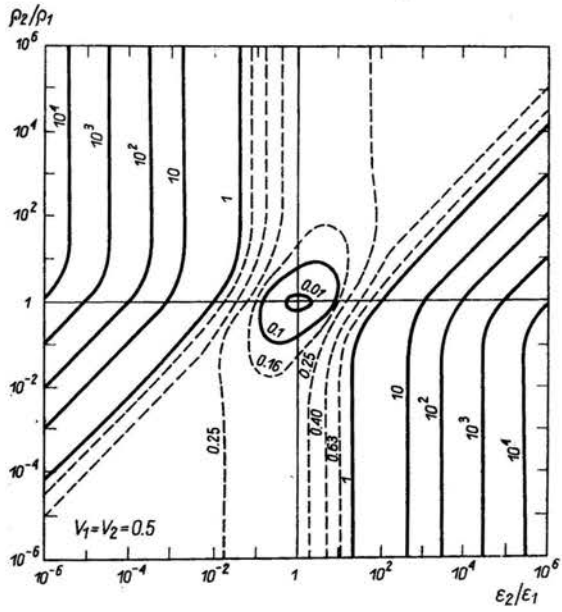

FIG. 3. Dispersion coefficient β of a two-component mixture of mixture ratio 0.5:0.5. Lines of constant β are plotted.

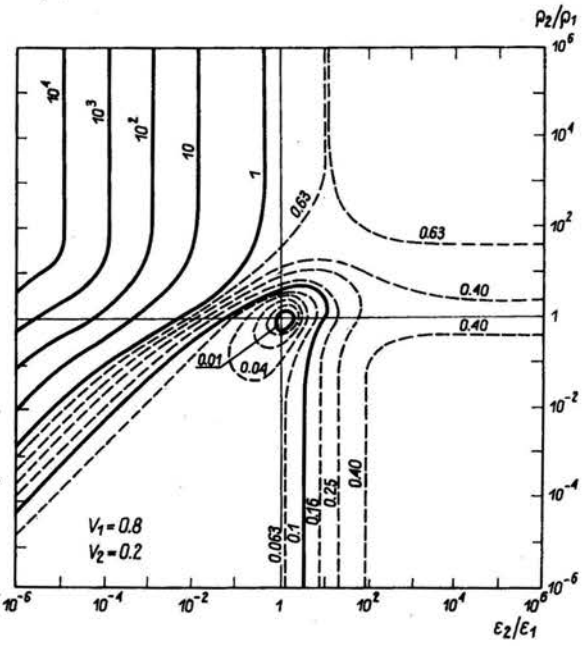

FIG. 4. Dispersion coefficient β of a two-component mixture of mixture ratio 0.2:0.8. Lines of constant β are plotted.

become very important. They give rise to a rapid energy transfer from the incident mean wave to incoherent scattered waves and, therefore, to a strong damping of the mean wave.

The dispersion β does not divide into parts depending on ϱ_{st} and ε_{st}, respectively. Therefore, in Figs. 3 and 4, β is represented as a function of both the heterogeneity parameters ϱ_2/ϱ_1 and $\varepsilon_2/\varepsilon_1$. On the plane of these coordinates, the curves of the constant β are plotted. Two typical mixture ratios 0.5:0.5 and 0.2:0.8 have been chosen. The most striking qualitative feature of these results is the divergence of β ocurring for strong heterogeneities $\varepsilon_2/\varepsilon_1 \to 0, \infty$ in materials whose mixture ratio lies above the percolation threshold (Fig. 3 and the left hand side of Fig. 4). Consequently, in such mixtures dispersion can be observed for wavelengths which largely exceed the characteristic length scale R of the microstructure.

The special case of porous media (where $\varrho_1 = 0$, $\varepsilon_1 = 0$ for the pores) results for $\varrho_2/\varrho_1 \gg \varepsilon_2/\varepsilon_1 \to \infty$. This limiting case, of course, shows a finite dispersion β which is represented in Fig. 3 by the region of the nearly constant β in the upper right hand corner (and, analogously, in the lower left hand corners of Figs. 3 and 4).

5. Concluding remarks

The aim of this paper has been to study the properties of mean wave propagation in a strongly heterogeneous material. The behaviour of the mean wave is mainly described by the effective dispersion relation which yields the velocity and the damping of the wave as functions of frequency.

In order to calculate the effective parameters from information about the heterogeneous structure of the material, a self-consistent method has been used which also permits to treat strongly heterogeneous media. As has been shown in the Appendix of I, in the limit of weak heterogeneity the method coincides with a perturbation treatment up to third order of the fluctuations of the material parameters.

For simplicity, the approach has been applied to a scalar model wave equation and to simple geometrical structures. It is to be hoped that the qualitative features of the results obtained above have general validity. But it would be desirable to treat in a similar manner more realistic cases. Especially, elastic instead of scalar waves and more general grain shapes are of interest for it is well known that the effective behaviour of strongly heterogeneous materials depends very sensitively on their morphology.

In principle, the approach pushed forward in the present paper can be extended to more complicated cases. But, unfortunately, attempts to do this encounter considerable computational difficulties. Thus, in order to achieve further progress in this area, it is perhaps inevitable to restrict oneself to special materials and to apply numerical procedures.

Appendix

Inserting the fields (3.6) into the first equation of the set (3.8) leads to the integrals

$$\text{(A.1)} \qquad \int d\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} \Theta_l \frac{J_{l+\frac{1}{2}}(k_l r)}{\sqrt{k_l r}} P_l(\zeta) = \int_0^{R_l} dr r^2 \int d\Omega e^{-i\mathbf{k}\cdot\mathbf{r}} \frac{J_{l+\frac{1}{2}}(k_i r)}{\sqrt{(k_i r)}} p_l(\zeta).$$

With the aid of the expansion (3.4), they are transformed into

$$\text{(A.2)}\quad \sum_{m=0}^{\infty}(-i)^m(2m+1)\sqrt{\frac{\pi}{2}}\int_0^{R_l} drr^2\,\frac{J_{l+\frac{1}{2}}(k_l r)J_{m+\frac{1}{2}}(kr)}{\sqrt{kk_l r^2}}\int d\Omega P_l(\zeta)P_m(\zeta)$$

$$= (-i)^l 4\pi\sqrt{\frac{\pi}{2}}\int_0^{R_l} drr^2\,\frac{J_{l+\frac{1}{2}}(kr)J_{l+\frac{1}{2}}(k_l r)}{\sqrt{kk_l r^2}}$$

$$= (-i)^l 4\pi\sqrt{\frac{\pi}{2}}\,\frac{J_{l+\frac{1}{2}}(kR_l)}{\sqrt{kR_l}}\,\frac{J_{l+\frac{1}{2}}(k_l R_l)}{\sqrt{k_l R_l}}\,\frac{kR_l[J,l,kR_l]-k_l R_l[J,l,kR_l]}{(kR_l)^2-(k_l R_l)^2}\,R_l^3.$$

On carrying out the integration over the space angle Ω, the orthogonality of Legendre's polynomials P_l has been taken into account. Further, suitable integral formulae for the Bessel functions [8] have been used.

The second equation of the set (3.8) contains the integral

$$\text{(A.3)}\quad \int d\mathbf{r}e^{-i\mathbf{k}\cdot\mathbf{r}}\Theta_l\frac{\partial}{\partial\mathbf{r}}(u_l+\langle u\rangle) = i\mathbf{k}\int d\mathbf{r}e^{-i\mathbf{k}\cdot\mathbf{r}}\Theta_l(u_l+\langle u\rangle)+\oiint_{r=R_l} d\mathbf{S}e^{-i\mathbf{k}\cdot\mathbf{r}}(u_l+\langle u\rangle),$$

which, by integration by parts, has been transformed into two contributions, the first of which coincides with the volume integral discussed just now.

For reason of symmetry, the surface integral in Eq. (A.3) must be parallel to the vector $\mathbf{k}$ so that it can be rewritten as

$$\text{(A.4)}\quad \oiint_{r=R_l} d\mathbf{S}e^{-i\mathbf{k}\cdot\mathbf{r}}(\langle u\rangle+u_l) = \frac{\mathbf{k}}{k}\oiint_{r=R_l} dS\frac{\mathbf{kr}}{kr}e^{-i\mathbf{k}\cdot\mathbf{r}}(\langle u\rangle+u_l)$$

$$= i\frac{\mathbf{k}}{k}R_l^2\oiint d\Omega(\langle u\rangle+u_l)\frac{\partial}{\partial(kr)}e^{-i\mathbf{k}\cdot\mathbf{r}}|_{r=R_l}.$$

Inserting again the expansions (3.4) and (3.6) and integrating over the space angle lead to

$$\text{(A.5)}\quad \oiint d\mathbf{S}e^{-i\mathbf{k}\cdot\mathbf{r}}(\langle u\rangle+u_l)$$

$$= i\frac{\mathbf{k}}{k}4\pi\sqrt{\frac{\pi}{2}}\,u_0R_l^2\sum_l(-i)^l B_l^i\frac{J_{l+\frac{1}{2}}(k_lR_l)}{\sqrt{k_lR_l}}\frac{\partial}{\partial(kR_l)}\frac{J_{l+\frac{1}{2}}(kR_l)}{\sqrt{kR_l}}$$

$$= i\frac{\mathbf{k}}{k}4\pi\sqrt{\frac{\pi}{2}}\,u_0R_l^2\sum_l(-i)^l B_l^i\frac{J_{l+\frac{1}{2}}(k_lR_l)}{\sqrt{k_lR_l}}\frac{J_{l+\frac{1}{2}}(kR_l)}{\sqrt{kR_l}}\{l-kR_l[J,l,kR_l]\}.$$

In the last expression, an abbreviation introduced in Eq. (3.7′) is used.

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