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The effect of shearing prestress on the response of a thick membrane strip
Part I. The static case

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THE STATIC response of a prestressed thick membrane (or thin plate) strip is analysed for the case when the bending rigidity is small. The full problem must necessarily include the case when a shearing prestress is applied parallel to the side walls. Singular perturbation analysis reveals that for such cases, unless the external load is sufficiently smooth, singular shear layers that run across the width of the strip are induced. The solutions in such layers as well as the usual core and edge layers are presented.

Badany jest statyczny problem dla wstępnie naprężonej grubej membrany (lub cienkiej płyty) w formie pasma, przy założeniu małej sztywności na zginanie. W ogólnym sformułowaniu występować musi przypadek, kiedy wstępne naprężenia ścinające przyłożone są równolegle do bocznych ścianek. Stosując metodę perturbacji osobliwych wykazano, że — z wyjątkiem przypadku kiedy obciążenie jest dostatecznie gładkie — powstają osobliwe warstwy ścinania biegnące w poprzek szerokości pasm. Podano rozwiązanie w tych warstwach, jak również w rdzeniu oraz warstwie brzegowej.

Исследуется статическая задача для предварительно напряженной толстой мембраны (или тонкой плиты) в форме полосы, принимая малую изгибную жесткость. В общей формулировке должен выступать случай, когда предварительные напряжения сдвига приложены параллельно к боковым стенкам. Применяя метод особых пертурбаций показано, что за исключением случая, когда нагрузка достаточно гладкая, возникают особые слои сдвига бегущие поперек ширины полосы. Дается решение в этих слоях, как тоже в сердечнике и в граничном слое.

1. Introduction

THE BEHAVIOUR of prestressed thick membranes is predicted on the competing influences of both the prestress mechanism and the bending rigidity of the membrane. In fact, the influence of the former is globally overriding except close to the boundary where the latter is just as important. The relative balancing of these forces makes the problem amenable to singular perturbation techniques. For this the entire domain must be divided into two regions viz. an outer region where the resistance to extensional deformation is of prime importance as well as an inner region where a balance is maintained between extensional and bending forces. This inner region usually takes the form of boundary or edge layers, but could sometimes appear within the core.

Singular perturbations have been successfully exploited in constructing solutions to thick membranes. In fact, SCHNEIDER (1972) determines the influence of the bending rigidity to the eigenfrequency of an isotropically prestressed rectangular membrane. The corresponding problem of the circular drum has been solved by HUTTER in (1972). HUTTER and OLUNLOYO (1974) extended the free vibration problem of rectangular membranes to include certain special cases of anisotropy in the prestress. Attempts to deal with membranes that were subjected to external loads (statically or dynamically) were thus far particularly successful for (infinitely) long strips (see HUTTER and OLUNLOYO (1974)).

The above mentioned problems are still too restrictive and they should be extended on several different levels. There are various possibilities to achieve such extensions. One possibility is to increase the complexity of the loadings, but such extensions do not lead to essentially new effects. Another possibility is to enlarge the complexity of the boundaries of the membranes.

The latter problem is to a certain extent akin to a change in the anisotropy conditions of prestress. Indeed, if the normal prestress in the x- and y-directions of a Cartesian coordinate system is denoted by N_x and N_y, respectively, and if the shearing prestress N_{xy} vanishes, then it is easy to determine the prestress tensor with respect to a rotated coordinate system $(\bar{x}, \bar{y})$. From such a calculation one concludes that in general $N_{\bar{x}\bar{y}} \neq 0$. It is thus interesting to investigate the influence of the shearing prestress in one of the above mentioned well-known problems.

For this purpose let us consider a membrane strip in the (x, y)-plane, bounded at $y = 0$ and $y = b$. At this stage we are not interested in the physical conditions of these boundaries, but we might mention that usual boundary conditions manifest themselves either as clamped edges or cylindrical hinges. The governing equations derive from the von Kármán-equations

$$\nabla^4 w = \frac{1}{D}(q + \langle \phi, w \rangle), \quad \nabla^4 \phi \cong 0, \tag{1.1}$$

where

$$\langle \phi, w \rangle = \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} \tag{1.2}$$

and where for our case of constant prestress conditions

$$\phi = \frac{1}{2}(N_x x^2 - 2N_{xy} xy + N_y y^2). \tag{1.3}$$

Therefore, Eq. (1.1) becomes

$$D\nabla^4 w = q(x, y) + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \tag{1.4}$$

which must be complemented by appropriate boundary conditions at $y = 0$ and $y = b$. In Eq. (1.4) D denotes the bending rigidity, w the transverse deflection, q the transverse loading and N_x, N_y, N_{xy} the (constant) prestress. Finally, ∇^4 is the bipotential operator which in Cartesian coordinates reads

$$\nabla^4 w \equiv \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}. \tag{1.5}$$

We non-dimensionalize the above equation (1.4) by introducing the transformation

$$(\hat{x}, \hat{y}, \chi) = (x/b, y/b, w/b). \tag{1.6}$$

Denoting

$$p = \frac{qb}{N_0}, \quad \beta_x^2 = \frac{N_x}{N_0}, \quad \beta_{xy}^2 = \frac{N_{xy}}{N_0}, \quad \beta_y^2 = \frac{N_y}{N_0}, \tag{1.7}$$

where N_0 is an appropriately chosen reference prestress (e.g. $N_0 = \max(N_x, N_y)$), and

$$\varepsilon^2 = \frac{D}{N_0 b^2} \tag{1.8}$$

we readily obtain from Eq. (1.4)

$$\varepsilon^2 \nabla^4_{(\hat{y},\hat{x})}\chi - \beta_x^2 \frac{\partial^2 \chi}{\partial \hat{x}^2} - 2\beta_{xy}^2 \frac{\partial^2 \chi}{\partial \hat{x}\partial \hat{y}} - \beta_y^2 \frac{\partial^2 \chi}{\partial \hat{y}^2} = p(\hat{x}, \hat{y}). \tag{1.9a}$$

In the following we shall only deal with Eq. (1.9a) and shall for brevity henceforth drop the hat and write (x, y) for $(\hat{x}, \hat{y})$. Moreover, we shall solve Eq. (1.9a) in the strip $0 \leqslant y \leqslant 1$ under the restriction

$$\varepsilon^2 \ll 1. \tag{1.9b}$$

This condition guaranties that the assumptions of matched asymptotic expansions are satisfied.

Usual boundary conditions that are accompanied with Eq. (1.9a) are those of built-in ends. For the strip under consideration they are

$$\chi(x, 0) = \chi(x, 1) = \frac{\partial \chi}{\partial y}(x, 0) = \frac{\partial \chi}{\partial y}(x, 1) = 0. \tag{1.9c}$$

The purpose of this paper is to demonstrate that the structure of the layers of the boundary value problem (1.9) depends on the operator

$$L \equiv \beta_x^2 \frac{\partial^2}{\partial x^2} + 2\beta_{xy}^2 \frac{\partial^2}{\partial x \partial y} + \beta_y^2 \frac{\partial^2}{\partial y^2}. \tag{1.10}$$

If $\gamma \equiv \beta_x^2 \beta_y^2 - \beta_{xy}^4 > 0$, then L is elliptic, otherwise hyperbolic. Using the Mohr circle arguments it is easy to show that the differential equation

$$\varepsilon^2 \nabla^4 \chi - L\chi = p \tag{1.11}$$

corresponds to pure prestress conditions when L is elliptic. If L is hyperbolic, there exists a distinct direction for L, the corresponding proper value of which is negative. Physically it means that the in-plane force in that direction is a pressure (prepressure). In the following we shall restrict ourselves to cases in which L is elliptic, but we shall not assume that the coordinates x and y are parallel to the principal directions of L. As we shall see, this implies that there exist not only boundary layers, but also free (shear) layers which are induced by the external loading $p(x, y)$. The demonstration of this latter phenomenon is the main goal of this paper.

2. The strip under static loading

Let $\mathscr{B} \subset \mathsf{R}^2$ be the open strip $[-\infty < x < \infty,\ 0 \leqslant y \leqslant 1]$. We are interested in solutions of the boundary value problem

$$\varepsilon^2\nabla^4\chi - L\chi = p, \quad \varepsilon \ll 1, \quad (x, y) \in \mathscr{B},$$

(2.1)

$$\left.\begin{aligned} \chi &= f_L, \\ \frac{\partial\chi}{\partial y} &= g_L, \end{aligned}\right\} y = 0, \qquad \left.\begin{aligned} \chi &= f_U, \\ \frac{\partial\chi}{\partial y} &= g_U, \end{aligned}\right\} y = 1,$$

where f_L, g_L, f_U and g_U are smooth functions on R. The known function $p(x, y)$ will be assumed to have the form

$$p(x, y) = p_0\delta(x - x_0),$$

where $\delta(x-x_0)$ is the Dirac distribution. The construction of solutions that are asymptotic approximations for small ε will be simplified if we resort to Fig. 1 which shows

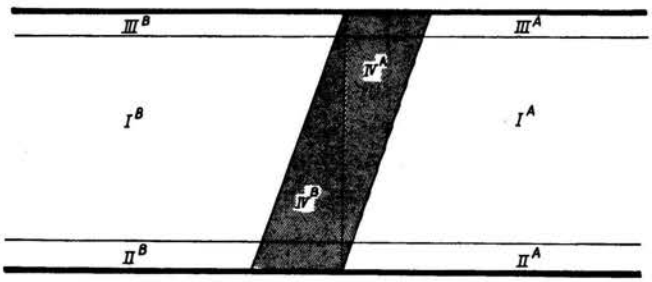

FIG. 1. Strip with core region and boundary and shear layers.

the strip together with the subdivision in various regions. In each of these regions different asymptotic approximations of the solution of Eq. (2.1) will hold. Solutions in the regions I^A and I^B will be called *outer* solutions, while those in the regions $\mathrm{II}^{A,B}$ and $\mathrm{III}^{A,B}$ are termed *inner* solutions or boundary layer solutions. Of special interest are the regions IV^A and IV^B.

2.1. Outer solutions (regions I^A and I^B)

Following the usual procedures in singular perturbation problems away from the boundaries, the solution is assumed to have the expansion

$$\chi(x, y; \varepsilon) = \sum_{\nu=0}^{\infty} \varepsilon^\nu \psi_\nu(x, y), \tag{2.2}$$

which, when substituted into Eq. $(2.1)_1$, gives

$$L\psi_\nu = \begin{cases} -p(x, y), & \nu = 0, \\ 0, & \nu = 1, \\ \nabla^4\psi_{\nu-2}, & \nu \geqslant 2. \end{cases} \tag{2.3}$$

At each stage a second-order differential equation has to be solved so that only two boundary conditions can be satisfied. It follows that in general the boundary conditions will be violated.

Solutions of Eq. (2.3) are most easily constructed by the Fourier-transform technique whereby we use the definition

$$\overline{(\cdot)} = \int_{-\infty}^{\infty} (\cdot) e^{-i\xi x} dx \tag{2.4}$$

with the inverse

$$(\cdot) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{(\cdot)} e^{i\xi x} d\xi. \tag{2.4'}$$

It is then straightforward to show that when L is elliptic,

$$\chi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{[(\overline{A}_0 + \varepsilon \overline{A}_1)\cosh(\delta\xi y) + (\overline{B}_0 + \varepsilon \overline{B}_1)\sinh(\delta\xi y)] e^{-i\alpha\xi y} + \overline{\psi}_0^p(\xi, y)\} e^{i\xi x} d\xi + 0(\varepsilon^2), \tag{2.5}$$

where $\overline{A}_\nu$ and $\overline{B}_\nu$ $(\nu = 0, 1)$ are as yet unknown functions of ξ, and where $\overline{\psi}_0^p(\xi, y)$ denotes the Fourier-transform of a particular solution of Eq. $(2.3)_1$ to the given p. Furthermore,

$$\alpha \equiv \frac{\beta_{xy}^2}{\beta_y^2}, \quad \delta \equiv \frac{\sqrt{\gamma}}{\beta_y^2} \equiv \frac{\sqrt{\beta_x^2\beta_y^2 - \beta_{xy}^4}}{\beta_y^2}. \tag{2.6}$$

The first two terms in Eq. (2.5) correspond to solutions of the homogeneous equation $L\psi_\nu = 0$ $(\nu = 0, 1)$. The arbitrary function $\overline{A}_\nu$ and $\overline{B}_\nu$ $(\nu = 0, 1)$ will be determined from subsequent matching in the neighbouring side layers.

2.2. Boundary layer solution near $y = 0$ (regions II^A and II^B) and $y = 1$ (regions III^A and III^B)

In order to balance out the two terms $\varepsilon^2 \nabla^4 \chi$ and $L\chi$, coordinate stretchings are needed. Of the two possibilities

$$Y = y/\varepsilon^{2/3} \quad \text{and} \quad Y = y/\varepsilon$$

the former does not allow matching with the outer solution, so that we introduce near $y = 0$ the coordinate transformation $Y = y/\varepsilon$ which, together with the expansions

$$\begin{aligned} \chi &= \sum_{\nu=0}^{\infty} \varepsilon^\nu \overset{\mathrm{II}}{\Xi}_\nu(x, Y), \\ p &= \sum_{\nu=0}^{\infty} \varepsilon^\nu p_\nu(x) Y^\nu, \end{aligned} \tag{2.7}$$

transforms Eq. (2.1) into the following hierarchy of differential equations:

$$\frac{\partial^4 \overset{\mathrm{II}}{\Xi}_{\nu+2}}{\partial Y^4} - \beta_y^2 \frac{\partial^2 \overset{\mathrm{II}}{\Xi}_{\nu+2}}{\partial Y^2} = 2\beta_{xy}^2 \frac{\partial^2 \overset{\mathrm{II}}{\Xi}_{\nu+1}}{\partial x \partial Y} + \beta_x^2 \frac{\partial^2 \overset{\mathrm{II}}{\Xi}_\nu}{\partial x^2} - 2\frac{\partial^4 \overset{\mathrm{II}}{\Xi}_\nu}{\partial x^2 \partial Y^2} + \overset{\mathrm{II}}{p}_\nu(x) Y^\nu - \frac{\partial^4 \overset{\mathrm{II}}{\Xi}_{\nu-2}}{\partial x^4}, \tag{2.8}$$

$$\nu = -2, -1, 0, 1, \ldots.$$

Here and everywhere else henceforth we shall adopt the convention that terms with negative indices be set to zero. The two lowest order solutions of Eq. (2.8) ($\nu = -2, -1$) assume the form

$$\overset{\mathrm{II}}{\Xi}_0(x, Y) = \overset{\mathrm{II}}{C}_0(x) + \overset{\mathrm{II}}{D}_0(x) Y + \overset{\mathrm{II}}{E}_0(x) e^{-\beta_y Y} \tag{2.9}$$

and

$$\overset{\mathrm{II}}{\Xi}_1(x, Y) = \overset{\mathrm{II}}{C}_1(x) + \overset{\mathrm{II}}{D}_1(x) Y + \overset{\mathrm{II}}{E}_1(x) e^{-\beta_y Y} - \frac{\beta_{xy}^2}{\beta_y^2} \frac{\partial \overset{\mathrm{II}}{D}_0}{\partial x} Y^2 + \frac{\beta_{xy}^2}{\beta_y^2} \frac{\partial \overset{\mathrm{II}}{E}_0}{\partial x} Y e^{-\beta_y Y}, \tag{2.10}$$

where exponentially growing terms have been suppressed since they would be unmatchable otherwise.

A similar stretching must be introduced near $y = 1$. The coordinate transformation here is

$$\tilde{Y} = (1-y)/\varepsilon$$

so that if we introduce the asymptotic expansions

$$\begin{aligned} \chi(x, y) &= \sum_{\nu=0}^{\infty} \varepsilon^{\nu} \overset{\mathrm{III}}{\Xi}_{\nu}(x, \tilde{Y}), \\ p(x) &= \sum_{\nu=0}^{\infty} \varepsilon^{\nu} \tilde{Y}^{\nu} \overset{\mathrm{III}}{p}_{\nu}(x), \end{aligned} \tag{2.11}$$

we obtain the recurrence relations

$$\begin{aligned} \frac{\partial^4 \overset{\mathrm{III}}{\Xi}_{\nu+2}}{\partial \tilde{Y}^4} - \beta_y^2 \frac{\partial^2 \overset{\mathrm{III}}{\Xi}_{\nu+2}}{\partial \tilde{Y}^2} = &-2\beta_{xy}^2 \frac{\partial^2 \overset{\mathrm{III}}{\Xi}_{\nu+1}}{\partial x \partial \tilde{Y}} + \beta_x^2 \frac{\partial^2 \overset{\mathrm{III}}{\Xi}_{\nu}}{\partial x^2} - 2 \frac{\partial^4 \overset{\mathrm{III}}{\Xi}_{\nu}}{\partial x^2 \partial \tilde{Y}^2} \\ &+ \overset{\mathrm{III}}{p}_{\nu} \tilde{Y} - \frac{\partial^4 \overset{\mathrm{III}}{\Xi}_{\nu-2}}{\partial x^4}, \quad \nu = -2, -1, 0, 1, \ldots \end{aligned} \tag{2.12}$$

with the solutions

$$\begin{aligned} \overset{\mathrm{III}}{\Xi}_0(x, \tilde{Y}) &= \overset{\mathrm{III}}{C}_0(x) + \overset{\mathrm{III}}{D}_0(x) \tilde{Y} + \overset{\mathrm{III}}{E}_0(x) e^{-\beta_y \tilde{Y}}, \\ \Xi_1(x, \tilde{Y}) &= \overset{\mathrm{III}}{C}_1(x) + \overset{\mathrm{III}}{D}_1(x) \tilde{Y} + \overset{\mathrm{III}}{E}_1(x) e^{-\beta_y \tilde{Y}} + \frac{\beta_{xy}^2}{\beta_y^2} \frac{\partial \overset{\mathrm{III}}{D}_0}{\partial x} \tilde{Y}^2 - \frac{\beta_{xy}^2}{\beta_y^2} \frac{\partial \overset{\mathrm{III}}{E}_0}{\partial x} \tilde{Y} e^{-{}_y \tilde{Y}}. \end{aligned} \tag{2.13}$$

2.3. Matching at the boundary layers

Next we invoke Van Dyke's matching principle to match the outer solution (2.5) wit the zeroth and first-order inner solutions, Eqs. (2.9) and (2.10). We then obtain the results

a) in the regions II^A and II^B, respectively:

$$(2.14)\quad \begin{aligned} &\overset{\mathrm{II}}{D}_0 = 0\\ &\bar{A}_0+\bar{\psi}^{\mathrm{II}}_{00} = \overline{\overset{\mathrm{II}}{C}}_0,\\ &\bar{A}_1-\overline{\overset{\mathrm{II}}{C}}_1 = 0,\\ &\bar{\psi}^{\mathrm{II}}_{01}+\bar{B}_0\delta\xi-i\alpha\xi\bar{A}_0 = \overline{\overset{\mathrm{II}}{D}}_1, \end{aligned}$$

where overhead bars denote Fourier-transforms and where the Taylor series expansion

$$(2.15)\qquad \bar{\psi}^p_\nu(\xi, Y) = \sum_{=0}^{\infty}\bar{\psi}^{\mathrm{II}}_{\nu\mu}(\xi)y^\mu = \sum_{\mu=0}^{\infty}\psi^{\mathrm{II}}_{\nu\mu}(\xi)\varepsilon^\mu Y^\mu$$

has been used.

b) in the regions III^A and III^B, respectively

$$(2.16)\quad \begin{aligned} &\overset{\mathrm{III}}{D}_0 = 0,\\ &\bar{A}_0\cosh(\delta\xi)+\bar{B}_0\sinh(\delta\xi) = -(\bar{\psi}^{\mathrm{III}}_{00}-\overline{\overset{\mathrm{III}}{C}}_0)e^{i\alpha\xi},\\ &\bar{A}_1\cosh(\delta\xi)+\bar{B}_1\sinh(\delta\xi) = \overline{\overset{\mathrm{III}}{C}}_1e^{i\alpha\xi},\\ &\bar{\psi}^{\mathrm{III}}_{01}e^{i\alpha\xi}+\bar{A}_0\xi\big(i\alpha\cosh(\delta\xi)-\delta\sinh(\delta\xi)\big)+\bar{B}_0\xi\big(i\alpha\sinh(\delta\xi)-\delta\cosh(\delta\xi)\big) = \overline{\overset{\mathrm{III}}{D}}_1e^{i\alpha\xi}, \end{aligned}$$

where the functions $\bar{\psi}^{\mathrm{III}}_{00}$ and $\bar{\psi}^{\mathrm{III}}_{01}$ are taken from the Taylor-series expansion

$$(2.17)\qquad \bar{\psi}^p_\nu(\xi, y) = \sum_{\mu=0}^{\infty}\bar{\psi}^{\mathrm{III}}_{\nu\mu}(\xi)(1-y)^\mu = \sum\bar{\psi}^{\mathrm{III}}_{\nu\mu}(\xi)\left(\frac{\tilde{Y}}{\varepsilon}\right)^\mu.$$

The results (2.14) and (2.16) constitute a system of 8 equations for 12 unknown functions. If in addition we introduce the four boundary conditions at the lower and upper boundaries, all unknown functions can then be uniquely determined. It is at this stage of the calculation that differences in the solutions in the regions II^A and II^B or III^A and III^B might emerge depending on the mode of bounding. In particular, if the upper and lower edge are the same in the regions A and B, there is no difference in the solutions. Indeed, for *clamped* edges Eq. (1.9c) must hold which implies

$$(2.18)\quad \begin{aligned} &\overset{\mathrm{II}}{C}_0 = \overset{\mathrm{II}}{E}_0 = 0, \quad \overset{\mathrm{II}}{D}_1 = -\beta_y\overset{\mathrm{II}}{C}_1, \quad \overset{\mathrm{II}}{E}_1 = -\overset{\mathrm{II}}{C}_1,\\ &\overset{\mathrm{III}}{C}_0 = \overset{\mathrm{III}}{E}_0 = 0, \quad \overset{\mathrm{III}}{D}_1 = -\beta_y\overset{\mathrm{III}}{C}_1, \quad \overset{\mathrm{III}}{E}_1 = -\overset{\mathrm{III}}{C}_1. \end{aligned}$$

It then follows, with the aid of Eqs. (2.14) and (2.16), from a tedious but straightforward calculation that

$$(2.19)\quad \begin{aligned} &\bar{A}_1 = -\bar{\psi}^{\mathrm{II}}_{00},\\ &\bar{B}_0 = (\bar{\psi}^{\mathrm{II}}_{00}\cosh(\delta\xi)-\bar{\psi}^{\mathrm{III}}_{00}e^{i\alpha\xi})/\sinh(\delta\xi),\\ &\bar{A}_1 = \{\delta\xi(\bar{\psi}^{\mathrm{III}}_{00}e^{i\alpha\xi}-\bar{\psi}^{\mathrm{II}}_{00}\cosh(\delta\xi))-\sinh(\delta\xi)(i\alpha\xi\bar{\psi}^{\mathrm{II}}_{00}+\bar{\psi}^{\mathrm{II}}_{01})\}\{\beta_y\sinh(\delta\xi)\}^{-1}, \end{aligned}$$

(2.19) [cont.]

$$\overline{B}_1 = \{\delta\xi[\overline{\psi}^{\mathrm{II}}_{00}(1+\cosh^2(\delta\xi))-2\overline{\psi}^{\mathrm{III}}_{00}e^{i\alpha\xi}\cosh(\delta\xi)]+i\alpha\xi\sinh(\delta\xi)(\overline{\psi}^{\mathrm{III}}_{00}e^{i\alpha\xi}+\overline{\psi}^{\mathrm{II}}_{00}\times\cosh(\delta\xi))-\sinh(\delta\xi)(\overline{\psi}^{\mathrm{III}}_{01}e^{i\alpha\xi}-\overline{\psi}^{\mathrm{II}}_{01}\cosh(\delta\xi))\}\{\beta_y\sinh^2(\delta\xi)\}^{-1},$$

$$\overset{\mathrm{II}}{\overline{C}}_1 = \overline{A}_1,$$

$$\overset{\mathrm{II}}{\overline{D}}_1 = -\beta_y\overline{A}_1,$$

$$\overset{\mathrm{II}}{\overline{E}}_1 = -\overline{A}_1,$$

$$\overset{\mathrm{III}}{\overline{C}}_1 = \{\delta\xi(\overline{\psi}^{\mathrm{II}}_{01}-\overline{\psi}^{\mathrm{III}}_{00}e^{i\alpha\xi}\cosh(\delta\xi))e^{-i\alpha\xi}+(i\alpha\xi\overline{\psi}^{\mathrm{III}}_{00}-\overline{\psi}^{\mathrm{III}}_{01})\sinh(\delta\xi)\}\{\beta_y\sinh(\delta\xi)\}^{-1},$$

$$\overset{\mathrm{III}}{\overline{D}}_1 = -\beta_y\overset{\mathrm{III}}{\overline{C}}_1,$$

$$\overset{\mathrm{III}}{\overline{E}}_1 = -\overset{\mathrm{III}}{\overline{C}}_1.$$

Thus it has been possible to determine all unknown coefficient functions. There still remains to construct a composite solution and to elaborate on the shear layer. The latter depends on the loading as seen by the fact that no existence has emerged so far. It is therefore advantageous to investigate an example first.

2.4. Example

Let us focus our attention on the line load

$$p(x,y) = p_0\delta(x-x_0)$$

for which the Fourier-transform obtains

$$\overline{p}(\xi) = p_0e^{-i\xi x_0}$$

so that from substituting into Eq. $(2.3)_1$ the following particular solution is deduced:

$$\overline{\psi}^p_0 = \frac{p_0e^{-i\xi x_0}}{\beta_x^2\xi^2} \tag{2.20}$$

and from this we subsequently find

$$\overline{\psi}^{\mathrm{II}}_{00} = \overline{\psi}^{\mathrm{III}}_{00} = \frac{p_0e^{-i\xi x_0}}{\beta_x^2\xi^2}, \qquad \overline{\psi}^{\mathrm{II}}_{\mu\nu} = \overline{\psi}^{\mathrm{III}}_{\mu\nu} = 0, \quad \text{if} \quad \mu \quad \text{or} \quad \nu\neq 0. \tag{2.21}$$

By substitution into Eqs. (2.14) and (2.16) the unknown coefficient functions may be determined and when this is done one obtains

$$\begin{aligned}
\overline{A}_0 &= -p_0e^{-i\xi x_0}\beta_x^{-2}\xi^{-2},\\
\overline{B}_0 &= p_0e^{-i\xi x_0}(\cosh(\delta\xi)-e^{i\alpha\xi})\beta_x^{-2}\xi^{-2}\sinh^{-1}(\delta\xi),\\
\overline{A}_1 &= \frac{p_0e^{-i\xi x_0}}{\beta_x^2\beta_y\xi^2\sinh(\delta\xi)}[\delta\xi(e^{i\alpha\xi}-\cosh(\delta\xi))-i\alpha\xi\sinh(\delta\xi)],\\
\overline{B}_1 &= \frac{p_0e^{-i\xi x_0}}{\beta_x^2\beta_y\xi^2\sinh^2(\delta\xi)}[\delta\xi(1+\cosh^2(\delta\xi)-2e^{i\alpha\xi}\cosh(\delta\xi))\\
&\qquad +i\alpha\xi(e^{i\alpha\xi}+\cosh(\delta\xi))\sinh(\delta\xi)],
\end{aligned} \tag{2.22}$$

(2.22) [cont.]

$$\overset{\text{II}}{E}_1 = -\bar{A}_1,$$

$$\overset{\text{III}}{E}_1 = \frac{p_0 e^{-i\xi x_0}}{\beta_x^2\beta_y\xi^2\sinh(\delta\xi)}\left[\delta\xi\left(\cosh(\delta\xi)-e^{-i\alpha\xi}\right)-i\alpha\xi\sinh(\delta\xi)\right].$$

The result obtained by inserting Eq. (2.22) into Eq. (2.5) can be written as

$$\chi = \chi_0+\varepsilon\xi\chi_1+0(\varepsilon^2), \tag{2.23}$$

where

$$\chi_0 = \frac{p_0}{2\pi\beta_x^2}\int\limits_{-\infty}^{\infty}\frac{e^{i\xi(x-x_0)}}{\xi^2}\left\{\left(1-e^{-i\alpha\xi y}\cosh(\delta\xi y)\right)\right. \left.+\frac{\left(\cosh(\delta\xi)e^{-i\alpha\xi y}-e^{i\alpha\xi(1-y)}\right)}{\sinh(\delta\xi)}\sinh(\delta\xi y)\right\}d\xi$$

and

$$\chi_1 = \frac{p_0}{2\beta_x^2\beta_y\pi}\int\limits_{-\infty}^{\infty}\frac{e^{i\xi(x-x_0)}}{\xi^2}\left\{\left[\delta\xi\left(e^{i\alpha\xi}-\cosh(\delta\xi)\right)-i\alpha\xi\sinh(\delta\xi)\right]\frac{\cosh(\delta\xi y)}{\sinh(\delta\xi)}\right. +\left[\delta\xi\left(1+\cosh^2(\delta\xi)-2e^{i\alpha\xi}\cosh(\delta\xi)\right)\right. \left.\left.+i\alpha\xi\left(e^{i\alpha\xi}+\cosh(\delta\xi)\right)\sinh(\delta\xi)\right]\frac{\sinh(\delta\xi y)}{\sinh^2(\delta\xi)}\right\}e^{-i\alpha\xi y}d\xi.$$

The above integral expressions are best evaluated by contour integration in the complex ξ-plane. The details are somewhat lengthy and we therefore refrain from presenting the pertinent calculations but rather list the results. They are

$$\chi_0 = \frac{p_0\delta}{\beta_x^2}\sum_{m=1}^{\infty}\frac{\sin(m\pi y)}{m^2\pi^2}\begin{cases}e^{-m\pi(x-x_0)/\delta}\left[e^{-m\pi\alpha y/\delta}-(-1)^m e^{m\pi\alpha(1-y)/\delta}\right], & x-x_0<0,\\ e^{m\pi(x-x_0)/\delta}\left[e^{m\pi\alpha y/\delta}-(-1)^m e^{-m\pi\alpha(1-y)/\delta}\right], & x-x_0>0;\end{cases} \tag{2.24}$$

$$\chi_1 = \begin{cases}\dfrac{p_0}{\beta_x^2\beta_y}\displaystyle\sum_{m=1}^{\infty}\left\{\left[2\left(-\frac{\alpha y}{m\pi}+\frac{x-x_0}{m\pi}-\frac{\delta}{m^2\pi^2}\right)\left(1-(-1)^m e^{+m\pi\alpha/\delta}\right)\right.\right.\\ \left.\left.-\dfrac{\alpha}{m\pi}\left((-1)^m e^{m\pi\alpha/\delta}-1\right)\right]\sin m\pi y+\dfrac{\delta}{m\pi}\left(1-(-1)^m e^{m\pi\alpha/\delta}\right)(2y-1)\cos m\pi y\right\}\\ \qquad\times e^{-m\pi[\alpha y-(x-x_0)]/\delta}, \quad x-x_0<0,\\ \dfrac{p_0}{\beta_x^2\beta_y}\displaystyle\sum_{m=1}^{\infty}\left\{\left[2\left(\frac{\alpha y}{m\pi}-\frac{x-x_0}{m\pi}-\frac{\delta}{m^2\pi^2}\right)\left(1-(-1)^m e^{-m\pi\alpha/\delta}\right)\right.\right.\\ \left.+\dfrac{\alpha}{m\pi}\left((-1)^m e^{-m\pi\alpha/\delta}-1\right)\right]\sin m\pi y+\dfrac{\delta}{m\pi}\left(1-(-1)^m e^{-m\pi\alpha/\delta}\right)\\ \qquad\left.\times(2y-1)\cos m\pi y\right\}e^{m\pi[\alpha y-(x-x_0)]/\delta}, \quad x-x_0>0.\end{cases} \tag{2.25}$$

The expressions (2.24) and (2.25) determine the outer solution of the deflection χ. It is seen from these formulae that they are not uniformly valid in x. Indeed, for the above series to be convergent the exponentials in the sums (2.24) and (2.25) must be smaller than 1. This implies

$$(2.26) \qquad (x-x_0) \geqslant \alpha y \quad \text{and} \quad (x-x_0) \leqslant -\alpha(1-y).$$

We thereby conclude that the expressions (2.24) and (2.25) are invalid in the parallelogram of Fig. 1. This implies that there is another layer apparently induced by the line load whose size depends on the coefficient α. As $\alpha \to 0$, this layer (of area α) becomes vanishingly small. According to Eq. (2.6) this means that the shearing prestress is small in comparison to the prestress in the y-direction. For $\alpha = 0$, the only case treated thus far, this layer disappears. Conversely, if α becomes large, then the shearing prestress N_{xy} is much larger than the prestress N_y. In the limit $N_y = 0$, α becomes infinitely large in which case the parallelogram of Fig. 1 covers the entire strip. This case corresponds to a membrane strip that carries vanishing prestress in the y-direction. In our earlier papers this situation was termed the *degenerate* case but could not be explained. On the other hand, when we set α to zero we easily recover results previously derived in our earlier paper (1974).

Another interesting feature of Eqs. (2.24) and (2.25) is the fact that the deflection χ is not symmetric with respect to $(x-x_0)$. Otherwise stated: $\chi(x-x_0) \neq \chi(x_0-x)$. This is due to the presence of the terms depending on α, even though for $\alpha = 0$ we recover symmetry. Of course it is physically obvious that a change in the direction of the shearing prestress must alternate the values of χ at the antipodal points $(x-x_0)$ and (x_0-x), respectively. Changing the sign of α must therefore alternate the expressions that are applicable in the formulae (2.24) and (2.25) for $(x-x_0) > 0$ and $(x-x_0) < 0$. This feature may also serve as partial check of these results.

The inducement of the shear layer is a most interesting fact and may be attributed to the high singular character of the line load distribution. It is therefore conceivable to presume that smoother loading functions might lead to zeroth and first-order outer solutions that are valid for all x. We have, however, found that this is not the case for strip-like loads and for "roof-shaped" loading functions. The results for these cases are easily derived from the above formulae by mere integration. They are presented in the appendix.

Before turning to these shear layers, let us determine the two functions $\overset{\mathrm{II}}{E}_1$ and $\overset{\mathrm{III}}{E}_1$ that govern the boundary layer solution. Their Fourier-transforms are listed in Eq. (2.22). Inversion gives

$$(2.27) \qquad \begin{aligned} \overset{\mathrm{II}}{E}_1 &= \frac{p_0\delta}{\beta_x^2\beta_y}\sum_{m=1}^{\infty}\frac{1}{m\pi}\left[1-(-1)^m e^{-m\pi\alpha/\delta}\right]e^{-m\pi|x-x_0|/\delta}, \\ \overset{\mathrm{III}}{E}_1 &= \frac{p_0\delta}{\beta_x^2\beta_y}\sum_{m=1}^{\infty}\frac{1}{m\pi}\left[1-(-1)^m e^{+m\pi\alpha/\delta}\right]e^{-m\pi|x-x_0|/\delta}. \end{aligned}$$

In contrast to the outer solution the boundary layer solutions are therefore symmetric with respect to $(x-x_0)$. However, in spite of the symmetry of the loading function in

y, the upper and lower boundary layer solutions are different. The reason is again due to the shearing prestress. The transformation $\alpha \to -\alpha$ should alter the solution of the upper boundary layer into that of the lower and *vice versa*. The result (2.27) is surely in agreement with this condition.

In the next subsection we shall need expressions for χ_0 and χ_1 as the lines

$$(x-x_0) = \alpha y \quad \text{and} \quad (x-x_0) = \alpha(y-1)$$

are approached. For $\alpha > 0$ a straightforward calculation shows that by the Taylor-series expansion one obtains

$$(2.28) \qquad \chi_0 \simeq \begin{cases} \dfrac{p_0\delta}{\beta_x^2}\displaystyle\sum_{m=1}^{\infty}\frac{\sin(\pi m y)}{m^2\pi^2}[1-(-1)^m e^{-m\pi\alpha/\delta}] \\ \qquad\qquad \times\left[1-\dfrac{m\pi}{\delta}\left((x-x_0)-\alpha y\right)+\ldots\right], \quad \text{if} \quad x-x_0 \sim \alpha y, \\ \dfrac{p_0\delta}{\beta_x^2}\displaystyle\sum_{m=0}^{\infty}\frac{\sin(m\pi y)}{m^2\pi^2}e^{-m\pi\alpha/\delta}[1-(-1)^m e^{m\pi\alpha/\delta}] \\ \qquad\qquad \times\left[1+\dfrac{m\pi}{\delta}\left((x-x_0)+\alpha(1-y)\right)+\ldots\right], \quad \text{if} \quad x-x_0 \sim \alpha(y-1), \end{cases}$$

and

$$(2.29) \qquad \chi_1 \simeq \begin{cases} \dfrac{p_0\delta}{\beta_x^2\beta_y}\displaystyle\sum_{m=1}^{\infty}\left[-\left(\frac{2}{m^2\pi^2}+\frac{\alpha/\delta}{m\pi}\right)\sin(m\pi y)+\frac{2y-1}{m\pi}\cos(m\pi y)\right] \\ \qquad\qquad \times[1-(-1)^m e^{-m\pi\alpha/\delta}]\left[1+0\left(|\alpha y-(x-x_0)|\right)\right], \quad \text{if} \quad (x-x_0) \geqslant 0, \\ \dfrac{p_0\delta}{\beta_x^2\beta_y}\displaystyle\sum_{m=1}^{\infty}\left[-\frac{2}{m^2\pi^2}\sin(m\pi y)+\frac{2y-1}{m\pi}\cos(m\pi y)\right]e^{-m\pi\alpha/\delta} \\ \qquad\qquad \times[1-(-1)^m e^{-m\pi\alpha/\delta}]\left[1+0\left(|\alpha(y-1)-(x-x_0)|\right)\right], \quad \text{if} \quad (x-x_0) \leqslant 0. \end{cases}$$

2.5. Shear layer solutions

We now turn to the determination of the solution in the layers IV^A and IV^B (see Fig. 1). To this end the governing differential equation must be subjected to a stretching transformation in the x-direction that accounts for the large changes occurring in the neighbourhood of $x = x_0$. We expect different solutions in the regions IV^A and IV^B as well as in the hatched regions (see Fig. 1).

a. Solution in Region IV^A

Introducing the coordinate stretching

$$(2.30) \qquad (x-x_0) = \varepsilon X$$

and the asymptotic expansion

$$(2.31) \qquad \chi = \sum_{\nu=0}^{\infty}\varepsilon^{\nu}\varphi_{\nu}^{\mathrm{IV}^A}(X, y),$$

the differential equation (1.11) with the loading function $p_0\delta(x-x_0)$ may be transformed into the following system of recurrence relations:

$$(2.32)\quad \frac{\partial^4\varphi_\nu^{\mathrm{IV}_A}}{\partial\chi^4}-\beta_x^2\frac{\partial^2\varphi_\nu^{\mathrm{IV}_A}}{\partial X^2}=+2\beta_{xy}^2\frac{\partial^2\varphi_{\nu-1}^{\mathrm{IV}_A}}{\partial X\partial y}+\beta_y^2\frac{\partial^2\varphi_{\nu-2}^{\mathrm{IV}_A}}{\partial y^2}-2\frac{\partial^4\varphi_{\nu-2}^{\mathrm{IV}_A}}{\partial X^2\partial y^2}-\frac{\partial^4\varphi_{\nu-4}^{\mathrm{IV}_A}}{\partial y^4},$$

$$\nu=0,1,\dots,\infty.$$

As before, we apply the convention that functions with a negative index vanish. The solutions to the zeroth and first-order equations read:

$$(2.33)\quad \begin{aligned}\varphi_0^{\mathrm{IV}^A}&=\mathscr{A}_0^A(y)+\mathscr{B}_0^A(y)X+\mathscr{C}_0^A(y)e^{-\beta_xX},\\ \varphi_1^{\mathrm{IV}^A}&=\mathscr{A}_1^A(y)+\mathscr{B}_1^A(y)X+\mathscr{C}_1^A(y)e^{-\beta_xX}-\frac{\beta_{xy}^2}{\beta_x^2}\frac{\partial\mathscr{B}_0^A}{\partial y}X^2+\frac{\beta_{xy}^2}{\beta_x^2}\frac{\partial\mathscr{C}_0^A}{\partial y}Xe^{-\beta_xX},\end{aligned}$$

where $\mathscr{A}_0^A$ through $\mathscr{C}_1^A$ are still to be determined.

b. Solution in region $\mathbf{IV}^B$

In this region we introduce the coordinate stretching

$$(2.34)\quad (x-x_0)=-\varepsilon\tilde{X}$$

together with the asymptotic expansion

$$(2.35)\quad \chi=\sum_{\nu=0}^{\infty}\varepsilon^\nu\varphi_\nu^{\mathrm{IV}^B}(\tilde{X},y),$$

which transforms Eq. (1.11) into the differential equations

$$(2.36)\quad \frac{\partial^4\varphi_\nu^{\mathrm{IV}^B}}{\partial\tilde{X}^4}-\beta_x^2\frac{\partial^2\varphi_\nu^{\mathrm{IV}^B}}{\partial\tilde{X}^2}=-2\beta_{xy}^2\frac{\partial^2\varphi_{\nu-1}^{\mathrm{IV}^B}}{\partial\tilde{X}\partial y}+\beta_y^2\frac{\partial^2\varphi_{\nu-2}^{\mathrm{IV}^B}}{\partial y^2}-2\frac{\partial^4\varphi_{\nu-2}^{\mathrm{IV}^B}}{\partial\tilde{X}^2\partial y^2}-\frac{\partial^4\varphi_{\nu-4}^{\mathrm{IV}^B}}{\partial y^4}$$

whose first-order solutions read

$$(2.37)\quad \begin{aligned}\varphi_0^{\mathrm{IV}^B}&=\mathscr{A}_0^B(y)+\mathscr{B}_0^B(y)\tilde{X}+\mathscr{C}_0^B(y)e^{-\beta_x\tilde{X}},\\ \varphi_\nu^{\mathrm{IV}^B}&=\mathscr{A}_1^B(y)+\mathscr{B}_1^B(y)\tilde{X}+\mathscr{C}_1^B(y)e^{-\beta_x\tilde{X}}+\frac{\beta_{xy}^2}{\beta_x^2}\frac{\partial\mathscr{B}_0^B}{\partial y}\tilde{X}^2-\frac{\beta_{xy}^2}{\beta_x^2}\frac{\partial\mathscr{C}_0^2}{\partial y}\tilde{X}e^{-\beta_x\tilde{X}}.\end{aligned}$$

Here again, $\mathscr{A}_0^B$ through $\mathscr{C}_1^B$ are still unknown functions of y.

c. Determination of the functions $\mathscr{A}_0^A$ *etc.*

We now turn to the determination of the unknown coefficient functions $\mathscr{A}_0^A$ ... ... $\mathscr{C}_1^A, \mathscr{A}_0^B, \dots, \mathscr{C}_1^B$. To this end, the systems (2.33) and (2.37) must be matched with the core solutions χ_0 and χ_1 as X and $\tilde{X}$ grow indefinitely. Moreover, the functions $\chi=\sum_{\nu=0}^{\infty}\varepsilon^\nu\varphi_\nu^{\mathrm{IV}^{A,B}}$ must be joined appropriately at $X=\tilde{X}=0$. The correct conditions are obtained, if one observes that χ as a function of x is of class C^2, while the third derivative suffers a finite jump at $x=x_0$. This jump condition reads

$$(2.38)\quad \varepsilon^2\left\{\frac{\partial^3\chi}{\partial x^3}\bigg|_{x=+x_0}-\frac{\partial^3\chi}{\partial x^3}\bigg|_{x=-x_0}\right\}=p_0$$

and is obtained from an integration of Eq. (1.11) between $-x_0$ and $+x_0$. Introducing the shear layer variables (2.30) and (2.34) and the expressions (2.31) and (2.35), respectively, the above condition requires that

$$\frac{\partial^3\varphi_\nu^{\mathrm{IV}^A}}{\partial X^3}+\frac{\partial^3\varphi_\nu^{\mathrm{IV}^B}}{\partial\tilde{X}^3}=\begin{cases}0, & \nu=0,\\ p_0, & \nu=1,\\ 0, & \nu\geqslant 2.\end{cases}\tag{2.39}$$

Since χ is of class C^2 at $x=x_0$, one also has

$$\begin{aligned}&\varphi_\nu^{\mathrm{IV}^A}-\varphi_\nu^{\mathrm{IV}^B}=0,\\ &\frac{\partial\varphi_\nu^{\mathrm{IV}^A}}{\partial X}+\frac{\partial\varphi_\nu^{\mathrm{IV}^B}}{\partial\tilde{X}}=0, \quad \text{for } X=\tilde{X}=0,\\ &\frac{\partial^2\varphi_\nu^{\mathrm{IV}^A}}{\partial X^2}-\frac{\partial^2\varphi_\nu^{\mathrm{IV}^B}}{\partial\tilde{X}^2}=0.\end{aligned}\tag{2.40}$$

Equations (2.39) and (2.40) form 4ν equations for the determination of the free coefficient functions $\mathscr{A}_\nu^{A,B}$ through $\mathscr{C}_\nu^{A,B}$. It is a simple matter to prove that they imply

$$\begin{aligned}&\mathscr{A}_0^A=\mathscr{A}_0^B, \quad \mathscr{B}_0^A=-\mathscr{B}_0^B, \qquad \mathscr{C}_0^A=\mathscr{C}_0^B=0,\\ &\mathscr{A}_1^A=\mathscr{A}_1^B, \quad \mathscr{B}_1^A=-\mathscr{B}_1^B-\frac{p_0}{\beta_x^2}, \quad \mathscr{C}_1^A=\mathscr{C}_1^B=-\frac{p_0}{2\beta_x^3}\end{aligned}\tag{2.41}$$

so that the zeroth and first-order solutions now read as follows:

$$\begin{aligned}&\varphi_0^{\mathrm{IV}^A}=\mathscr{A}_0^A+\mathscr{B}_0^A X,\\ &\varphi_1^{\mathrm{IV}^A}=\mathscr{A}_1^A+\mathscr{B}_1^A X-\frac{p_0}{2\beta_x^3}e^{-\beta_x X}-\frac{\beta_{xy}^2}{\beta_x^2}\frac{\partial\mathscr{B}_0^A}{\partial y}X^2,\end{aligned}\tag{2.42}$$

and

$$\begin{aligned}&\varphi_0^{\mathrm{IV}^B}=\mathscr{A}_0^A-\mathscr{B}_0^A\tilde{X},\\ &\varphi_1^{\mathrm{IV}^B}=\mathscr{A}_1^A-\left\{\mathscr{B}_1^A+\frac{p_0}{\beta_x^2}\right\}\tilde{X}-\frac{p_0}{2\beta_x^3}e^{-\beta_x\tilde{X}}-\frac{\beta_{xy}^2}{\beta_x^2}\frac{\partial\mathscr{B}_0^A}{\partial y}\tilde{X}^2.\end{aligned}\tag{2.43}$$

In deriving Eqs. (2.41) and (2.42) we have also assumed that $p_0\neq p_0(y)$.

It remains to match the above solutions with the outer solutions of regions I^A and I^B, respectively. In particular, this matching must be carried out as $\tilde{X}\to\infty$ along the line $(x-x_0)=\alpha y$ at the edge of region I^A and as $\tilde{X}\to\infty$ along the line $(x-x_0)=-\alpha(1-y)$ at the edge of region I^B. The first few terms of the outer expansions near these lines are listed in Eqs. (2.28) and (2.29). A straightforward two-term matching along these lines then gives

$$\mathscr{B}_0^A\equiv 0,\tag{2.44}$$

$$\mathscr{A}_0^A=\frac{p_0\alpha(1-y)y}{\beta_x^2}+\frac{\delta p_0}{\beta_x^2}\sum_{m=2,4,6}^{\infty}\frac{2y\sin(m\pi y)}{m^2\pi^2}\left(e^{-m\pi\alpha/\delta}-1\right)+\frac{\delta p_0}{\beta_x^2}\sum_{m=1,3,5}^{\infty}\frac{\sin(m\pi y)}{m^2\pi^2}\left(1-(-1)^m e^{-m\pi\alpha/\delta}\right),$$

$$\mathscr{A}_1^A = -\frac{p_0}{\beta_x^2\beta_y}\sum_{m=1}^{\infty}\left(1-(-1)^m e^{-m\pi\alpha/\delta}\right)\left[\left(\frac{2\delta}{m^2\pi^2}+\frac{\alpha}{m\pi}\right)\sin(m\pi y)\right.$$

$$\left.-\frac{\delta}{m\pi}(2y-1)\cos(m\pi y)\right],$$

$$\mathscr{B}_1^A = -\frac{p_0(1-y)}{\beta_x^2} - \frac{\delta p_0}{\alpha\beta_x^2}\sum_{m=2,4,6}^{\infty}\frac{2\sin(m\pi y)}{m^2\pi^2}\left(e^{-m\pi\alpha/\delta}-1\right).$$

This completes the construction of the solution up to order ε-terms.

Concluding remarks

In this paper we have investigated the response of a thick membrane strip to static loadings for the case when the membrane forces contain a contribution due to shearing prestress. Our main interest was to determine the influence of a small bending rigidity and the mathematical technique to account for it was the method of matched asymptotic expansions. We found that the prestress conditions dictate to a large degree the boundary layer structure. In fact, we found that the existence of shearing prestress parallel to the side walls resulted in what we called shear layers. These layers are induced by the external loading function and occur away from the boundaries. The prestress conditions considered here embrace all cases of constant prestress from isotropic prestress to uniaxial prestress in a preferred direction. If the direction of the latter is parallel to the strip wall, the entire strip consists of the shear layer and the solution becomes invalid. This degeneracy was already observed earlier.

Of course, the problem treated in this paper is to a certain extent academical; it only deals with static solutions and excludes dynamic effects. From a practical point of view such effects are more interesting. Their treatment is complex, however, so that we shall present the corresponding solutions in a different paper (Part II).

Appendix

The purpose of this appendix is to demonstrate that the existence of the shear layer is preserved even if the loading functions are of class C^1. We shall list the zeroth and first-order solutions to strip-like loading functions and to roof-shaped loading functions. The solutions are obtained by merely integrating Eqs. (2.24) and (2.25). The calculations are tedious, even though they are straightforward and for that reason we only list the results:

(i) for the strip-like loading function

$$p(x) = p = \text{constant}, \quad |x| \leqslant a$$

the solution reads

$$
\chi_0 = \begin{cases}
\dfrac{2p_0\delta^2}{\pi^3\beta_x^2}\displaystyle\sum_{m=1}^{\infty}\left\{\dfrac{\sin(m\pi y)\,\mathrm{Sh}\left(\dfrac{m\pi a}{\delta}\right)}{m^3}\left[e^{-\frac{m\pi\alpha y}{\delta}}-(-1)^m e^{+\frac{m\pi\alpha(1-y)}{\delta}}\right]\right\}e^{\frac{m\pi x}{\delta}}, \quad x<-a,\\[2ex]
\dfrac{2p_0\delta^2}{\pi^3\beta_x^2}\displaystyle\sum_{m=1}^{\infty}\dfrac{\sin(m\pi y)}{m^3}\Bigg\{\mathrm{Ch}\left(\dfrac{m\pi\alpha y}{\delta}\right)-(-1)^m\mathrm{Ch}\left(\dfrac{m\pi\alpha}{\delta}(1-y)\right)\\
\qquad -e^{-\frac{m\pi\alpha}{\delta}}\left[\mathrm{Ch}\left(\dfrac{m\pi}{\delta}(x-\alpha y)\right)-(-1)^m\mathrm{Ch}\left(\dfrac{m\pi}{\delta}(x+\alpha(1-y))\right)\right]\Bigg\},\\
\qquad\qquad -a\leqslant|x|\leqslant+a,\\[2ex]
\dfrac{2p_0\delta^2}{\pi^2\beta_x^2}\displaystyle\sum_{m=1}^{\infty}\left\{\dfrac{\sin(m\pi y)\,\mathrm{Sh}\left(\dfrac{m\pi a}{\delta}\right)}{m^3}\left[e^{\frac{m\pi\alpha y}{\delta}}-(-1)^m e^{-\frac{m\pi\alpha}{\delta}(1-y)}\right]\right\}e^{-\frac{m\pi x}{\delta}}, \quad x>a,
\end{cases}
$$

$$
\chi_1 = \begin{cases}
\dfrac{p_0\delta}{\beta_x^2\beta_y}\displaystyle\sum_{m=1}^{\infty}\Bigg\{\Bigg\{\Bigg[-2\left(\dfrac{\alpha y}{m\pi}+\dfrac{\delta}{m^2\pi^2}\right)\left(1-(-1)^m e^{\frac{m\pi\alpha}{\delta}}\right)-\dfrac{\alpha}{m\pi}\left((-1)^m e^{\frac{m\pi\alpha}{\delta}}-1\right)\Bigg]\\
\quad\times\sin(\pi m y)+\dfrac{\delta}{m\pi}\left(1-(-1)^m e^{\frac{m\pi\alpha}{\delta}}\right)(2y-1)\cos(m\pi y)\Bigg\}\,2\exp\left(-\dfrac{m\pi}{\delta}(\alpha y-x)\right)\\
\quad\times\mathrm{Sh}\left(\dfrac{m\pi a}{\delta}\right)+\dfrac{2}{m\pi}\left(1-(-1)^m e^{-\frac{m\pi\alpha}{\delta}}\right)\sin(m\pi y)\,e^{-\frac{m\pi\alpha y}{\delta}}\Bigg\{(x-a)\,e^{\frac{m\pi}{\delta}(x-a)}\\
\qquad -(x-a)\,e^{\frac{m\pi}{\delta}(x-a)}+\dfrac{\delta}{m\pi}e^{\frac{m\pi x}{\delta}}\,\mathrm{Sh}\left(\dfrac{m\pi a}{\delta}\right)\Bigg\}\Bigg\}, \quad x<-a,\\[2ex]
\dfrac{p_0}{\beta_x^2\beta_y}\Bigg\langle\displaystyle\sum_{m=1}^{\infty}\dfrac{2\delta}{m^2\pi^2}\sin(m\pi y)\Bigg\{\Bigg\{\alpha(2y-1)\,\mathrm{Sh}\left(\dfrac{m\pi\alpha y}{\delta}\right)+(-1)^m\alpha(2y-1)\\
\quad\times\mathrm{Sh}\left(\dfrac{m\pi\alpha(1-y)}{\delta}\right)+2\,\mathrm{Ch}\left(\dfrac{m\pi\alpha y}{\delta}\right)\left[\dfrac{\delta}{m\pi}-1\right]+2(-1)^m\mathrm{Ch}\left(\dfrac{m\pi\alpha(1-y)}{\delta}\right)\left[\dfrac{\delta}{m\pi}+1\right]\Bigg\}\\
\qquad +e^{-\frac{m\pi a}{\delta}}\Bigg\{-\alpha(2y+1)\,\mathrm{Sh}\left(\dfrac{m\pi}{\delta}(\alpha y-x)\right)-(-1)^m\alpha(2y+1)\\
\qquad\times\mathrm{Sh}\left(\dfrac{m\pi}{\delta}(\alpha(1-y)+x)\right)+2\,\mathrm{Ch}\left(\dfrac{m\pi}{\delta}(\alpha y-x)\right)\left[\dfrac{\delta}{m\pi}+1\right]+2(-1)^m\\
\quad\times\mathrm{Ch}\left(\dfrac{m\pi}{\delta}(\alpha(1-y)+x)\right)\left[\dfrac{\delta}{m\pi}-1\right]+x\left[\mathrm{Ch}\left(\dfrac{m\pi}{\delta}x\right)-(-1)^m\mathrm{Ch}\left(\dfrac{m\pi}{\delta}(x+\alpha)\right)\right]\\
\qquad -a\left[\mathrm{Sh}\left(\dfrac{m\pi}{\delta}x\right)-(-1)^m\mathrm{Sh}\left(\dfrac{m\pi}{\delta}(x+\alpha)\right)\right]\Bigg\}\Bigg\}
\end{cases}
$$

$$+\sum_{m=1}^{\infty}\frac{2\delta^2}{m^2\pi^2}(2y-1)\cos(m\pi y)\left\{\mathrm{Ch}\left(\frac{m\pi\alpha}{\delta}y\right)-(-1)^m\mathrm{Sh}\left(\frac{m\pi\alpha}{\delta}(1-y)\right)\right.$$

$$\left.\left.-e^{-\frac{m\pi a}{\delta}}\left[\mathrm{Sh}\left(\frac{m\pi}{\delta}(\alpha y+x)\right)+(-1)^m\mathrm{Sh}\left(\frac{m\pi}{\delta}(\alpha(1-y)+x)\right)\right]\right\}\right\rangle,\quad -a\leqslant x\leqslant a$$

$$\frac{p_0\delta}{\beta_x^2\beta_y}\sum_{m=1}^{\infty}\left\{\left\{\left[2\left(\frac{\alpha y}{m\pi}-\frac{\delta}{m^2\pi^2}\right)\left(1-(-1)^m e^{-\frac{m\pi\alpha}{\delta}}\right)+\frac{\alpha}{m\pi}\left((-1)^m e^{-\frac{m\pi\alpha}{\delta}}-1\right)\right]\right.\right.$$

$$\left.\times\sin(m\pi y)+\frac{\delta}{m\pi}\left(1-(-1)^m e^{-\frac{m\pi\alpha}{\delta}}\right)(2y-1)\cos(m\pi y)\right\}2\exp\left(\frac{m\pi}{\delta}(\alpha y-x)\right)$$

$$\times\mathrm{Sh}\left(\frac{m\pi a}{\delta}\right)-\frac{2}{m\pi}\left(1-(-1)^m e^{-\frac{m\pi\alpha}{\delta}}\right)\sin(m\pi y)e^{\frac{m\pi\alpha y}{\delta}}\left\{(x-a)e^{-\frac{m\pi}{\delta}(x-a)}\right.$$

$$\left.\left.-(x+a)e^{-\frac{m\pi}{\delta}(x+a)}+2\frac{\delta}{m\pi}e^{\frac{m\pi x}{\delta}}\mathrm{Sh}\left(\frac{m\pi a}{\delta}\right)\right\}\right\},\quad x>a.$$

By mere inspection we see that the above series expansions are convergent everywhere in the strip except in the shear layers

$$\left\{\begin{matrix}(x-a)-\alpha y\leqslant a\\ x>a\end{matrix}\right\}\quad\text{and}\quad\left\{\begin{matrix}(x+a)-\alpha(1-y)\leqslant 0\\ x<-a\end{matrix}\right\};$$

(ii) for the roof-shaped loading function

$$p=\begin{cases}p_0\left(1-\dfrac{x}{a}\right), & 0\leqslant x\leqslant a,\\[2mm] p_0\left(1+\dfrac{x}{a}\right), & 0\geqslant x\geqslant -a\end{cases}$$

no new features are observed and it is indeed easy to show that the shear layers lie in the same region.

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