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On the formation of simplified theories of solid mechanics

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BY THE SIMPLIFIED theories of mechanics we usually mean the theories in which all basic unknowns 1) either depend on only one or two material coordinates, 2) are only time-depedent functions or 3) constitute finite sets of numbers. Thus, to the simplified theories of solid mechanics belong all theories of structural mechanics (shell, plate or rod theories), mechanics of rigid and discretized bodies, finite element approaches, the approximate methods of the Galerkin type, etc. We are to show that all simplified theories of solid mechanics are special cases of the field theory based on the concept of abstract constraints, [1]. Such any approach gives a new interpretation of the well-known theories of structural mechanics and approximative methods as well as enable us to formulate new schemes of describing different problems of mechnics.

Uproszczonymi teoriami w mechanice nazywamy zwykle teorie, w których wszystkie podstawowe niewiadome funkcje albo zależą od tylko jednej bądź dwóch współrzędnych materialnych, albo są funkcjami tylko czasu lub sprowadzają się do zbiorów liczb. Takimi teoriami są teorie dźwigarów powierzchniowych, ciał sztywnych lub dyskretyzowanych, podejścia oparte na metodzie elmentów skończonych oraz innych metodach przybliżonych. W pracy pokazano, że wszystkie teorie uproszczone są przypadkami szczególnymi jednej teorii otrzymanej z mechaniki ciała stałego przy wykorzystaniu więzów abstrakcyjnych, [1]. Przedstawione podejście umożliwia nową interpretację tak znanych teorii mechaniki konstrukcji jak i różnych metod przybliżonych oraz ułatwia formułowanie nowych sposobów opisu różnych zagadnień mechaniki.

Упрощенными теориями в механике называем обычно теории, в которых все основные неизвестные функции или зависят только от одной или двух материальных координат, или являются функциями только времени, или сводятся к множествам чисел. Такими теориями являются теории поверхностных балок, жестких или дискретизированных тел, подходы, опирающиеся на метод конечных элементов и на другие приближенные методы. В работе показане, что все упрощенные теории являются частными случаями одной теории, полученной из механики твердого тела при использовании абстрактных связей [1]. Представленный подход дает возможность новой интерпретации так известных теорий механики конструкций, как и разных приближенных методов, а также облегчает формулировку новых способов описания разных вопросов механики.

1. Abstract D-constraints for operators

THE CONCEPT of constraints which up to now has been applied almost exclusively in mechanics or thermomechanics, has a more general sense. In this section we shall introduce the concept of constraints independently of any problem of theoretical physics.

Let A be the known mapping with the domain $D(A)$ in the linear space X and with the range $R(A)$ in the linear space Y: $A(\mathbf{x}) = \mathbf{y}$.

DEFINITION 1. *The relation $D_A \subset D(A)$, where D_A is the known non-empty subset of X, will be called the D-constraints relation for A. If $A(D_A)$ is a proper subset of $R(A)$, then the D-constraints relation for A will be called strong, otherwise it will be called weak. If $D_A = D(A)$, then it will be called trivial.*

If the D-constraints relation is strong, then the right hand side of $A(\mathbf{x}) = \mathbf{y}$, provided that only the elements $\mathbf{x}$ belonging to D_A are taken into account, is restricted by the condition $\mathbf{y} \in A(D_A)$, where $R(A) \setminus A(D_A) \neq \phi$. Now assume that in the problem under consideration we have to deal with all the elements $\mathbf{y}$, $\mathbf{y} \in R(A)$, and with the elements $\mathbf{x}$, $\mathbf{x} \in D_A$ (the latter will be called admissible by the constraints for A). To reconcile these two opposite requirements we must modify the mapping $A(\mathbf{x}) = \mathbf{y}$ to the new form of a certain relation between the elements of D_A and these of $R(A)$. To this aid we shall introduce the concept of the realization of D-constraints.

DEFINITION 2. *The multi-function*: r_D: $D_A \ni \mathbf{x} \to Y_{\mathbf{x}} \subset Y$, $Y_{\mathbf{x}} \neq \phi$, *such that for every* $\mathbf{y}$, $\mathbf{y} \in R(A)$, *there exists at least one pair* $(\mathbf{x}, \mathbf{r}) \in D_A \times Y_{\mathbf{x}}$, *satisfying the relation* $A(\mathbf{x}) = \mathbf{y} + \mathbf{r}$ *and such that* $\mathbf{y} \in A(D_A)$ *implies* $\mathbf{r} = \mathbf{0}$, *will be called the realization of D-constraints for A.*

We can easily prove that the following statement holds:

PROPOSITION. For every $\mathbf{x} \in D_A$ there is $\mathbf{0} \in Y_{\mathbf{x}}$. If $D_A = \{\mathbf{x}_0\}$, then $Y_{\mathbf{x}_0} = Y$. If the D-constraints relation for A is weak, then $Y_{\mathbf{x}} = \{\mathbf{0}\}$ for every $\mathbf{x} \in D_A$.

COROLLARY. The problems with the weak constraints relation are governed by $A(\mathbf{x}) = \mathbf{y}$ and $\mathbf{x} \in D_A \subset D(A)$.

Now suppose that $X = X(\Omega)$, $Y = Y(\Omega)$ are linear spaces of the vector-valued functions defined on the manifold Ω and that $A(\mathbf{x}) = \mathbf{y}$ describes a certain field theory of physics, (the field equation of this theory). Assuming that a certain class of problems of this theory requires that only the fields $\mathbf{x}$, $\mathbf{x} \in D_A$, be taken into account, we obtain what will be called the D-constrained field theory governed by

$$A(\mathbf{x}) = \mathbf{y} + \mathbf{r}, \quad \mathbf{x} \in D_A, \quad \mathbf{r} \in Y_{\mathbf{x}}, \tag{1.1}$$

where $D_A \subset D(A)$, $\mathbf{y} \in R(A) \subset Y(\Omega)$ and $Y_{\mathbf{x}} \subset Y(\Omega)$ for every $\mathbf{x} \in D_A$. Every field $\mathbf{r}$, $\mathbf{r} \in Y_{\mathbf{x}}$, will be called the reaction which can maintain the admissible field $\mathbf{x}$, $\mathbf{x} \in D_A$.

Independently of the concept of D-constraints for A we can define R-constraints for A (introducing the inclusion $R_A \subset R(A)$ and the suitable realization of R-constraints), D-R-constraints for A and constraints for an arbitrary binary relation, cf. [1].

2. Simplified field theories of mechanics

Let B be the continuous body, $\varkappa_t$ its time-dependent configuration and $\varkappa_R$ its reference configuration. We denote by $\mathbf{u}(\mathbf{\theta}, t)$, $\mathbf{\theta} \in \overline{\varkappa_R(B)}$, $t \in I \subset R$, where I is the known time interval, the displacement field from the configuration $\varkappa_t$. Denoting by $\overline{\Omega} \equiv \overline{\varkappa_R(B)} \times I$ the domain of the definition of $\mathbf{u}$, let us assume that the smooth bijection $\overline{\Omega} \to \overline{\Pi} \times \overline{\Delta}$ is given where, Π is either the n-th dimensional differentiable manifold, $n \leqslant 3$, or a one point set. Moreover, let us take into account the body B for which the equations of motion, the kinetic boundary conditions, the constitutive equations and the equations defining the external loads in terms of the motion lead to the system of three equation in Ω and three boundary conditions almost everywhere on $\partial\Omega \setminus \partial I$, for the field $\mathbf{u}(\mathbf{\theta}, t)$. This system of six equations can be written down in the form

$$A(\mathbf{u}) = \mathbf{y}, \tag{2.1}$$

where $\mathbf{y}(\mathbf{\theta}, t)$ stands for three volume forces and three boundary tractions which are independent of the motion of B. Equation (2.1) includes the governing equations of bodies made of an arbitrary simple material (then we can put $\varkappa_t = \varkappa_R$ for every $t \in I$) as well as the equations of the incremental theories of elastic-plastic bodies (then $\mathbf{u}(\boldsymbol{\theta}, t)$ is the small increment of the deformation function).

Now the problem arises how to formulate the simplified theory of the problems governed by Eq. (2.1) in which all unknown functions are defined exclusively on the manifold $\overline{\Pi}$. This problem has an exact solution (which is of course not unique) if we use the concept of the constrained field theory, which was developed in Sect. 1. We shall see that the simplified theory we are going to formulate is the D-constrained field theory for the operator A, with the special case of the D-constraints relation and a special realization of these constraints. To this end we shall denote by $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ the points of $\overline{\Pi}$ and $\overline{\Delta}$, respectively, (if $\Pi = \{\boldsymbol{\xi}_0\}$ then $\boldsymbol{\xi} \equiv \boldsymbol{\xi}_0$) and we assume that the following objects are known:

1. The Banach spaces $X(\overline{\Pi})$, $Y(\overline{\Pi})$ of the vector functions $\boldsymbol{\psi} \equiv (\psi_a(\boldsymbol{\xi}))$, $a = 1, \ldots, n$, and $\bar{\mathbf{y}} = (\bar{y}_A(\boldsymbol{\xi}))$, $A = 1, \ldots, N$, respectively, $\boldsymbol{\xi} \in \overline{\Pi}$.
2. The smooth mapping $\boldsymbol{\Phi}$: $X(\overline{\Pi}) \supset D(\boldsymbol{\Phi}) \ni \boldsymbol{\psi} \to \mathbf{u} \in D(A) \subset X(\overline{\Omega})$.
3. The multi-function $D(\boldsymbol{\Phi}) \ni \boldsymbol{\psi} \to Y_{\boldsymbol{\psi}} \subset Y(\overline{\Pi})$.
4. The Banach space $Y(\Delta)$ and the system of N linear independent functionals $f_A^* \in Y^*(\overline{\Delta})$, $A = 1, \ldots, N$.

Now let the D-constraints relation for the operator A be assumed in the form

$$D(A) \supset D_A := \{\mathbf{u} | \mathbf{u} = \boldsymbol{\Phi}(\boldsymbol{\psi}) \quad \text{for some } \boldsymbol{\psi} \in D(\boldsymbol{\Phi}) \subset X(\overline{\Pi})\}, \tag{2.2}$$

and the realization of D-constraints for A be given by

$$Y_{\mathbf{u}} := \{\mathbf{r} | \langle \mathbf{r}_{\boldsymbol{\xi}}, f_A^* \rangle = \bar{r}_A, \quad A = 1, \ldots, N, \quad \text{for some} \quad \bar{\mathbf{r}} = (\bar{r}_A) \in Y_{\boldsymbol{\psi}}\}, \tag{2.3}$$

where we have denoted $\mathbf{y}_{\boldsymbol{\xi}} \equiv \mathbf{y} | \{\boldsymbol{\xi}\} \times \overline{\Delta}$, for an arbitrary $\mathbf{y} \in Y(\overline{\Omega})$, $\boldsymbol{\xi} \in \overline{\Pi}$. Let us also denote

$$\begin{aligned} \bar{A}_A(\boldsymbol{\psi}) &\equiv \langle (A \circ \boldsymbol{\Phi}(\boldsymbol{\psi}))_{\boldsymbol{\xi}}, f_A^* \rangle, \quad \bar{A} \equiv (\bar{A}_A), \\ \bar{y}_A &\equiv \langle \mathbf{y}_{\boldsymbol{\xi}}, f_A^* \rangle, \quad \boldsymbol{\xi} \in \Pi, \quad A = 1, \ldots, N. \end{aligned} \tag{2.4}$$

Thus we obtain finally the following system of relations:

$$\begin{aligned} \bar{A}_A(\boldsymbol{\psi}) &= \bar{y}_A + \bar{r}_A, \quad A = 1, \ldots, N, \\ \boldsymbol{\psi} \equiv (\psi_a) \in D(\Phi) \subset X(\overline{\Pi}), &\quad \bar{\mathbf{r}} = (\bar{r}_A) \in Y_{\boldsymbol{\psi}} \subset Y(\overline{\Pi}), \end{aligned} \tag{2.5}$$

in which for every $\bar{\mathbf{y}} \in Y(\overline{\Pi})$ there exists $\boldsymbol{\psi} \in D(\boldsymbol{\Phi})$ and $\bar{\mathbf{r}} \in Y_{\boldsymbol{\psi}}$ such that Eqs. (2.5) hold. Moreover, if $\bar{\mathbf{y}} \in \bar{A}(D(\boldsymbol{\Phi}))$, then $\bar{\mathbf{r}} = \theta$ (cf. Sect. 1). Equations (2.5) represent the D-constrained field theory of mechanics which, at the same time, constitutes the simplified field theory. All fields in Eqs. (2.5) are defined exclusively on the manifold $\overline{\Pi}$, i.e. on a certain sub-manifold of $\overline{\Omega}$ (if Π is one, two or three-dimensional differentiable manifold) or are finite systems of numbers (if $\overline{\Pi}$ is a single point set, $\overline{\Pi} = \{\boldsymbol{\xi}_0\}$). In the latter case Eqs. (2.5) are the algebraic relations for the system of numbers ψ_a, $a = 1, \ldots, n$. We have either $n = N$ (if ψ_a are numbers or only time-dependent functions) or $N = 2n$ in other cases, when we deal with the field equations and the boundary conditions.

If $D(A) = D(\boldsymbol{\Phi})$, then $\bar{\mathbf{r}} = \theta$ and the problem reduces to

$$\bar{A}_A(\boldsymbol{\psi}) = \bar{y}_A, \quad A = 1, \ldots, N, \tag{2.6}$$

which has to have at least one solution for every $\bar{\mathbf{y}} \in Y(\overline{\Pi})$. If $Y(\overline{\Delta})$ is the Hilbert space with the scalar product $(\,|\,)$ and if $\mathbf{\Phi}(\boldsymbol{\psi}) = \sum_{a=1}^{n} \boldsymbol{\alpha}^a \psi_a$, where $\boldsymbol{\psi} = (\psi_a) \in D(\mathbf{\Phi}) \subset X(\overline{\Pi})$ and $\boldsymbol{\alpha}^a \in X(\overline{\Delta})$, then Eq. (2.6) has the form

$$\left(A\left(\sum_{a=1}^{n}\right)\boldsymbol{\alpha}^a \psi_a)_{\boldsymbol{\xi}}|\mathbf{f}_A\right) = (\mathbf{y}_{\boldsymbol{\xi}}|\mathbf{f}_A), \quad \boldsymbol{\xi} \in \overline{\Pi}, \quad A = 1, \dots, N. \tag{2.7}$$

Here $\mathbf{f}_A$ are linear independent elements from $Y(\overline{\Delta})$ (they can also depend on $\boldsymbol{\xi}$, $\boldsymbol{\xi} \in \overline{\Pi}$). If $\overline{\Delta} = \overline{\Omega}$ and $N = n$ (i.e. ψ_a, $a = 1, \dots, n$, are numbers), then putting $\mathbf{f}_A = A(\boldsymbol{\alpha}^A)$ we obtain Eq. (2.7) as the system of equations of the least square method. Moreover, if $X(\overline{\Delta}) = Y(\overline{\Delta})$ and $\boldsymbol{\alpha}^a = \delta^{aA}\mathbf{f}_A$, then Eq. (2.7) constitutes the equations of the Bubnov-Galerkin method. If $\Pi = \Pi_0 \times I$ and Π_0 is the two-dimensional differentiable manifold (one dimensional manifold), then Eqs. (2.5) include all well-known shell theories (rod theories). If $\Pi = I$, then from Eq. (2.7) we obtain the system of n ordinary differential equations representing different theories of the discretized bodies (including finite element and finite difference methods as well as the mechanics of rigid bodies).

In the foregoing analysis the positive integers n, N have been fixed. If $n \to \infty$, but n/N is the fixed number and the objects defined above as 1–4 are known, for every n, then we deal with a certain sequence of the simplified theories of mechanics. It must be stressed that even if Eq. (2.1) has the unique solution for the fixed $\mathbf{y} = \mathbf{y}_0$, then the solutions of Eq. (2.6) for $\bar{y}_A = \langle \mathbf{y}_0, f_A^* \rangle$, $A = 1, \dots, N$, may not exist. Moreover, if for each N such a solution exists and is unique, then the sequence $\mathbf{x}_n = \mathbf{\Phi}_n(\boldsymbol{\psi})$, $\boldsymbol{\psi} = (\psi_1, \dots, \psi_n)$, $n = 1, 2, \dots$, may not be convergent to the solution $\mathbf{x}$ of Eq. (2.1). In nonlinear continuum mechanics all these problems seem to be open.

Now let us return back to Eqs. (2.5) and give some examples of the sets $Y_{\boldsymbol{\psi}}$, $\boldsymbol{\psi} \in \Psi \equiv D(\Phi)$, of reactions to constraints $\Psi \subset D(\overline{A})$.

Let the functional $\bar{y}^0 \in X^*(\overline{\Pi})$ be assigned to every $\bar{\mathbf{y}} \in Y(\overline{\Pi})$ in such a way that $\langle v, \bar{y}^0 \rangle$ is the work of the kinetic field $\bar{\mathbf{y}}$ on an arbitrary kinematic field $\mathbf{v} \in X(\overline{\Pi})$. If $X(\overline{\Pi}) = Y(\overline{\Pi})$, then $\bar{y}^0 = \bar{y}^*$. Moreover, let $C_{\boldsymbol{\psi}}$ be for every $\boldsymbol{\psi} \in \Psi$ the known subset of $X(\overline{\Pi})$. The realization of constraints $\Psi \subset D(A)$ which is given by

$$Y_{\boldsymbol{\psi}} := \{\bar{\mathbf{r}} | \langle \mathbf{h}, \bar{r}^0 \rangle \geqslant 0 \quad \text{for every} \quad \mathbf{h} \in C_{\boldsymbol{\psi}}\} \tag{2.8}$$

will be calle dquasi-ideal; elements of $C_{\boldsymbol{\psi}}$ are said to be the quasi-virtual fields. From Eq. (2.8) and Eqs. (2.5) we obtain the fuctional equation of the simplified theory of mechanics:

$$\langle \mathbf{h}, \overline{A}^0(\boldsymbol{\psi}) \rangle \geqslant \langle \mathbf{h}, \bar{y}^0 \rangle \quad \text{for every} \quad \mathbf{h} \in C_{\boldsymbol{\psi}}, \tag{2.9}$$

where $\bar{\mathbf{y}} \equiv (\bar{y}_A)$, $\overline{A} \equiv (\overline{A}_A(\boldsymbol{\psi}))$ and $\overline{A}^0(\boldsymbol{\psi}) \equiv (\overline{A}(\boldsymbol{\psi}))^0$.

If Ψ is the convex set in $X(\overline{\Pi})$, then assuming $C_{\boldsymbol{\psi}} := \{\mathbf{h} | \mathbf{h} = \overline{\boldsymbol{\psi}} - \boldsymbol{\psi}$ for some $\overline{\boldsymbol{\psi}} \in \Psi\}$ we obtain

$$Y_{\boldsymbol{\psi}} := \{\bar{\mathbf{r}} | \langle \overline{\boldsymbol{\psi}} - \boldsymbol{\psi}, \bar{r}^0 \rangle \geqslant 0 \quad \text{for every } \overline{\boldsymbol{\psi}} \in \Psi\}. \tag{2.10}$$

The realization of constraints given by Eq. (2.10) will be called ideal. If Ψ is the subspace of $X(\overline{\Pi})$, then putting $\overline{\boldsymbol{\psi}} - \boldsymbol{\psi} \in \Psi$ we also have $-\mathbf{h} \in \Psi$ and $Y_{\boldsymbol{\psi}} := \{\bar{r} | \langle \mathbf{h}, r^0 \rangle = 0$ for every $\mathbf{h} \in \Psi\}$.

Now let $B_{\boldsymbol{\psi}}$ be, for every $\boldsymbol{\psi} \in \Psi$, the known linear mapping from $X(\overline{\Pi})$ to a certain Banach space $Z(\overline{\Pi})$. Then the sets $C_{\boldsymbol{\psi}}$ in Eq. (2.8) can be defined by $C_{\boldsymbol{\psi}} = \operatorname{Ker} B_{\boldsymbol{\psi}}$. Moreover, if the constraints relation $\Psi \subset D(A)$ is determined by $\Psi = \operatorname{Ker} M$, M being the differentiable mapping from $X(\overline{\Pi})$ to $Z(\overline{\Pi})$, and $B_{\psi} = M'(\psi)$, then the realization is also said to be ideal, provided that ψ is the regular point of M.

Let $\Lambda(\overline{\Pi})$ be the Banach space of vector functions λ and let L_{ψ} be for every $\psi \in \Psi$ the known linear operator from $\Lambda(\overline{\Pi})$ to $Y(\overline{\Pi})$. If

$$Y_{\psi} := \{\bar{\mathbf{r}} | \bar{\mathbf{r}} = L_{\psi}(\boldsymbol{\lambda}) \quad \text{for some} \quad \boldsymbol{\lambda} \in \Lambda(\overline{\Pi})\}, \tag{2.11}$$

then the realization will be referred to as defined by the constraints functions λ, $\lambda \in \Lambda(\overline{\Pi})$. If $\langle \mathbf{h}, (L_{\psi}(\lambda))^{0} \rangle \equiv 0$ for every $\mathbf{h} \in C_{\psi}$ and if $C_{\psi} = \operatorname{Ker} M'(\psi)$, $\Psi = \operatorname{Ker} M$, then Eq. (2.11) determines the ideal realization of constraints. The example of such realization will be given in Sect. 3.

3. Examples: structural mechanics and mechanics of discretized bodies

By structural mechanics we shall mean such a constrained field theory of mechanics (such simplified theory) in which $\Pi = \Pi_0 \times I$, where Π_0 is one or two-dimensional differentiable manifold (which can be immersed in R^3), and in which the constraints relation for the operator A and the realization of constraints are ideal (cf. below) and local in time. Thus structural mechanics includes the rod and shell theories. In what follows we shall give the governing equations of structural mechanics. To this aid we shall assume that $\overline{\varkappa_R(B)} = \overline{\Pi}_0 \times \overline{\Delta}$, $\Pi = \Pi_0 \times I$, and $\boldsymbol{\eta} \equiv (\theta^K) \in \overline{\Pi}_0$, $\boldsymbol{\zeta} \equiv (\theta^R) \in \overline{\Delta}$, $t \in I$, where either $K = 1, 2$ and $R = 3$ or $K = 1$ and $R = 2, 3$. We shall also use the denotations

$$\oint_{\partial\Delta} (\cdot)\, dl \equiv [\cdot]_{\theta^3 = 0, b} \quad \text{if} \quad \Delta(a, b) \quad \text{and} \quad \oint_{\partial\Delta} (\cdot)\, dl \equiv [\cdot]_{\theta^1 = 0,} \quad \text{if} \quad \Pi_0 = (0, h).$$

The D-constraints relation for the operator A, i.e. the relation (2.2), will be postulated in the form

$$\begin{gathered} u_k(\boldsymbol{\theta}, t) - \Phi_k(\boldsymbol{\theta}, \boldsymbol{\psi}(\boldsymbol{\eta}, t)) = 0, \quad \boldsymbol{\theta} \in \overline{\varkappa_R(B)}, \\ h_\nu(t, \boldsymbol{\eta}, \boldsymbol{\psi}, \nabla\boldsymbol{\psi}) = 0, \quad \nu = 1, \ldots, \overline{N}, \quad \boldsymbol{\eta} \in \Pi_0, \\ R_\pi(t, \boldsymbol{\eta}, \boldsymbol{\psi}) = 0, \quad \pi = 1, \ldots, \overline{P} \leqslant n, \quad \boldsymbol{\eta} \in \partial\Pi_0, \end{gathered} \tag{3.1}$$

where $\Phi_k(\cdot)$, $h_\nu(\cdot)$, $R_\pi(\cdot)$, are known independent differentiable functions, $\det \Phi_{k,\alpha} > 0$, and $\boldsymbol{\psi} = (\psi_a(\eta, t))$, $a = 1, \ldots, n$, are unknown differentiable functions. Putting $\mathbf{r} = (r^k, s^k)$, where r^k are the volume and s^k the boundary reaction to constraints, we shall postulate the realization (2.3) of the D-constraints (3.1) in the form of the condition

$$\begin{gathered} \int_{\Delta} r^k \frac{\partial \Phi_k}{\partial \psi_a} J d\Delta + \oint_{\partial\Delta} s^k \frac{\partial \Phi_k}{\partial \psi_a} j dS = r^a \quad \eta \in \Pi_0, \\ \int_{\Delta} s^k \frac{\partial \Phi_k}{\partial \psi_a} j d\Delta = s^a, \quad \eta \in \partial\Pi_0, \quad \bar{\mathbf{r}} = (r^a, s^a) \in Y_{\psi}, \end{gathered} \tag{3.2}$$

where r^a, s^a, $a = 1, \ldots, n$, stand for r_A, $A = 1, \ldots, 2n$, and $J \equiv dv/dv_R$, $j \equiv ds/ds_R$, where dv, dv_R, ds, ds_R, are elements of $\chi(B, t)$, $\varkappa_R(B)$, $\partial\chi(B, t)$, $\partial\varkappa_R(B)$, respectively, $\boldsymbol{\chi} \equiv \boldsymbol{\varkappa}_t + \mathbf{u}$ being the deformation function. To define the set Y_{ψ} in Eq. (3.2) let us first denote by $\psi^a(\sigma, t)$ the boundary value of $\psi_a(\boldsymbol{\eta}, t)$ on $\partial\Pi_0$ (only if $\Pi_0 \subset R^2$), where $\sigma \in \langle 0, l \rangle$ is the coordinate on $\partial\Pi_0$. Moreover, let $\overline{h}_\varrho$ be the boundary values of these functions $h_\nu(\cdot)$, which at $\boldsymbol{\eta} \in \partial\Pi_0$ can be represented in the form $\overline{h}_\varrho(\boldsymbol{\eta}, \boldsymbol{\psi}, \boldsymbol{\psi}_{,\sigma})$. Let Y_ψ be the set of all $\bar{\mathbf{r}} = (r^a, s^a)$, $a = 1, \ldots, n$, which have the form (cf. Eq. (2.8)):

$$
\begin{aligned}
r^a &= -\lambda^\nu \frac{\partial h_\nu}{\partial \psi_a} + \left(\lambda^\nu \frac{\partial h_\nu}{\partial \psi_{a,K}}\right)_{,K}, \qquad \boldsymbol{\eta} \in \Pi_0,\\
s^a &= -\lambda^\nu \frac{\partial h_\nu}{\partial \psi_{a,K}} n_K + \overline{\mu}^\varrho \frac{\partial \overline{h}_\varrho}{\partial \psi_a} - \left(\overline{\mu}^\varrho \frac{\partial \overline{h}_\varrho}{\partial \psi_{a,\sigma}}\right)_{,\sigma} + \mu^\varkappa \frac{\partial R_\varkappa}{\partial \psi_a}, \qquad \boldsymbol{\eta} \in \partial\Pi_0,
\end{aligned} \tag{3.3}
$$

where $\lambda^\nu(\boldsymbol{\eta}, t)$, $\boldsymbol{\eta} \in \overline{\Pi}_0$, $\overline{\mu}^\varrho(\boldsymbol{\eta}, t)$, $u^\varkappa(\boldsymbol{\eta}, t)$, $\boldsymbol{\eta} \in \partial\overline{\Pi}_0$, are unknown constraints functions and $\mathbf{n} = (n_K)$ is either the unit vector normal to $\partial\Pi_0$ (if $\Pi_0 \subset R^2$) or $n_1 = \pm 1$ (if $\Pi_0 \subset R$). If $\Pi_0 \subset R^2$, then the functions $\overline{\mu}^\varrho$ have to satisfy the extra conditions

$$
\left[\overline{\mu}^\varrho \frac{\partial \overline{h}_\varrho}{\partial \psi_{a,\sigma}}\right] = 0, \tag{3.4}
$$

in all points of $\partial\Pi_0$ in which the function μ^ϱ is uncontinuous (the square bracket denotes the jump of the function).

It can be easily proved that the realization of constraints given by Eqs. (3.2) to (3.4) is ideal, i.e. the work of the reaction forces on an arbitrary virtual displacement field is equal to zero:

$$
\oint_{\partial\chi(B,t)} s^k \delta\chi_k ds + \int_{\chi(B,t)} r^k \delta\chi_k dv = 0
$$

for an arbitrary $\delta\chi_k$ defined by

$$
\delta\chi_k = \delta u_k = \frac{\partial \psi_k}{\partial \psi_a} \delta\psi_a, \qquad \delta\psi_{a,R} = 0, \qquad \boldsymbol{\theta} \in \varkappa_K(B),
$$

$$
\frac{\partial h_\nu}{\theta \psi_a} \delta\psi_a + \frac{\partial h_\nu}{\partial \psi_{a,K}} \delta\psi_{a,K} = 0, \qquad \eta \in \Pi_0, \qquad \frac{\partial R_\varkappa}{\partial \psi_a} \delta\psi_a = 0, \qquad \boldsymbol{\eta} \in \partial\Pi_0.
$$

To obtain Eq. (2.5) we shall assume that $\mathbf{y} = (f^\alpha, p^\alpha)$, $\mathbf{r} = (r^\alpha, s^\alpha)$, and we shall write down the equation $A(\mathbf{u}) = \mathbf{y} + \mathbf{r}$ in the explicit form

$$
\begin{aligned}
\varrho \ddot{\chi}^k \theta^\alpha_{,k} - T^{\alpha\beta}|_\beta - f^\alpha_0 &= f^\alpha + r^\alpha, \qquad \boldsymbol{\theta} \in \varkappa_R(B),\\
T^{\alpha\beta} n_\beta - p^\alpha_0 &= p^\alpha + s^\alpha, \qquad \boldsymbol{\theta} \in \partial\varkappa_R(B),
\end{aligned} \tag{3.5}
$$

where $T^{\alpha\beta} = T^{\alpha\beta}(\mathbf{u})$, $f^\alpha_0 = f^\alpha_0(\mathbf{u})$, $p^\alpha_0 = p^\alpha_0(\mathbf{u})$, are the known functionals of the constitutive type. Substituting $r^k = \chi^k_{,\alpha} r^\alpha$, $s^k = \chi^k_{,\alpha} s^\alpha$, into Eqs. (3.2) and using Eqs. (3.5), we arrive at

$$
\begin{aligned}
&\int_\Delta (\varrho \ddot{\chi}^k - T^{\alpha\beta}|_\beta \chi^k_{,\alpha}) \frac{\partial \Phi_k}{\partial \psi_a} J d\Delta + \oint_{\partial\Delta} T^{\alpha\beta} n_\beta \chi^k_{,\alpha} \frac{\partial \Phi_k}{\partial \psi_a} j dS\\
&\qquad - \left(\int_\Delta f^k_0 \frac{\partial \Phi_k}{\partial \psi_a} J d\Delta + \oint_{\partial\Delta} p^k_0 \frac{\partial \Phi_k}{\partial \psi_a} j dS\right) = r^a, \qquad \boldsymbol{\eta} \in \Pi_0,\\
&\int_\Delta T^{\alpha\beta} n_\beta \chi^k_{,\alpha} \frac{\partial \Phi_k}{\partial \psi_a} j d\Delta - \int_\Delta p^k_0 \frac{\partial \Phi_k}{\partial \psi_a} j d\Delta = s^a, \qquad \boldsymbol{\eta} \in \partial\Pi_0.
\end{aligned} \tag{3.6}
$$

Then, taking into account Eqs. (3.3) and introducing the quantities

$$
\begin{aligned}
&\varrho^{ab} \equiv \int_{\Delta} \varrho_R \frac{\partial \Phi^k}{\partial \psi_a} \frac{\partial \Phi_k}{\partial \psi_b}\, d\Delta, \quad \varrho^{abc} \equiv \int_{\Delta} \varrho_R \frac{\partial^2 \Phi^k}{\partial \psi_b \partial \psi_c} \frac{\partial \Phi_k}{\partial \psi_a}\, d\Delta,\\
&f^a \equiv \int_{\Delta} f_0^k \frac{\partial \Phi_k}{\partial \psi_a} J d\Delta + \oint_{\partial\Delta} p_0^k \frac{\partial \Phi_k}{\partial \psi_a} j dS, \quad w^a \equiv \int_{\Delta} \ddot{\varkappa}_t^k \frac{\partial \Phi_k}{\partial \psi_a} J d\Delta,\\
&H^{aK} \equiv \int_{\Delta} \Phi^k_{,\alpha} T^{\alpha K} \frac{\partial \Phi_k}{\partial \psi_a} J d\Delta, \quad h^a \equiv -\int_{\Delta} \Phi^k_{,\alpha} T^{\alpha\beta} \left(\frac{\partial \Phi_k}{\partial \psi_a}\right)_{,\beta} J d\Delta, \quad \boldsymbol{\eta} \in \overline{\Pi}_0,\\
&p^a \equiv \int_{\Delta} p_0^k \frac{\partial \Phi_k}{\partial \psi_a} j d\Delta, \quad \boldsymbol{\eta} \in \partial \Pi_0,
\end{aligned}
\tag{3.7}
$$

after rather complicated calculations (cf. [2]), we obtain finally

$$
\left(H^{aK} + \lambda^\nu \frac{\partial h_\nu}{\partial \psi_{a,K}}\right)_{,K} + h^a - \lambda^\nu \frac{\partial h_\nu}{\partial \psi_a} + f_0^a = w^a + \varrho^{ab}\ddot{\psi}_b + \varrho^{abc}\dot{\psi}_b\dot{\psi}_c, \quad \boldsymbol{\eta} \in \Pi_0
\tag{3.8}
$$

and

$$
\left(H^{aK} + \lambda^\nu \frac{\partial h_\nu}{\partial \psi_{a,K}}\right) n_K = \overline{\mu}^\rho \frac{\partial \overline{h}_\rho}{\partial \psi_a} - \left(\overline{\mu}^\rho \frac{\partial \overline{h}_\rho}{\partial \psi_{a,\sigma}}\right)_{,\sigma} + \mu^\varkappa \frac{\partial R_\varkappa}{\partial \psi_a} + p_0^a, \quad \boldsymbol{\eta} \in \partial \Pi_0.
\tag{3.9}
$$

The equations of motion (3.8) and the kinetic boundary condition (3.9) constitute the explicit form of Eq. (2.5) and together with Eqs. $(3.1)_{2,3}$ and (3.4) describe the basic system of relations of structural mechanics. Let us observe that by virtue of Eq. $(3.1)_1$, all integrands in Eqs. (3.7) are known functions of the argument $\boldsymbol{\zeta} \in \overline{\Delta}$ (we have tacitly assumed that $\boldsymbol{\varkappa}_t$ is known) and all integrals can be calculated.

If the material the body is made of is hyperelastic, $\mathbf{T} = 2\varrho\partial\sigma/\partial\mathbf{C}$ where $\sigma = \sigma(\boldsymbol{\theta}, \mathbf{C})$ is the strain energy function and $\mathbf{C} = \nabla \boldsymbol{\chi}^T \nabla \boldsymbol{\chi} (\boldsymbol{\chi} = \boldsymbol{\varkappa}_t + \boldsymbol{\Phi}(\cdot)$ in the presence of constraints $(3.1)_1$), then after introducing the functions

$$
\begin{aligned}
&\mathscr{L} \equiv \frac{1}{2} \int_{\Delta} \varrho_R (\dot{\varkappa}_t^k + \dot{\Phi}^k)(\dot{\varkappa}_{tk} + \dot{\Phi}_k) d\Delta - \int_{\Delta} \varrho_R \sigma d\Delta - \lambda^\nu h_\nu = \mathscr{L}(\boldsymbol{\eta}, t, \boldsymbol{\psi}, \nabla\boldsymbol{\psi}, \lambda^\nu),\\
&\delta \equiv \overline{\mu}^\rho \overline{h}_\rho + \mu^\varkappa R_\varkappa = \delta(\boldsymbol{\eta}, \boldsymbol{\psi}, \boldsymbol{\psi}_{,\sigma}, \overline{\mu}^\rho, \mu^\varkappa),
\end{aligned}
\tag{3.10}
$$

we can transform Eqs. (3.8) to the form

$$
\frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \dot{\psi}_a} + \left(\frac{\partial \mathscr{L}}{\partial \psi_{a,K}}\right)_{,K} - \frac{\partial \mathscr{L}}{\partial \psi_a} = f_0^a, \quad \boldsymbol{\eta} \in \Pi_0.
\tag{3.11}
$$

The kinetic boundary conditions (3.9) at the same time will be given by

$$
\left(\frac{\partial \delta}{\partial \psi_{a,\sigma}}\right)_{,\sigma} - \frac{\partial \delta}{\partial \psi_a} = p_0^a + \frac{\partial \mathscr{L}}{\partial \psi_{a,K}} n_K, \quad \boldsymbol{\eta} \in \partial \Pi_0.
\tag{3.12}
$$

Equations (3.11), (3.12), $(3.1)_{2,3}$ and (3.4) are the governing equations of the structural mechanics of hydroelastic bodies.

If $\Pi_0 \subset R^2$, i.e. $\boldsymbol{\eta} = (\theta^1, \theta^2)$, then the equations of structural mechanics embrace different theories of the shell-like bodies, if $\Pi_0 \subset R$, i.e. $\eta = \theta^1$, then they include different theories of the rod-type bodies. Assuming that $\Pi_0 = \{\eta_0\}$ is the one-point set, $\partial\Pi_0 = \phi$, $\Delta = \varkappa_R(B)$, and using the same approach as before, we arrive at the system of equations

$$h^a - \lambda^\nu \frac{\partial h_\nu}{\partial \psi_a} + f_0^a = w^a + \varrho^{ab}\ddot{\psi}_b + \varrho^{abc}\dot{\psi}_b\dot{\psi}_c \tag{3.13}$$

for the time-dependent functions $\psi_a(t)$, $a = 1, \ldots, n$. For the hyperelastic bodies Eqs. (3.13) reduce to

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{\psi}_a} - \frac{\partial \mathscr{L}}{\partial \psi_a} = f_0^a, \tag{3.14}$$

where

$$\mathscr{L} = \mathscr{L}(t, \boldsymbol{\psi}, \lambda). \tag{3.15}$$

Equations (3.13) or (3.14) together with $h_0(t, \boldsymbol{\psi}) = 0$ represent what is called mechanics of discretized bodies.

Structural mechanics and mechanics of discretized bodies are two special examples of the simplified theories of mechanics; a general approach to these problems was outlined in Sect. 2.

It must be stressed here that there are known shell or rod theories which can be obtained from the general scheme of Sect. 2 but are not included in the special scheme of Eqs. (3.8) and (3.9).

At the end of the section let us give an example of the functional equation (2.12). From Eqs. (3.3) and (3.4) it follows that

$$\int_{\Pi_0} r^a \delta\psi_a d\Pi_0 + \oint_{\partial\Pi_0} s^a \delta\psi_a dS_0 = 0 \tag{3.16}$$

for every $\delta\psi_a$ defined by

$$\frac{\partial h_\nu}{\partial \psi_a}\delta\psi_a + \frac{\partial h_\nu}{\partial \psi_{a,K}}\delta\psi_{a,K} = 0, \quad \boldsymbol{\eta} \in \Pi_0, \quad \frac{\partial R_\pi}{\partial \psi_a}\delta\psi_a = 0, \quad \boldsymbol{\eta} \in \partial\Pi_0. \tag{3.17}$$

Equation (3.16) is the example of $\langle \mathbf{h}, \bar{r}^0 \rangle = 0$ where $\bar{\mathbf{r}} = (r^a, s^a)$ and $\mathbf{h} = (\delta\psi_a)$, cf. (2.10). Substituting into Eq. (3.16) $r^a = w^a + \varrho^{ab}\ddot{\psi}_b + \varrho^{abc}\dot{\psi}_b\dot{\psi}_c - f_0^a - H^{aK}_{,K} - h^a$ and $s^a = H^{aK}n_K - p_0^a$, we arrive at the special case of Eq. (2.12) which can be called the functional equation of structural mechanics:

$$\oint_{\partial\Pi_0} p_0^a \delta\psi_a dS_0 + \int_{\Pi_0} (f_0^a - w^a - \varrho^{ab}\ddot{\psi}_b - \varrho^{abc}\dot{\psi}_b\dot{\psi}_c)\delta\psi_a d\Pi_0 = \int_{\Pi_0} (H^{aK}\delta\psi_{a,K} - h^a\delta\psi_a) d\Pi_0. \tag{3.18}$$

For the unilateral constraints, i.e. if instead of $R_\pi = 0$ we have $R_\pi \geqslant 0$, the sign $R = 0$ in Eq. (3.18) has to be replaced by $R \geqslant 0$.

Final remarks

Among the features of the simplified theories of mechanics based on the concept of abstract constraints we can mention the following: 1. All simplified theories are consistent with the classical theories (in classical theories we deal as a rule with the boundary constraints and very special cases of internal constraints). 2. The constraints approach comprises in one formal scheme many theories which up to now have been treated separately. 3. Abstract constraints enable us to interpret different approximate methods of mechanics as exact constrained field theories with the special case of constraints. 4. Abstract constraints constitute the formal tool for setting up simplified theories in thermomechanics and other branches of theoretical physics. Moreover, the constraints approach to the formation of simplified theories of solid mechanics makes it possible to compare different simplified theories by comparing the corresponding reaction forces maintaining the constraints, cf. [2].

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