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High speed wire drawing

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THE PROBLEM of high speed wire drawing is considered. Since high rates of strains are involved, as in [3] a viscoplastic constitutive equation of the Bingham type is used. However, the method used in this paper is different. One starts from the local balance laws and the boundary value problem in the viscoplastic region is solved using a perturbation series method. Finally, a formula for the drawing stress which takes into account the speed of the process as well is given and from this the optimum drawing regime can be found. Numerical examples are given. A comparison with the results obtaines with another method [3, 4] gives practically the same results.

Rozpatruje się problem ciągnienia drutu z dużymi prędkościami. Ponieważ wiąże się to z dużymi napięciami, więc stosuje się, podobnie do pracy [3], lepkoplastyczny związek konstytutywny typu Binghama. Jednakże metoda stosowana w niniejszej pracy jest inna. Wychodzi się od praw równowagi lokalnej, a problem brzegowy w obszarze lepkoplastycznym rozwiązuje się przy zastosowaniu metody rozwinięć perturbacyjnych. Wreszcie, podaje się wzór na naprężenia ciągnienia z uwzględnieniem szybkości procesu, skąd można określić optymalne warunki ciągnienia. Podano przykłady numeryczne. Rezultaty praktycznie pokrywają się z rezultatami otrzymanymi przy zastosowaniu innej metody w pracach [3, 4].

Рассматривается задача скоростного волочения проволоки. Т. к. это связано с большими напряжениями, применяется, аналогично работе [3], вязкопластическое определяющее соотношение типа Бингема. Однако метод применяемый в настоящей работе совсем другой. Исходится из законов локального равновесия, а краевая задача в вязкопластической области решается при применении метода пертурбационных разложений. Наконец, приведена формула для напряжения волочения с учетом скорости процесса, откуда можно определить оптимальный режим волочения. Даются численные примеры. Результаты практически совпадают с результатами полученными при применении другого метода в работах [3, 4].

1. Introduction

THE PROBLEM of wire drawing has been considered starting from the classical plasticity theory [1, 2]. However, it is not possible to take into account within this theory the influence of the drawing speed on the drawing force and on other drawing parameters.

The influence of the drawing speed on the whole drawing process has been recently considered in [3, 4, 5] using the Bingham type constitutive equation and the principle of total power. These results are in good agreement with the experimental data.

Recently [6], starting from the local balance laws, the problem of the strip drawing at high speeds was solved as a boundary value problem in the viscoplastic flow region, using the perturbation series method with respect to a small parameter, the Bingham number. In the present paper the same method will be used for the problem of high speed wire drawing.

2. Statement of the problem

Let R_1 be the original radius of the wire which is reduced by drawing through a conical converging die to R_2. The desired reduction can be obtained by changing the semi-angle α of the die.

Let us assume that the mechanical properties of the material can be described by a Bingham rigid-viscoplastic model. The region occupied by the material is divided into three zones (Fig. 1). The material in zones I and II has a rigid body motion in the negative

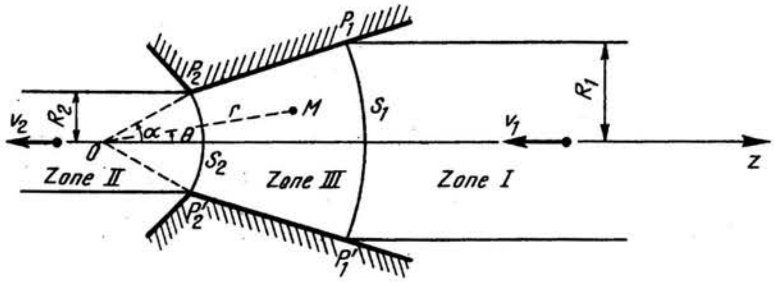

FIG. 1. Geometry in wire drawing.

Oz direction while zone III bounded by the die wall and by two surfaces S_1 and S_2 (which are to be determined) is the domain where the viscoplastic deformation takes place.

Assuming a stationary incompressible axi-symmetrical motion, in the absence of body forces, the governing equations in spherical coordinates (r, θ, φ) are:

the Cauchy's equations of motion

$$(2.1)\quad \begin{aligned} &\frac{\partial t_{rr}}{\partial r}+\frac{1}{r}\frac{\partial t_{r\theta}}{\partial \theta}+\frac{1}{r}(2t_{rr}-t_{\theta\theta}-t_{\varphi\varphi}+\operatorname{ctg}\theta\, t_{r\theta}) = \varrho\left(v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}\right),\\ &\frac{\partial t_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial t_{\theta\theta}}{\partial \theta}+\frac{1}{r}\Big[3t_{r\theta}+\operatorname{ctg}\theta(t_{\theta\theta}-t_{\varphi\varphi})\Big] = \varrho\left(v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}\right);\end{aligned}$$

the equation of local conservation of mass

$$(2.2)\quad \frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{2v_r}{r}+\frac{v_\theta \operatorname{ctg}\theta}{r} = 0;$$

the constitutive equations

$$(2.3)\quad \begin{aligned} t_{rr} &= -p+\left(2\eta+\frac{k}{\sqrt{II_d}}\right)d_{rr},\\ t_{\theta\theta} &= -p+\left(2\eta+\frac{k}{\sqrt{II_d}}\right)d_{\theta\theta},\\ t_{\varphi\varphi} &= -p+\left(2\eta+\frac{k}{\sqrt{II_d}}\right)d_{\varphi\varphi},\\ t_{r\theta} &= \left(2\eta+\frac{k}{\sqrt{II_d}}\right)d_{r\theta},\\ t_{\theta\varphi} &= t_{r\varphi} = 0,\end{aligned}$$

where $\mathbf{d}$ is the strain rate tensor, given by

(2.4)
$$\begin{aligned} d_{rr} &= \frac{\partial v_r}{\partial r}, \\ d_{\theta\theta} &= \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}, \\ d_{\varphi\varphi} &= \frac{v_r}{r} + \frac{v_\theta \operatorname{ctg}\theta}{r}, \\ d_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right), \\ d_{\theta\varphi} &= d_{r\varphi} = 0, \end{aligned}$$

and

(2.5)
$$II_d = \frac{1}{2}(d_{rr}^2 + d_{\theta\theta}^2 + d_{\varphi\varphi}^2 + 2d_{r\theta}^2)$$

is the second invariant of the strain rate tensor, and v_r, v_θ are the components of the velocity vector.

The previous equations are valid in the region $0 \leqslant \theta < \alpha$, $r_2(\theta) < r < r_1(\theta)$, where $r_1 = r_1(\theta)$ and $r_2 = r_2(\theta)$ are the equations of surfaces S_1 and S_2, respectively.

Introducing the dimensionless variables, denoted by index 0

(2.6)
$$r = r^0 R_2, \quad v_r = v_r^0 v_2, \quad v_\theta = v_\theta^0 v_2, \quad p = p^0 \frac{\eta v_2}{R_2}$$

the system of equations (2.1)–(2.4) becomes

(2.1′)
$$\begin{aligned} &\frac{\partial t_{rr}^0}{\partial r^0} + \frac{1}{r^0}\frac{\partial t_{r\theta}^0}{\partial \theta} + \frac{1}{r^0}(2t_{rr}^0 - t_{\theta\theta}^0 - t_{\varphi\varphi}^0 + \operatorname{ctg}\theta \cdot t_{r\theta}^0) = R_e\left(v_r^0 \frac{\partial v_r^0}{\partial r^0} + \frac{v_\theta^0}{r^0}\frac{\partial v_r^0}{\partial \theta} - \frac{v_\theta^{0\,2}}{r^0}\right), \\ &\frac{\partial t_{r\theta}^0}{\partial r^0} + \frac{1}{r^0}\frac{\partial t_{\theta\theta}^0}{\partial \theta} + \frac{1}{r^0}[3t_{r\theta}^0 + \operatorname{ctg}\theta(t_{\theta\theta}^0 - t_{\varphi\varphi}^0)] = R_e\left(v_r^0 \frac{\partial v_\theta^0}{\partial r^0} + \frac{v_\theta^0}{r^0}\frac{\partial v_\theta^0}{\partial \theta} + \frac{v_r^0 v_\theta^0}{r^0}\right); \end{aligned}$$

(2.2′)
$$\frac{\partial v_r^0}{\partial r^0} + \frac{1}{r^0}\frac{\partial v_\theta^0}{\partial \theta} + \frac{2v_r^0}{r^0} + \frac{v_\theta^0 \operatorname{ctg}\theta}{r^0} = 0;$$

(2.3′)
$$\begin{aligned} t_{rr}^0 &= -p^0 + \left(2 + \frac{B_g}{\sqrt{II_{d^0}}}\right) d_{rr}^0, \\ t_{\theta\theta}^0 &= -p^0 + \left(2 + \frac{B_g}{\sqrt{II_{d^0}}}\right) d_{\theta\theta}^0, \\ t_{\varphi\varphi}^0 &= -p^0 + \left(2 + \frac{B_g}{\sqrt{II_{d^0}}}\right) d_{\varphi\varphi}^0, \\ t_{r\theta}^0 &= \left(2 + \frac{B_g}{\sqrt{II_{d^0}}}\right) d_{r\theta}^0; \end{aligned}$$

(2.4′)
$$d_{rr}^0 = \frac{\partial v_r^0}{\partial r^0},$$

(2.4′) [cont.]

$$d^0_{\theta\theta} = \frac{1}{r^0}\frac{\partial v^0_\theta}{\partial\theta} + \frac{v^0_r}{r^0},$$

$$d^0_{\varphi\varphi} = \frac{v^0_r}{r^0} + \frac{v^0_\theta \operatorname{ctg}\theta}{r^0},$$

$$d^0_{r\theta} = \frac{1}{2}\left(\frac{1}{r^0}\frac{\partial v^0_r}{\partial\theta} + \frac{\partial v^0_\theta}{\partial r^0} - \frac{v^0_\theta}{r^0}\right),$$

where

$$R_e = \frac{\varrho v_2 R_2}{\eta} \tag{2.7}$$

is the "Reynolds number" and

$$B_g = \frac{kR_2}{\eta v_2} \tag{2.8}$$

is the "Bingham number".

In what follows we assume that $R_e \ll 1$ and $B_g < 1$, and therefore in the system (2.1′) we neglect the inertial terms.

3. Solution of the problem

Introducing the potential function $\psi = \psi(r, \theta) = R_2^2 v_2 \psi^0(r^0, \theta)$ the equation (2.2′) eads to

$$\begin{aligned} v^0_r &= -\frac{1}{r^{02}\sin\theta}\frac{\partial\psi^0}{\partial\theta}, \\ v^0_\theta &= \frac{1}{r^0\sin\theta}\frac{\partial\psi^0}{\partial r^0}. \end{aligned} \tag{3.1}$$

Expanding the functions $\psi^0(r^0, \theta)$ and $p^0(r^0, \theta)$ in power series of the form

$$\begin{aligned} \psi^0(r^0, \theta) &= \psi^0_0(\theta) + B_g \psi^0_1(r^0, \theta) + \frac{B_g^2}{2}\psi^0_2(r^0, \theta) + \dots, \\ p^0(r^0, \theta) &= p^0_0(r^0, \theta) + B_g p^0_1(r^0, \theta) + \frac{B_g^2}{2} p^0_2(r^0, \theta) + \dots, \end{aligned} \tag{3.2}$$

we get

$$\begin{aligned} II_{d^0} = \frac{1}{2}\left(d^{02}_{rr} + d^{02}_{\theta\theta} + d^{02}_{\varphi\varphi} + 2d^{02}_{r\theta}\right) = \frac{1}{2r^{06}\sin^2\theta}\Bigg\{6\left(\frac{d\psi^0_0}{d\theta}\right)^2 \\ + \frac{1}{2}\left(\operatorname{ctg}\theta\frac{d\psi^0_0}{d\theta} - \frac{d^2\psi^0_0}{d\theta^2}\right)^2 + 2B_g\Bigg[3\frac{d\psi^0_0}{d\theta}\left(2\frac{\partial\psi^0_1}{\partial\theta} - r^0\frac{\partial^2\psi^0_1}{\partial r^0\partial\theta}\right) \\ + \frac{1}{2}\left(\operatorname{ctg}\theta\frac{d\psi^0_0}{d\theta} - \frac{d^2\psi^0_0}{d\theta^2}\right)\left(\operatorname{ctg}\theta\frac{\partial\psi^0_1}{\partial\theta} - \frac{\partial^2\psi^0_1}{\partial\theta^2} + r^{02}\frac{\partial^2\psi^0_1}{\partial r^{02}} - 2r^0\frac{\partial\psi^0_1}{\partial r^0}\right)\Bigg] + \dots\Bigg\}. \end{aligned} \tag{3.3}$$

We substitute Eqs. (3.2) and (3.3) via Eqs. (2.3′) and (2.4′) in the equilibrium equations (2.1′) and terms of equal order in B_g are equated. If we choose the function $\psi_1^0(r_1^0, \theta)$ in the form

$$\psi_1^0(r^0, \theta) = r^{03}\varphi(\theta) \tag{3.4}$$

and introduce the functions

$$u(\theta) = \frac{1}{\sin\theta}\frac{d\psi_0^0}{d\theta}, \tag{3.5}$$

$$v(\theta) = \frac{1}{\sin\theta}\frac{d\varphi}{d\theta}, \tag{3.6}$$

we get the following equations for u and v:

$$u''' + \mathrm{ctg}\theta u'' + \left(6 - \frac{1}{\sin^2\theta}\right)u' = 0, \tag{3.7}$$

and

$$v''' + \mathrm{ctg}\theta v'' + \left(6 - \frac{1}{\sin^2\theta}\right)v' = \frac{d}{d\theta}\left(\frac{12u - \mathrm{ctg}\theta u'}{2\sqrt{3u^2 + \frac{u'^2}{4}}}\right) - \frac{d^2}{d\theta^2}\left(\frac{u'}{2\sqrt{3u^2 + \frac{u'^2}{4}}}\right). \tag{3.8}$$

The regular solutions of these equations are

$$u(\theta) = a + b\cos 2\theta, \tag{3.9}$$

$$v(\theta) = \frac{A}{6} + B\left(\frac{1}{3} + \cos 2\theta\right) + K_1(\theta)\left(\frac{1}{3} + \cos 2\theta\right) + K_2(\theta)\left[\left(\frac{1}{3} + \cos 2\theta\right)\ln\left(\mathrm{tg}\frac{\theta}{2}\right) - (1 - 3\cos\theta)(1 + \cos\theta)\right], \tag{3.10}$$

where

$$K_1(\theta) = -\frac{9}{16}\int_0^\theta f(t)\sin t\left[\left(\frac{1}{3} + \cos 2t\right)\ln\left(\mathrm{tg}\frac{t}{2}\right) - (1 - 3\cos t)(1 + \cos t)\right]dt, \tag{3.11}$$

$$K_2(\theta) = \frac{9}{16}\int_0^\theta f(t)\sin t\left(\frac{1}{3} + \cos 2t\right)dt$$

and

$$f(\theta) = \frac{12u - u'\mathrm{ctg}\theta}{2\sqrt{3u^2 + \frac{u'^2}{4}}} - \frac{d}{d\theta}\left(\frac{u'}{2\sqrt{3u^2 + \frac{u'^2}{4}}}\right). \tag{3.12}$$

We also get

$$p_0^0 = -\frac{2b}{r^{03}}\left(\frac{1}{3} + \cos 2\theta\right) + c, \tag{3.13}$$

and

$$p_1^0 = -A\ln r^0 + h(\theta), \tag{3.14}$$

where

$$h'(\theta) = v' + \frac{6\varphi(\theta)}{\sin\theta} - \frac{d}{d\theta}\left(\frac{u}{\sqrt{3u^2+\frac{u'^2}{4}}}\right) - \frac{3u'}{2\sqrt{3u^2+\frac{u'^2}{4}}}. \tag{3.15}$$

Here a, b, c, A, B are constants.

Returning to the dimensional quantities, we have

$$\psi(r,\theta) = R_2^2 v_2 \psi_0^0(\theta) + \frac{k}{\eta} r^3 \varphi(\theta) + O(B_g^2) \tag{3.16}$$

and

$$\begin{aligned} v_r(r,\theta) &= -\frac{R_2^2 v_2}{r^2} u(\theta) - \frac{k}{\eta} r v(\theta) + O(B_g^2), \\ v_\theta(r,\theta) &= 3\frac{k}{\eta} r \frac{\varphi(\theta)}{\sin\theta} + O(B_g^2). \end{aligned} \tag{3.17}$$

The components of the stress tensor are

$$\begin{aligned} t_{rr} &= \frac{2\eta R_2^2 v_2 b}{r^3}\left(\frac{1}{3}+\cos 2\theta\right) + \frac{4\eta R_2^2 v_2 a}{r^3} - h(\theta) - c \\ &\quad + k\left(A\ln\frac{r}{R_2} - 2v(\theta) + \frac{2u(\theta)}{\sqrt{3u^2+\frac{u'^2}{4}}}\right) + O(B_g^2), \\ t_{\theta\theta} &= \frac{2\eta R_2^2 v_2}{r^3}\left(\frac{b}{3}-a\right) - h(\theta) - c + k\left(A\ln\frac{r}{R_2} + 4v(\theta)\right. \\ &\quad \left. - 6\,\mathrm{ctg}\,\theta\,\frac{\varphi(\theta)}{\sin\theta} - \frac{u}{\sqrt{3u^2+\frac{u'^2}{4}}}\right) + O(B_g^2), \\ t_{\varphi\varphi} &= \frac{2\eta R_2^2 v_2}{r^3}\left(\frac{b}{3}-a\right) - h(\theta) - c + k\left(A\ln\frac{r}{R_2} - 2v(\theta)\right. \\ &\quad \left. + 6\,\mathrm{ctg}\,\theta\,\frac{\varphi(\theta)}{\sin\theta} - \frac{u}{\sqrt{3u^2+\frac{u'^2}{4}}}\right) + O(B_g^2), \\ t_{r\theta} &= \frac{2\eta R_2^2 v_2 b}{r^3}\sin 2\theta - k\left(v'(\theta) + \frac{u'(\theta)}{2\sqrt{3u^2+\frac{u'^2}{4}}}\right) + O(B_g^2). \end{aligned} \tag{3.18}$$

Let

$$r = r(\theta), \quad \theta \in [0, \alpha] \tag{3.19}$$

be the equation of a smooth surface of revolution S in spherical coordinates. Denoting by $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi$ the unit vectors tangent to the coordinate curves and by $\mathbf{n}$ the unit normal vector to this surface, we get

$$\mathbf{n} = \frac{1}{\sqrt{r^2+r'^2}}(r\mathbf{e}_r - r'\mathbf{e}_\theta). \tag{3.20}$$

Across the surfaces S the dynamic conditions of compatibility [7]

$$[v_n] = 0, \tag{3.21}$$

$$[t_{kl}n_k] - \varrho v_n[v_l] = 0 \tag{3.22}$$

must be satisfied. Here v_n is the normal component of the velocity on the singular surface.

Since the second term in Eq. (3.22) is of the order of the Reynolds number, we neglect it with respect to the first term.

The condition (3.21) written for the singular surface S_1, of the equation $r = r_l(\theta)$, has the form

$$r_1 v_r - r_1' v_\theta = -v_1 \frac{d}{d\theta}(r_1 \sin\theta) \tag{3.23}$$

if the terms of $O(B_\theta^2)$ are neglected. Substituting here the expressions (3.17) for v_r and v_θ with $r = r_1(\theta)$ and integrating with respect to θ, we get the equation of the surface S_1 in the form

$$\frac{v_1}{2} r_1^2(\theta)\sin^2\theta - R_2^2 v_2 \left[(b-a)\cos\theta - \frac{2b}{3}\cos^3\theta + a - \frac{b}{3}\right] - \frac{k}{\eta} r_1^3(\theta)\varphi(\theta) = 0. \tag{3.24}$$

In a similar way, for the surface S_2 we find

$$\frac{v_2}{2} r_2^2(\theta)\sin^2\theta - R_2^2 v_2 \left[(b-a)\cos\theta - \frac{2b}{3}\cos^3\theta + a - \frac{b}{3}\right] - \frac{k}{\eta} r_2^3(\theta)\varphi(\theta) = 0. \tag{3.25}$$

In what follows, the stress resultant on a surface $r = r(\theta)$ with $\theta \in [0, \alpha]$ situated in the zone III will be determined.

The stress vector $\mathbf{t}_n$ is given by

$$\begin{aligned}\mathbf{t}_n = t_{nr}\mathbf{e}_r + t_{n\theta}\mathbf{e}_\theta + t_{n\varphi}\mathbf{e}_\varphi = (t_{nr}\sin\theta\cos\varphi + t_{n\theta}\cos\theta\cos\varphi - t_{n\varphi}\sin\varphi)\mathbf{i} \\ + (t_{nr}\sin\theta\sin\varphi + t_{n\theta}\cos\theta\sin\varphi + t_{n\varphi}\cos\varphi)\mathbf{j} + (t_{nr}\cos\theta - t_{n\theta}\sin\theta + t_{n\varphi}\cos\varphi)\mathbf{k},\end{aligned} \tag{3.26}$$

where

$$\begin{aligned} t_{nr} &= t_{rr}n_r + t_{r\theta}n_\theta, \\ t_{n\theta} &= t_{r\theta}n_r + t_{\theta\theta}n_\theta, \\ t_{n\varphi} &= 0. \end{aligned} \tag{3.27}$$

Introducing Eqs. (3.20) and (3.27) in Eq. (3.26), we get

$$\begin{aligned} t_{nx} &= \frac{1}{\sqrt{r^2+r'^2}}[(rt_{rr} - r't_{r\theta})\sin\theta + (rt_{r\theta} - r't_{\theta\theta})\cos\theta]\cos\varphi, \\ t_{ny} &= \frac{1}{\sqrt{r^2+r'^2}}[(rt_{rr} - r't_{r\theta})\sin\theta + (rt_{r\theta} - r't_{\theta\theta})\cos\theta]\sin\varphi, \\ t_{nz} &= \frac{1}{\sqrt{r^2+r'^2}}[(rt_{rr} - r't_{r\theta})\cos\theta - (rt_{r\theta} - r't_{\theta\theta})\sin\theta]. \end{aligned} \tag{3.28}$$

The element of area of the surface $r = r(\theta)$ is given by

$$d\sigma = \sqrt{r^2+r'^2}\, r\sin\theta\, d\theta\, d\varphi. \tag{3.29}$$

Let (X, Y, Z) denote the components of the stress resultant in the Ox, Oy and Oz directions, respectively. After integration we easily get

$$X = Y = 0,$$

$$\begin{aligned} Z &= 2\pi \int_0^\alpha [r^2\cos\theta t_{rr}+rr'\sin\theta t_{\theta\theta}-(rr'\cos\theta+r^2\sin\theta)t_{r\theta}]\sin\theta\, d\theta \\ &= 2\pi \Bigg\{\frac{2\eta R_2^2 v_2}{r}\sin^2\alpha\left[a+b\left(2\cos^2\alpha-\frac{1}{3}\right)\right]-\frac{c}{2}r^2\sin^2\alpha \\ &\quad -\frac{r^2\sin^2\alpha}{2}h(\alpha)+\frac{k}{2}\Bigg[Ar^2\sin^2\alpha\left(\ln\frac{r}{R_2}-\frac{1}{2}\right) \\ &\quad -r^2\sin^2\alpha\frac{u(\alpha)}{\sqrt{3u^2+\frac{u'^2}{4}}}+\frac{r^2\sin 2\alpha}{4}\frac{u'(\alpha)}{\sqrt{3u^2+\frac{u'^2}{4}}}+4r^2\sin^2\alpha v(\alpha) \\ &\quad +\frac{r^2\sin 2\alpha}{2}v'(\alpha)\Bigg]\Bigg\}. \end{aligned} \tag{3.30}$$

Using the relation (3.22) with the above mentioned simplification, the stress resultants acting in the zones I and II on the surfaces S_1 and S_2 are obtained. The following relations are evident:

$$r_1(\alpha)\sin\alpha = R_1, \quad r_2(\alpha)\sin\alpha = R_2. \tag{3.31}$$

We also have

$$R_1^2 v_1 = R_2^2 v_2, \tag{3.32}$$

which expresses the global conservation of the mass.

Denoting by Z^I and Z^{II} the stress resultants on S_1 and S_2, we have

$$\begin{aligned} \frac{Z^I}{\pi R_1^2} &= \frac{4\eta R_2^2 v_2}{R_1^3}\sin^3\alpha\left[a+b\left(2\cos^2\alpha-\frac{1}{3}\right)\right]-c-h(\alpha) \\ &\quad +k\Bigg[A\left(\ln\frac{R_1}{R_2\sin\alpha}-\frac{1}{2}\right)+\frac{1}{2}\operatorname{ctg}\alpha\frac{u'(\alpha)}{\sqrt{3u^2+\frac{u'^2}{4}}}-\frac{u(\alpha)}{\sqrt{3u^2+\frac{u'^2}{4}}} \\ &\quad +4v(\alpha)+\operatorname{ctg}\alpha v'(\alpha)\Bigg], \end{aligned} \tag{3.33}$$

(3.33) [cont.]
$$\frac{Z^{II}}{\pi R_2^2} = -\frac{4\eta v_2}{R_2}\sin^3\alpha\left[a+b\left(2\cos^2\alpha-\frac{1}{3}\right)\right]+c+h(\alpha)$$
$$+k\left[-A\left(\ln\frac{1}{\sin\alpha}-\frac{1}{2}\right)-\frac{1}{2}\operatorname{ctg}\alpha\frac{u'(\alpha)}{\sqrt{3u^2+\frac{u'^2}{4}}}+\frac{u(\alpha)}{\sqrt{3u^2+\frac{u'^2}{4}}}\right.$$
$$\left.-4v(\alpha)-\operatorname{ctg}\alpha v'(\alpha)\right].$$

Let us calculate the stress resultant on the die surface $\theta = \alpha$, $r_2(\alpha) \leqslant r \leqslant r_1(\alpha)$. We obtain

(3.34)
$$\mathbf{T} = \int_{r_2(\alpha)}^{r_1(\alpha)}\int_0^{2\pi} t_{r\theta}\big|_{\theta=\alpha} r\sin\alpha(\sin\alpha\cos\varphi\mathbf{i}+\sin\alpha\sin\varphi\mathbf{j}+\cos\alpha\mathbf{k})\,dr\,d\varphi =$$
$$= -\pi R_1^2\sin 2\alpha\left[-2\eta v_2\sin 2\alpha\sin\alpha\cdot b\frac{R_2^2}{R_1^2}\frac{R_1-R_2}{R_1R_2}\right.$$
$$\left.+k\frac{1}{2\sin^2\alpha}\left(1-\frac{R_2^2}{R_1^2}\right)\left(v'(\alpha)+\frac{u'(\alpha)}{2\sqrt{3u^2+\frac{u'^2}{4}}}\right)\right]\mathbf{k},$$

and

(3.35)
$$\mathbf{N} = \int_{r_2(\alpha)}^{r_1(\alpha)}\int_0^{2\pi} t_{\theta\theta}\big|_{\theta=\alpha} r\sin\alpha(\cos\alpha\cos\varphi\mathbf{i}+\cos\alpha\sin\varphi\mathbf{j}-\sin\alpha\mathbf{k})\,dr\,d\varphi$$
$$= -\pi R_1^2\left\{4\eta v_2\sin^3\alpha\left(\frac{b}{3}-a\right)\frac{R_2^2}{R_1^2}\frac{R_1-R_2}{R_1R_2}-c\left(1-\frac{R_2^2}{R_1^2}\right)\right.$$
$$-h(\alpha)\left(1-\frac{R_2^2}{R_1^2}\right)+k\left[A\left(\ln\frac{R_1}{R_2\sin\alpha}-\frac{R_2^2}{R_1^2}\ln\frac{1}{\sin\alpha}-\frac{1}{2}\left(1-\frac{R_2^2}{R_1^2}\right)\right)\right.$$
$$\left.\left.+\left(4v(\alpha)-\frac{6\cos\alpha}{\sin^2\alpha}\varphi(\alpha)-\frac{u(\alpha)}{\sqrt{3u^2+\frac{u'^2}{4}}}\right)\left(1-\frac{R_2^2}{R_1^2}\right)\right]\right\}\mathbf{k}.$$

It is easy to check that the condition

(3.36)
$$\mathbf{Z}^I+\mathbf{Z}^{II}+\mathbf{T}+\mathbf{N} = 0$$

is satisfied. This means that the equilibrium condition of all the resultant forces acting on the boundary of zone III is satisfied.

4. The kinematic and boundary conditions which determine the parameters a, b, c, A, B

We shall assume the following conditions:

a) the v_θ component of the velocity is zero along the die surfaces;

b) the discontinuity surface S_1 passes through the points P_1, P_1';

c) a friction condition on the die surface is prescribed;

d) the force resultant Z^I is given.

Using Eqs. (3.6), (3.10) and (3.17), the condition (a) can be written as

$$(4.1)\quad \frac{A}{6}(1-\cos\alpha)+\frac{2}{3}\cos\alpha\sin^2\alpha B+\int_0^\alpha \sin\theta\left\{K_1(\theta)\left(\frac{1}{3}+\cos 2\theta\right)\right. \\ \left.+K_2(\theta)\left[\left(\frac{1}{3}+\cos 2\theta\right)\ln\left(\operatorname{tg}\frac{\theta}{2}\right)-(1-3\cos\theta)(1+\cos\theta)\right]\right\}d\theta = 0.$$

By imposing the condition that the surface S_1 should pass through points P_1 and P_1', it yields also that the surface S_2 is passing through points P_2 and P_2'. From Eqs. (3.24) and (3.25) we get the unique condition

$$(4.2)\quad \frac{1}{2}-(b-a)\cos\alpha+\frac{2}{3}b\cos^3\alpha-a+\frac{b}{3}=0.$$

The friction condition on the die surface is taken in the form [8]

$$(4.3)\quad t_{r\theta}\big|_{\theta=\alpha}=m\sqrt{II_{t'}}\big|_{\theta=\alpha},$$

where $II_{t'}$ is the second invariant of the deviatoric part of the stress tensor obtained from

$$(4.4)\quad \sqrt{II_{t'}}=k+2\eta\sqrt{II_d},$$

and m is the constant friction factor which satisfies the condition

$$(4.5)\quad 0<m<1.$$

From Eqs. (2.6), (3.3), (3.4), (3.5) and (3.6) we obtain easily

$$(4.6)\quad \sqrt{II_d}=\frac{R_2^2 v_2}{r^3}\sqrt{3u^2+\frac{u'^2}{4}}\left[1+\frac{kr^3}{4\eta R_2^2 v_2}\,\frac{u'v'-6uv}{3u^2+\frac{u'^2}{4}}+\ldots\right].$$

Using Eqs. $(3.18)_4$, (4.4), (4.6) in Eq. (4.3), we find

$$(4.7)\quad b\sin 2\alpha=m\sqrt{3u^2(\alpha)+\frac{u'^2(\alpha)}{4}},$$

$$(4.8)\quad v'(\alpha)+\frac{u'(\alpha)}{2\sqrt{3u^2+\frac{u'^2}{4}}}=-m-m\frac{u'(\alpha)v'(\alpha)-6u(\alpha)v(\alpha)}{2\sqrt{3u^2+\frac{u'^2}{4}}}.$$

Finally, if the back force resultant Z^I is given, then the condition $(3.33)_1$ is the fifth relationship between the unknown parameters.

The conditions (4.1), (4.2), (4.7), (4.8) and $(3.33)_1$ determine the five parameters as follows:

From Eqs. (4.2) and (4.7) it results

$$a = \frac{\sqrt{3}(\sin 2\alpha - \gamma\sqrt{3}\cos 2\alpha)}{2(1-\cos\alpha)\left[2\gamma(1-\cos\alpha)(1+2\cos\alpha)+\sqrt{3}\sin 2\alpha\right]}, \tag{4.9}$$

$$b = \frac{3\gamma}{2(1-\cos\alpha)\left[2\gamma(1-\cos\alpha)(1+2\cos\alpha)+\sqrt{3}\sin 2\alpha\right]},$$

where

$$\gamma = \frac{m}{\sqrt{1-m^2}}. \tag{4.10}$$

From Eqs. (4.1) and (4.8) via Eqs. (3.9), (3.10) and (4.9) we get

$$A = 6\,\frac{-\frac{m}{3}\cos\alpha\sin^2\alpha+\sin\alpha\left(\gamma\sin\alpha-\frac{2\cos\alpha}{\sqrt{3}}\right)I_1}{(1-\cos\alpha)\left[\sin 2\alpha-\frac{\gamma}{\sqrt{3}}(1-\cos\alpha)(1+2\cos\alpha)\right]} + 6\,\frac{\dfrac{\gamma\left[\sqrt{3}\sin 2\alpha+6\gamma\cos 2\alpha-4\gamma\right]I_2+4\gamma^2 I_3}{\sqrt{3}\sin 2\alpha+6\gamma\sin^2\alpha}}{(1-\cos\alpha)\left[\sin 2\alpha-\frac{\gamma}{\sqrt{3}}(1-\cos\alpha)(1+2\cos\alpha)\right]}, \tag{4.11}$$

where

$$I_k = \int_0^\alpha \frac{\sin^{2k-1}\theta\,d\theta}{\sqrt{1-\sin^2\theta(\lambda_1-\lambda_2\sin^2\theta)}}, \quad k = 1, 2, 3 \tag{4.12}$$

and

$$\lambda_1 = \frac{3\gamma\left(\sqrt{3}\sin 2\alpha-3\gamma\cos 2\alpha+2\gamma\right)}{\sin^2\alpha\left(\sqrt{3}\cos\alpha+3\gamma\sin\alpha\right)^2}, \tag{4.13}$$

$$\lambda_2 = \frac{6\gamma^2}{\sin^2\alpha\left(\sqrt{3}\cos\alpha+3\gamma\sin\alpha\right)^2}.$$

The expressions for the parameters B and c were no more given since they are not involved in the final formula for the drawing stress.

In the limiting case $m \to 1$, we have

$$a^* = -\frac{3\cos\alpha}{4(1-\cos\alpha)^2(1+2\cos\alpha)}, \tag{4.14}$$

$$b^* = \frac{3}{4(1-\cos\alpha)^2(1+2\cos\alpha)}$$

and

$$A^* = -6\frac{3\sin^4\alpha I_1^* + (3\cos 2\alpha - 2)I_2^* + 2I_3^*}{\sqrt{3}\sin^2\alpha(1-\cos\alpha)^2(1+2\cos\alpha)}, \tag{4.15}$$

$$\lambda_1^* = \frac{2-3\cos 2\alpha}{3\sin^4\alpha}, \quad \lambda_2^* = \frac{2}{3\sin^4\alpha}, \tag{4.16}$$

where "*" indicates the corresponding limiting values.

Using Eq. $(3.17)_1$ it is easy to verify in this case that $v_r(r, \alpha) = 0$, i.e. the material adheres to the die surface.

In the other limiting case, $m \to 0$, i.e. in the case of no friction, we have

$$a^{**} = \frac{1}{2(1-\cos\alpha)}, \quad b^{**} = 0, \quad A^{**} = -2\sqrt{3}, \quad \lambda_1^{**} = \lambda_2^{**} = 0. \tag{4.17}$$

For the drawing force Z^{II}, from Eq. (3.33) we get

$$\frac{Z^{II}}{\pi R_2^2} = -\frac{Z^I}{\pi R_1^2} + \frac{4\eta v_2}{R_2}\left(\frac{R_2^3}{R_1^3} - 1\right)\sin^3\alpha\left[a + b\left(2\cos^2\alpha - \frac{1}{3}\right)\right] + Ak\ln\frac{R_1}{R_2}. \tag{4.18}$$

The magnitude of this drawing force is

$$\frac{|Z^{II}|}{\pi R_2^2} = \frac{|Z^I|}{\pi R_1^2} + \frac{2\eta v_2}{R_2}\left(1 - \frac{R_2^3}{R_1^3}\right)F(\alpha, m) + 2k\ln\left(\frac{R_1}{R_2}\right)^3 G(\alpha, m), \tag{4.19}$$

where

$$F(\alpha, m) = \frac{\cos^3\frac{\alpha}{2}\left(2\gamma + \sqrt{3}\sin 2\alpha\right)}{\gamma\sin\frac{\alpha}{2}(1+2\cos\alpha) + \sqrt{3}\cos\alpha\cos\frac{\alpha}{2}}, \tag{4.20}$$

$$G(\alpha, m) = \frac{\frac{m}{3}\cos\alpha\sin^2\alpha - \sin\alpha\left(\gamma\sin\alpha - \frac{2}{\sqrt{3}}\cos\alpha\right)I_1}{(1-\cos\alpha)\left[\sin 2\alpha - \frac{\gamma}{\sqrt{3}}(1-\cos\alpha)(1+2\cos\alpha)\right]} - \frac{\dfrac{\gamma\left[\sqrt{3}\sin 2\alpha + 6\gamma\cos 2\alpha - 4\gamma\right]I_2 + 4\gamma^2 I_3}{\sqrt{3}\sin 2\alpha + 6\gamma\sin^2\alpha}}{(1-\cos\alpha)\left[\sin 2\alpha - \frac{\gamma}{3}(1-\cos\alpha)(1+2\cos\alpha)\right]}. \tag{4.21}$$

In the two limiting cases, $m \to 1$ and $m \to 0$, we have

$$F^*(\alpha) = \frac{2\cos^3\frac{\alpha}{2}}{\sin\frac{\alpha}{2}(1+2\cos\alpha)}, \qquad G^*(\alpha) = \frac{3\sin^4\alpha I_1^* + (3\cos 2\alpha - 2)I_2^* + 2I_3^*}{\sqrt{3}\sin^2\alpha(1-\cos\alpha)^2(1+2\cos\alpha)} \tag{4.22}$$

and

$$F^{**}(\alpha) = 2\sin\alpha\cos^2\frac{\alpha}{2},$$

(4.23)

$$G^{**}(\alpha) = \frac{1}{\sqrt{3}},$$

respectively.

Introducing the notations

(4.24) $$\sigma_{z1} = \frac{Z^I}{\pi R_1^2}, \quad \sigma_{z2} = \frac{Z^{II}}{\pi R_2^2},$$

and the mean yield stress in tension $\sigma_y = \sqrt{3}\,k$, the relation (4.19) becomes

(4.25) $$\frac{|\sigma_{z2}|}{\sigma_y} = \frac{|\sigma_{z1}|}{\sigma_y} + \frac{2}{\sqrt{3}B_g}\left[\left(1-\frac{R_3^2}{R_1^3}\right)F(\alpha, m)+B_g\ln\left(\frac{R_1}{R_2}\right)^3 G(\alpha, m)\right].$$

Taking $\sigma_{z1} = 0$ in Eq. (4.25), the variation of drawing stresses with respect to α, for different reductions in the area $r\% = 100\left(1-\frac{R_2^2}{R_1^2}\right)$ for two values of the B_g number and

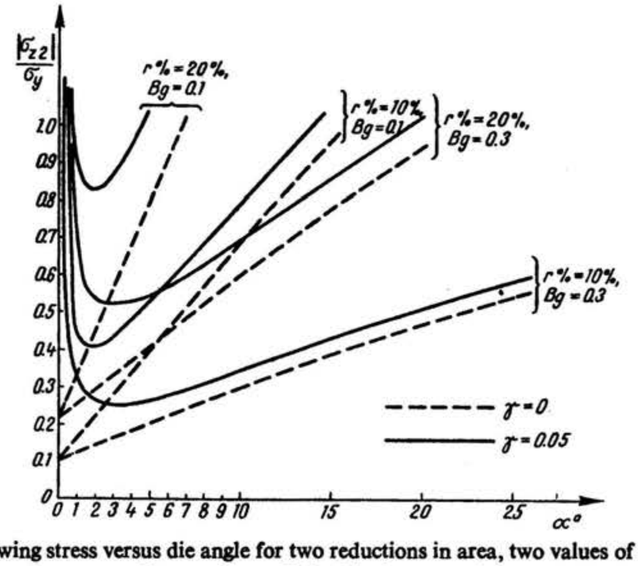

FIG. 2. Relative drawing stress versus die angle for two reductions in area, two values of the friction factor, and two values of the Bingham number.

for $\gamma = 0$ and $\gamma = 0.05$ (corresponding to the friction factor $m = 0.0499$), is shown in Fig. 2. Each of these curves exhibits a minimum which is the optimal angle of the die for a certain combination of the process variables.

5. Comparison with another method

In papers [3, 4] another formula for the drawing stress was established, starting from a radial kinematic admissible velocity field for the flow of a viscoplastic body through a conical die and using the principle of total power. This formula, in which the dissipation due to the presence of the discontinuity surfaces is neglected (since the Reynolds number is small), is of the form

$$(5.1)\qquad \frac{|\sigma_{z2}|}{\sigma_y} = \frac{2}{\sqrt{3}B_g}\left[\left(1-\frac{R_2^3}{R_1^3}\right)\mathcal{F}(\alpha, m)+B_g\ln\left(\frac{R_1}{R_2}\right)^3\mathcal{G}(\alpha, m)\right],$$

where

$$(5.2)\qquad \mathcal{F}(\alpha, m) = \frac{m}{3}\cos\alpha\sqrt{1+11\cos^2\alpha}+\frac{1}{3\sin\alpha}\left(\frac{14}{3}-\cos\alpha-\frac{11}{3}\cos^3\alpha\right),$$

$$(5.3)\qquad \mathcal{G}(\alpha, m) = \frac{m}{3}\operatorname{ctg}\alpha-\frac{\sqrt{11}}{6\sin^2\alpha}\left[\cos\alpha\sqrt{\frac{1}{11}+\cos^2\alpha}-\sqrt{\frac{12}{11}}\right.$$
$$\left.+\frac{1}{11}\ln\frac{\cos\alpha+\sqrt{\cos^2\alpha+\frac{1}{11}}}{1+\sqrt{\frac{12}{11}}}\right].$$

The formula (5.1) is of the same form as Eq. (4.25). In order to compare the predictions for the drawing stress as given by Eqs. (5.1) and (4.25), it is sufficient to compare the numerical values of the functions $F(\alpha, m)$ obtained using Eq. (4.20) with $\mathcal{F}(\alpha, m)$ from Eq. (5.2), and $G(\alpha, m)$ from Eq. (4.21) with $\mathcal{G}(\alpha, m)$ obtained using Eq. (5.3). In the following Tables 1 and 2 this comparison is done for $\gamma = 0.05$ and for various values

Table 1

α	$F(\alpha, m)$	$\mathcal{F}(\alpha, m)$	α	$F(\alpha, m)$	$\mathcal{F}(\alpha, m)$
1°	0.0925716	0.0925487	8°	0.3329874	0.3338127
2	0.1273378	0.1273770	9	0.3665293	0.3675794
3	0.1620073	0.1621238	10	0.3997939	0.4011024
4	0.1965536	0.1967655	15	0.5610998	0.5643093
5	0.2309507	0.2312786	20	0.7118981	0.7182991
6	0.2651726	0.2656395	25	0.8494882	0.8607876
7	0.2991934	0.2998253	30	0.9715071	0.9898914

Table 2

α	$G(\alpha, m)$	$\mathcal{G}(\alpha, m)$	α	$G(\alpha, m)$	$\mathcal{G}(\alpha, m)$
1°	1.5307271	1.5309953	8°	0.6960114	0.6960278
2	1.0538124	1.0540403	9	0.6827402	0.6827473
3	0.8956561	0.8950054	10	0.6721172	0.6721237
4	0.8153284	0.8154560	15	0.6401979	0.6403163
5	0.7676158	0.7677053	20	0.6241699	0.6246091
6	0.7358003	0.7358575	25	0.6144947	0.6154844
7	0.7141481	0.7131004	30	0.6079916	0.6097919

of α. These tables show for the values of the drawing parameters considered an excellent agreement between the predictions for the drawing stress as given by Eq. (4.25) and by Eq. (5.1), though these formulas were established using two distinct methods.

6. Conclusions

For high speed drawing of wires a formula giving the drawing stress as function of drawing speed, friction, die angle, plastic and viscous property of the metal is obtained. The formula makes possible the finding of optimum regimes for the drawing process. A comparison with a formula obtained earlier with another method [3, 4] gives an excellent agreement.

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