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Displacement description of dislocation lines
II. Application of cyclic functions

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CYCLIC functions introduced in [1] are applied to the description of dislocation lines. Displacement equations of a medium are formulated in the class of cyclic functions. Two essentially different solutions are given for the problem of moving dislocations, corresponding to the cyclic functions $\Omega_{(3)}(\mathbf{x}, t)$ (t being a parameter) and $\Omega_{(4)}(\mathbf{x}, t)$ defined in a time-space. It may be demonstrated that the known solutions [7] and [8], treated as representants of a cyclic functions, are particular cases of a general solution (4.24), (4.23), while the formula given in [6] may be written in the form (4.18).

Podano zastosowanie funkcji cyklicznych wprowadzonych w pracy [1] do opisu linii dyslokacji. Sformułowano przemieszczeniowe równania ośrodka w klasie funkcji cyklicznych. Podano dwa istotnie różne rozwiązania dla dyslokacji ruchomych, odpowiadające funkcjom cyklicznym $\Omega_{(3)}(\mathbf{x}, t)$, gdzie czas jest parameterm oraz funkcji $\Omega_{(4)}(\mathbf{x}, t)$ określonej w czasoprzestrzeni. Można wykazać, że dotychczas znane rozwiązania [7 i 8] traktowane jako reprezentanty funkcji cyklicznej są szczególnymi przypadkami ogólnego rozwiązania danego wzorami (4.24) i (4.23), zaś wzór z pracy [6] można przedstawić w postaci (4.18).

Дается применение циклических функций, введенных в работе [1], для описания линии дислокаций. Сформулированы уравнения среды в перемещениях в классе циклических функций. Приведены два существенно разные решения для подвижных дислокаций, отвечающие циклическим функциям $\Omega_{(3)}(\mathbf{x}, t)$, где время является параметром и $\Omega_{(4)}(\mathbf{x}, t)$, определенной в пространстве-времени. Можна показать, что известные до сих пор решения [7] и [8], трактованные как представители циклической функции, являются частными случаями общего решения данного формулами (4.24) и (4.23), формулу же из работы [6] можно представить в виде (4.18).

1. Introduction

THIS PAPER is a continuation of paper [1], which will be referred to as Part I. The cyclic function Ω constructed there satisfies in a n-dimensional metric space the commutativity condition of the mixed second derivatives, i.e.

$$(1.1) \qquad \in^{\alpha_1\alpha_2\ldots\alpha_n}\nabla_{\alpha_{n-1}}\nabla_{\alpha_n}\Omega_{(n)} = \oint\limits_{S_{(n-2)}} dS^{\alpha_1\alpha_2\ldots\alpha_{n-2}}\delta(\mathbf{x}-\boldsymbol{\zeta}) = J^{\alpha_1\ldots\alpha_{n-2}}.$$

Here $S_{(n-2)}$ is the $(n-2)$-dimensional closed surface constituting the boundary of an oriented $(n-1)$-dimensional surface. These funtions will be used for constructing the displacement field produced by a dislocation line in a linear elastic medium.

2. Displacement equations

The equations describing the action of a dislocation line in a linear elastic body consists of the homogeneous equations of motion of the medium

$$(2.1) \qquad \nabla_j\sigma^{ij} - \varrho\ddot{u}^i = 0,$$

the constitutive equations

$$\sigma^{ij} = C^{ijkl}\varepsilon_{kl}, \tag{2.2}$$

the geometric equations

$$\varepsilon_{kl} = \nabla_{(k}u_{l)}, \tag{2.3}$$

and the Burgers condition

$$\oint_B d\mathbf{u} = \mathbf{b}, \quad \oint_B \nabla_i u_l dx^i = b_l. \tag{2.4}$$

The differential counterpart of the Burgers condition $(2.4)_2$ is obtained by applying the Stokes theorem and the identity

$$\int_S dS_i(x) \int_L \delta(x-x')dL^i(x') = \begin{cases} 1 & \text{if } L \text{ pierces } S \text{ in positive direction,} \\ -1 & \text{if } L \text{ pierces } S \text{ in negative direction,} \\ 0 & \text{if } L \text{ does not pierce } S. \end{cases} \tag{2.5}$$

Assuming $L = D$, $S = S_B$ where S_B is an arbitrary open surface based on the Burgers circuit B, and D — the dislocation loop, we obtain from Eq. (2.4)

$$\in^{kij}\nabla_i\nabla_j u_l = b_l \oint_D d\zeta^k \delta(\mathbf{x}-\boldsymbol{\zeta}) = b_l t^k. \tag{2.6}$$

Inserting Eq. (2.3) into Eq. (2.2) and the latter into Eq. (2.1), we should remember that the solutions are sought for in the class of cyclic functions whose second derivatives do not commute, the displacement equations of motion will then take the form

$$C^{ijkl}\nabla_i\nabla_k u_l - \varrho\ddot{u}^j = 0. \tag{2.7}$$

These equations together with the Burgers condition (2.6) constitute a complete set of equations describing the dislocation line D in a linear elastic medium. Contrary to what could be expexted, by replacing Eq. (2.4) with Eq. (2.6) we do not introduce any new conditions (three equations (2.4) are replaced with nine equations (2.6)) since with $\mathbf{u} = \mathbf{b}\Omega + \mathring{\mathbf{u}}$ (Ω — cyclic function, $\mathring{u}^i$ — generalized functions) the condition (2.6) is reduced to

$$\in^{ijk}\nabla_i\nabla_j\Omega = t^k. \tag{2.8}$$

This condition is satisfied identically, e.g. by the cyclic function $\Omega = ||\omega_{S(k)}|| \cdot \frac{1}{4\pi}$; its representant is given by the formula (I.2.8) (Eq. (2.8) in Part I).

In the theory of elasticity of isotropic bodies the vectorial form of the Lamé equations (2.7) is frequently encountered and, namely,

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\,\text{grad div}\,\mathbf{u} - \varrho\ddot{\mathbf{u}} = 0,$$

(2.9) or

$$(\lambda+\mu)\,\text{grad div}\,\mathbf{u} - \mu\,\text{rot rot}\,\mathbf{u} - \varrho^{\cdot\cdot} = 0,$$

which is equivalent to Eqs. (2.7) for

$$C^{ijkl} = \lambda g^{ij}g^{kl} + \mu(g^{ik}g^{jl} + g^{il}g^{jk}). \tag{2.10}$$

After substituting Eq. (2.10) in Eq. (2.7) we obtain

$$L_j = \mu\nabla^2 u_j + \lambda\nabla_j\nabla_k u^k + \mu\nabla_k\nabla_j u^k - \varrho\ddot{u}_j = 0.$$

In the indicial notation Eqs. (2.9) have the form

$$\bar{L}_j = \mu\nabla^2 u_j + (\lambda+\mu)\nabla_j\nabla_k u^k - \varrho\ddot{u}_j = 0,$$

$$(2.11) \quad \bar{\bar{L}}_j = (\lambda+2\mu)\nabla_j\nabla_k u^k - \mu g^{qs}\epsilon_{jqp}\epsilon^{plm}\nabla_s\nabla_l u_m - \varrho\ddot{u}_j$$
$$= (\lambda+2\mu)\nabla_j\nabla_k u^k - \mu(\delta^l_j g^{sm} - \delta^m_j g^{ls})\nabla_s\nabla_l u_m - \varrho\ddot{u}_j$$
$$= \mu\nabla^2 u_j + (\lambda+2\mu)\nabla_j\nabla_k u^k - \mu\nabla_k\nabla_j u^k - \varrho\ddot{u}_l = 0.$$

It is now seen that if $\mathbf{u}$ is a cyclic function, not only Eqs. (2.9) are not equivalent to Eqs. (2.7), but also the two equations (2.9) are not equivalent to each other. The following relation holds true:

$$(2.12) \quad \bar{L}_j = L_j - \mu(\nabla_j\nabla_k - \nabla_k\nabla_j)u^k = \bar{\bar{L}}_j - 2\mu(\nabla_j\nabla_k - \nabla_k\nabla_j)u^k$$
$$= L_j - \mu\epsilon_{jkl}b^k t^l = \bar{\bar{L}}_j - 2\mu\epsilon_{jkl}b^k t^l.$$

One property of the field equations (2.7) is important for computational reasons. Since the Burgers condition (2.6) is not explicitly dependent on time or the material constants, the knowledge of only one cyclic function Ω satisfying Eq. (2.8), i.e. $\epsilon^{ljk}\nabla_l\nabla_j\Omega = t^k$, reduces the problem of solution of the Lamé system of equations (2.7) to the classical problem of elasticity in the domain of generalized functions. Namely, once the "statical" cyclic function $\Omega(\mathbf{x}; D)$ and the motion of dislocation line $D(t)$, $\boldsymbol{\zeta} = \boldsymbol{\zeta}(l, t)$ are known, the cyclic function $\Omega(\mathbf{x}, t)$ (time being a parameter) is constructed by means of variation of the $\boldsymbol{\zeta}$ as the function of time

$$(2.13) \qquad \Omega(\mathbf{x}) = \oint_D \boldsymbol{\omega}(\mathbf{x}, \boldsymbol{\zeta}(l)) \cdot \mathbf{d}\boldsymbol{\zeta}, \qquad \Omega(\mathbf{x}; t) = \oint_{D(t)} \boldsymbol{\omega}(\mathbf{x}, \boldsymbol{\zeta}(l, t)) \cdot \mathbf{d}\boldsymbol{\zeta}.$$

This function satisfies the condition (2.8). The solution of Eqs. (2.7) is assumed in the form

$$(2.14) \qquad \mathbf{u}(\mathbf{x}, t) = \mathbf{b}\Omega(\mathbf{x}; t) + \mathring{\mathbf{u}}(\mathbf{x}, t).$$

Here $\mathring{\mathbf{u}}(\mathbf{x}, t)$ is the distribution to be determined from the equation obtained by substituting Eqs. (2.14) into Eqs. (2.7)

$$(2.15) \qquad C^{ijkl}\nabla_i\nabla_k\mathring{u}_l - \varrho\partial_t^2\mathring{u}^j = A^j,$$

where

$$(2.16) \qquad A^j = -C^{ijkl}b_l\nabla_i\nabla_k\Omega + b^j\partial_t^2\Omega$$

is a known distribution. The solution of Eq. (2.15) is possible if the Green tensor of the Lamé operator is known.

In the approach presented here the cyclic component of the displacement produced by a moving dislocation is independent of the line motion history. Displacement $\mathbf{u}$ consists of two parts, the first one representing a "photograph" of the actual state which depends exclusively on the configuration of the pair: "observer $\mathbf{x}$ — line $D(t)$", while the other, distributional part $\mathring{\mathbf{u}}$ depends on the entire motion history

$$(2.17) \qquad \mathring{\mathbf{u}}(\mathbf{x}, t) = -\int_{-\infty}^{t} d\tau \int_{V\infty} dV(\boldsymbol{\xi})\mathbf{G}(\mathbf{x}-\boldsymbol{\xi}, t-\tau) \cdot \mathbf{A}(\boldsymbol{\xi}, \tau),$$

$\mathbf{G}$ is here the dynamical Green tensor of the Lamé operator $\mathbf{L}$.

The displacement accompanying the dislocation may assume another form if Ω is replaced with a four-dimensional cyclic function $\Omega_{(4)}$, this requires a new interpretation and new forms of individual terms in the displacement. This problem will be dealt with in the following section.

3. Dislocation description in time-space

In order to describe a dislocation in a four-dimensional Minkowskian time-space V_4, we must, first of all, define it properly. In the three-dimensional description, when t is a parameter, the motion of the dislocation loop (configuration of $D(t)$ at each instant t) is prescribed. The loop $D(t)$ is a dislocation loop if the corresponding displacement of the medium satisfies the Burgers condition

$$\oint_B \mathbf{du} = \mathbf{b} \quad \text{for every } t. \tag{3.1}$$

In the four-dimensional description we must prescribe, instead of the configuration $D(t)$, the two-dimensional surface $S_{(2)}$ representing in V_4 the motion history of the loop $D(t)$. $S_{(2)}$ is determined by two tangent vectors $\mathbf{l}$ and $\boldsymbol{\lambda}$, $\mathbf{l}$ being a vector tangent to the line $D(t)$ and lying in E^3, and $\boldsymbol{\lambda}$ representing the four-velocity vector of the points of the surface $S_{(2)}$.

The Burgers condition has in this case the same form as (3.1); it should be remembered that the Burgers circuit is a closed curve in V_4 embracing once the surface $S_{(2)}$. The curve B in the case of a plane surface $S_{(2)}$ lying in the hyperplane $x^3 = a$ (Fig. 1) is shown in Fig. 2 on the cross-section $x^1 = \mathring{x}^1$. The differential $\mathbf{du}$ occurring in Eq. (3.1) has now the form $du_i = \nabla_\alpha u_i dx^\alpha$, and hence the Burgers condition may be written as [2]

$$\oint_B \nabla_\alpha u_i dx^\alpha = b_i. \tag{3.2}$$

The differential counterpart of this condition has the form

$$\epsilon^{\alpha\beta\gamma\delta}\nabla_\gamma\nabla_\delta u_i = b_i J^\alpha \tag{3.3}$$

with

$$J^{\alpha\beta} = \oint_{S_{(2)}} dS^{\alpha\beta}\delta_{(4)}(\mathbf{x}-\boldsymbol{\zeta}). \tag{3.4}$$

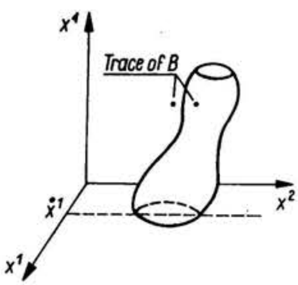

FIG. 1.

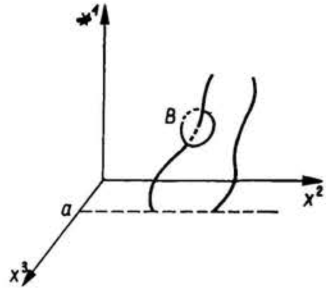

FIG. 2.

With $\alpha = 4$ Eq. (3.3) yields

$$\in^{ljk}\nabla_j\nabla_k u_l = b_l t^l, \tag{3.5}$$

what is the condition (2.6) from the three-dimensional description, and with $\alpha = l, \beta = j$

$$\in^{ljk}(\nabla_k\nabla_4 - \nabla_4\nabla_k)u_l = b_l J^{lj}. \tag{3.6}$$

Equation (3.6) corresponds to the well-known compatibility condition for distortion $\boldsymbol{\beta}$ and velocities $\mathbf{v}$ of the points of the medium:

$$\partial_t\beta_{ik} - \nabla_k v_i = \hat{J}_{ik}.$$

Here $\hat{\mathbf{J}}$ is the dislocation current tensor.

Assuming that $\mathbf{u} = \mathbf{b}\Omega_{(4)} + \mathring{\mathbf{u}}$, $\Omega_{(4)}$ being the cyclic function determined by Eq. (I.3.11), it may be seen that the Burgers condition (3.3) transforms into Eq. (1.1) for $n = 4$. Since the assumed cyclic function $\Omega_{(4)}$ satisfies the condition identically, it remains to determine the distribution $\mathring{\mathbf{u}}$ either from the field equations (2.7), or from Eq. (2.17), where now $A^j = -C^{ijkl}b_l\nabla_i\nabla_k\Omega_{(4)} - \varrho b^j\partial_t^2\Omega_{(4)}$.

From Eq. (3.3) it is seen that the tensor $\mathbf{J}$ must satisfy the equation

$$\nabla_\beta J^{\alpha\beta} = 0 \tag{3.7}$$

since

$$dS^{\alpha\beta} = \in^{\alpha\beta\gamma\delta}n_\gamma m_\delta dS_{(2)} = \frac{1}{2}\in^{\alpha\beta\gamma\delta}\in_{\gamma\delta\mu\nu}l^\mu\lambda^\nu d\tau dl. \tag{3.8}$$

Inserting this into Eq. (3.4) we obtain

$$\begin{aligned} J^{ij} &= \frac{1}{c}\oint_{D(t)}\in^{kij}\in_{kmn}\dot{\zeta}^m\delta\big(\mathbf{x}-\boldsymbol{\zeta}(l,t)\big)d\zeta^n, \\ J^{4j} &= \oint_{D(t)}d\zeta^j\delta\big(\mathbf{x}-\boldsymbol{\zeta}(l,t)\big). \end{aligned} \tag{3.9}$$

Equation (3.7) is known in the dislocation theory as the so-called continuity equation for the dislocation density tensor and the dislocation current tensor.

Let us now write Eqs. (3.7) for $\alpha = 4$ and $\alpha = i$ by introducing the classical notations for the dislocation density tensor $\boldsymbol{\alpha}$ and the dislocation current tensor $\hat{\mathbf{J}}$:

$$\alpha^{ij} = b^i t^j,$$

$$\hat{J}^{ij} = b^i\in^{jmn}\oint_{D(t)}\dot{\zeta}_m\delta(\mathbf{x}-\boldsymbol{\zeta})d\zeta_n.$$

With these notations Eq. (3.7) for $\alpha = 4$ takes the form

$$\nabla_j\alpha^{ij} = 0 \quad \text{then} \quad t^j(\nabla_j b^i) + b^i(\nabla_j t^j) = 0. \tag{3.10}$$

This condition states that for a constant Burgers vector ($\nabla_j b_i = 0$) the dislocation line must be either closed ($\operatorname{div}\mathbf{t} = 0$) or terminate at the boundary of the body. And, conversely, the assumption that the dislocation line is closed leads to the conclusion that the Burgers vector $\mathbf{b}$ is constant.

With $\alpha = i$ we obtain from Eq. (3.7)

$$\frac{\partial\alpha_{i\cdot}^{\cdot j}}{\partial t} - \in^{jmk}\nabla_m\hat{J}_{ik} = 0 \tag{3.11}$$

what, together with the condition $\mathbf{b} = \text{const}$, yields

$$\frac{\partial t^i}{\partial t} - \epsilon^{ijk}\nabla_j j_k = 0, \quad j_k = \oint\limits_{D(t)} \epsilon_{kmn}\dot{\zeta}^m \delta(\mathbf{x}-\boldsymbol{\zeta})d\zeta^n. \tag{3.12}$$

This equation ensures the condition that in the half-space $t \leqslant t'$ (t' — actual time) the surface $S_{(2)}$ intersects the hyperplanes $x^i = \text{const}$ along closed curves. The geometric sense of this condition reduces to the conclusion that the dislocation has always existed and will never vanish. In other words, the model assumed does not involve the process of creation or annihilation of dislocations. In the case when the loop moves in a certain finite time interval (t_1, t_2), the world tube of the dislocation loop is shown in Fig. 3. Figure 4 corresponds to a statical dislocation.

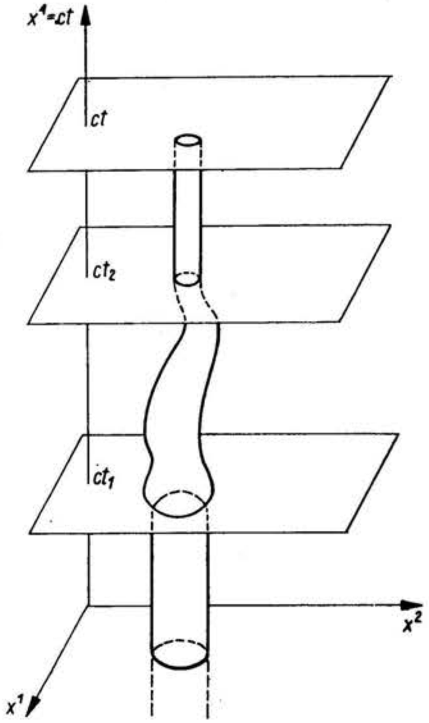

Fig. 3.

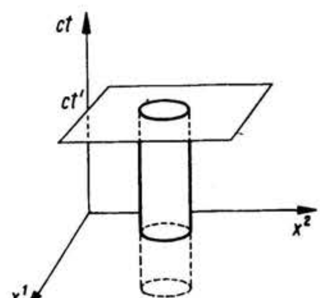

Fig. 4.

4. Comparison of the spatial and time-spatial descriptions

From the foregoing considerations it is known that displacement of the medium produced by a discrete dislocation line may be described by means of the cyclic function $\Omega_{(3)}$ defined in $E^3 \times T$ ($t \in T$):

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{b}\Omega_{(3)}(\mathbf{x}; t) + \mathring{\mathbf{u}}(\mathbf{x}, t) \tag{4.1}$$

and the cyclic function $\Omega_{(3)}$ satisfies identically the condition

$$\epsilon^{ijk}\nabla_j\nabla_k\Omega_{(3)} = t^i = \oint\limits_{D(t)} d\zeta^i \delta(\mathbf{x}-\boldsymbol{\zeta}). \tag{4.2}$$

Applying the cyclic function $\Omega_{(4)}$ defined in the time-space V_4, we obtain

(4.3) $$\mathbf{u}(\mathbf{x}, t) = \mathbf{b}\Omega_{(4)}(\mathbf{x}, t)+\hat{\mathbf{u}}(\mathbf{x}, t).$$

The function Ω_4 satisfies identically the condition

(4.4) $$\epsilon^{\alpha\beta\gamma\delta}\nabla_\gamma\nabla_\delta\Omega_{(4)} = J^{\alpha\beta} = \int_{S_{(2)}} dS^{\alpha\beta}\delta_{(4)}(\mathbf{x}-\boldsymbol{\zeta}).$$

Writing this equation explicitly for $\alpha = l, \beta = j$ we obtain [1]

(4.5) $$(\nabla_l\partial_t-\partial_t\nabla_l)\Omega_{(4)} = \epsilon_{ljk}\int \dot{\zeta}^j\delta_{(3)}(\mathbf{x}-\boldsymbol{\zeta})d\zeta^k.$$

Let us demonstrate that the condition (4.5) is fulfilled identically also by the cyclic function $\Omega_{(3)}$. Using the representation of $\Omega_{(3)}$ in the form (I.2.8)

$$\Omega_{(3)}(x;\ t) = \frac{1}{4\pi}\oint_{D(t)}\frac{\epsilon_{ljk}k_j r_k d\zeta_l}{r(r-\mathbf{r}\cdot\mathbf{k})}$$

the derivatives $\partial_t\Omega_{(3)}$ and $\nabla_l\Omega_{(3)}$ are calculated from (I.2.3); the following distributions are obtained:

$$\partial_t\Omega_{(3)} = \frac{1}{4\pi}\oint_{D(t)}\epsilon_{kpj}\dot{\zeta}_p\nabla_j\left(\frac{1}{r}\right)d\zeta_k,$$

$$\nabla_l\Omega_{(3)} = -\frac{1}{4\pi}\oint_{D(t)}\epsilon_{ljk}\nabla_j\left(\frac{1}{r}\right)d\zeta_k.$$

Let us now calculate the expression $(\nabla_l\partial_t-\partial_t\nabla_l)\Omega_{(3)}(\mathbf{x}; t)$.

$$\nabla_l(\partial_t\Omega_{(3)}) = \frac{1}{4\pi}\oint_{D(t)}\epsilon_{kpj}\dot{\zeta}_p\nabla_l\nabla_j\left(\frac{1}{r}\right)d\zeta,$$

$$\partial_t(\nabla_l\Omega_{(3)}) = -\frac{1}{4\pi}\oint_{D(t)}\epsilon_{ljk}\left[\nabla_j\left(\frac{1}{r}\right)\dot{\overline{d\zeta_k}}+\dot{\zeta}_p\frac{\partial}{\partial\zeta_p}\nabla_j\left(\frac{1}{r}\right)d\zeta_k\right]$$

$$= -\frac{1}{4\pi}\oint_{D(t)}\epsilon_{ljk}\left[\nabla_j\left(\frac{1}{r}\right)\frac{\partial\dot{\zeta}_k}{\partial\zeta_p}d\zeta_p-\dot{\zeta}_p\nabla_p\nabla_j\left(\frac{1}{r}\right)d\zeta_k\right]$$

$$= -\frac{1}{4\pi}\oint_{D(t)}\epsilon_{ljk}\left[\dot{\zeta}_k\nabla_p\nabla_j\left(\frac{1}{r}\right)d\zeta_p-\dot{\zeta}_p\nabla_p\nabla_j\left(\frac{1}{r}\right)d\zeta_k\right]-\frac{1}{4\pi}\oint_{D(t)}\frac{\partial}{\partial\zeta_p}\left(\dot{\zeta}_k\nabla_j\frac{1}{r}\right)d\zeta_p$$

$$= \frac{1}{4\pi}\oint_{D(t)}(\epsilon_{ljk}\dot{\zeta}_p\nabla_p-\epsilon_{ljp}\dot{\zeta}_p\nabla_k)\nabla_j\left(\frac{1}{r}\right)d\zeta_k,$$

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[1] In order to pass from $J^{\alpha\beta} = \int_{S_{(2)}} dS^{\alpha\beta}\delta_{(4)}(\mathbf{x}-\varepsilon)$ to the integral $\oint \dot{\zeta}^j\delta_{(3)}(x-\zeta)d\zeta^k$, the tensor $d\hat{S}^{\alpha\beta}$ dual to $dS^{\alpha\beta}$ should be used; they are related by the formulae [3] $d\hat{S}_{\alpha\beta} = \frac{1}{2!}\epsilon_{\alpha\beta\mu\nu}dS^{\mu\nu}$, $dS^{\alpha\beta} = \frac{1}{2!}\epsilon^{\alpha\beta\mu\nu}d\hat{S}_{\mu\nu}$. Since $dS^{\alpha\beta} = \epsilon^{\alpha\beta\gamma\delta}n_\gamma m_\delta dS$, $\mathbf{m}$ and $\mathbf{n}$ being the unit vectors normal to $S_{(2)}$ in V_4 and $dS^{\alpha\beta}\tau_\alpha s_\beta d\tau ds = d_{(4)}V$, $dS_{\mu\nu} = \epsilon_{\mu\nu\varkappa\eta}\tau^\varkappa s^\eta dS$, where $\boldsymbol{\tau}$ and $\mathbf{s}$ are unit tangent vectors lying in the surface $S_{(2)}$, we obtain the final result:

$$dS^{\alpha\beta} = \frac{1}{2!}\varepsilon^{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta\mu\nu}\tau^\mu s^\nu d\tau ds = (\tau^\alpha s^\beta-\tau^\beta s^\alpha)d\tau ds.$$

In our case $\boldsymbol{\tau} = \dot{\boldsymbol{\zeta}}$, $\mathbf{s} = \mathbf{l}$, $d\boldsymbol{\zeta} = \mathbf{l}dl$.

$$(\nabla_l \partial_t - \partial_t \nabla_l) \Omega_{(3)}(\mathbf{x}; t) = \frac{1}{4\pi} \oint_{D(t)} \dot{\zeta}_p (\epsilon_{kpj} \nabla_l - \epsilon_{ljk} \nabla_p + \epsilon_{ljp} \nabla_k) \nabla_j \left(\frac{1}{r}\right) d\zeta_k$$

$$= \frac{1}{4\pi} \oint_{D(t)} \dot{\zeta}_p (\epsilon_{klj} \nabla_p - \epsilon_{klp} \nabla_j - \epsilon_{ljk} \nabla_p) \nabla_j \left(\frac{1}{r}\right) d\zeta_k$$

$$= -\frac{1}{4\pi} \oint_{D(t)} \dot{\zeta}_p \epsilon_{klp} \nabla^2 \left(\frac{1}{r}\right) d\zeta_k = \oint_{D(t)} \epsilon_{lpk} \dot{\zeta}_p \delta(\mathbf{x} - \boldsymbol{\zeta}) d\zeta_k.$$

The cyclic function $\Omega_{(3)}(\mathbf{x}, t)$ obtained from the statical cyclic function $\Omega_{(3)}(\mathbf{x})$ according to Eq. (2.13) is found to fulfill the same comutative condition (4.5) as $\Omega_{(4)}$.

In further considerations use will be made of the cyclic functions determined by Eqs. (I.2.8), (I.3.4) and (I.3.15)

$$\Omega_{(3)}(\mathbf{x}; t) = \frac{1}{4\pi} \oint_{D(t)} \frac{(\mathbf{k} \times \mathbf{r}) \cdot \mathbf{d}\boldsymbol{\zeta}}{r(r - \mathbf{r} \cdot \mathbf{k})}, \tag{4.6}$$

$$\overset{(-)}{\Omega}_{(4)}(\mathbf{x}, t) = \frac{1}{4\pi} \oint_{-\infty}^{t} d\tau \oint_{D(\tau)} dl \epsilon^{ijk} l_k \dot{\zeta}_j \nabla_i \frac{H(c_2\theta - r)}{r}, \tag{4.7}$$

$$\overset{(+)}{\Omega}_{(4)}(\mathbf{x}, t) = \frac{-c_2}{4\pi} \oint_{-\infty}^{t} d\tau \oint_{D(\tau)} dl \epsilon^{ijk} l_k k_j \left(\nabla_i + \frac{1}{c_2^2} \dot{\zeta}_i \partial_t\right) \frac{H(c_2\theta - r)}{\sqrt{c_2^2\theta^2 - r^2 + (\mathbf{r} \cdot \mathbf{k})^2}}, \tag{4.8}$$

$$\theta = t - \tau.$$

The cyclic function $\Omega_{(3)}(\mathbf{x})$ given by Eq. (4.6) represents the solid angle subtended by the loop D with the vertex at $\mathbf{x}$. The corresponding formula for the solid angle expressed in terms of a line integral was given originally by Z. Wesołowski (private communication):

$$\omega = \oint_{D} \frac{1 - \cos\vartheta}{r^2 \sin^2\vartheta} (\mathbf{k} \times \mathbf{r}) \cdot \mathbf{dr} = \oint \frac{(\mathbf{k} \times \mathbf{r}) \cdot \mathbf{dl}}{r^2(1 + \cos\vartheta)}, \tag{4.9}$$

$\mathbf{k}$ is an arbitrary unit vector and ϑ the angle between the vectors $\mathbf{r}$ and $\mathbf{k}$.

M. O. PEACH and J. S. KOEHLER gave in [4] another formula for the solid angle $\tilde{\omega}$:

$$\tilde{\omega}(\mathbf{x}) = \frac{1}{3} \oint_{D} dl \left[l_1 \frac{YZ}{r} \left(\frac{1}{X^2 + Z^2} - \frac{1}{X^2 + Y^2} \right) + l_2 \frac{XZ}{r} \left(\frac{1}{X^2 + Y^2} - \frac{1}{Y^2 + Z^2} \right) + l_3 \frac{XY}{r} \left(\frac{1}{Y^2 + Z^2} - \frac{1}{X^2 + Z^2} \right) \right], \tag{4.10}$$

with the notations $X = \xi(l) - x$, $Y = \eta(l) - y$, $Z = \zeta(l) - z$.

According to F. R. N. NABARRO [5], this expression depends on the choice of coordinates. Writing Eq. (4.10) in terms of the vector $\mathbf{k}$ and the angle ϑ, we obtain

$$\tilde{\omega} = -\oint_{D} \frac{\cos\vartheta}{r^2 \sin^2\vartheta} (\mathbf{k} \times \mathbf{r}) \cdot \mathbf{dr}.$$

While the expression $(\mathbf{k}\times\mathbf{r})/(1+\cos\vartheta)$ in the integrand of (4.9) tends to zero with $\vartheta\to 0$, the term $[\cos\vartheta(\mathbf{k}\times\mathbf{r})]/\sin^2\vartheta\to\infty$ with $\vartheta\to 0$. Absence of the term $1/r^2\sin^2\vartheta$ in the formula for the solid angle $\tilde{\omega}$ makes the displacement given in [4] erroneous.

Let us now pass to the evaluation of the displacement fields produced by a moving dislocation loop $D(t)$, corresponding to the cyclic functions $\Omega_{(3)}(\mathbf{x},t)$ and $\Omega_{(4)}(\mathbf{x},t)$. In order to utilize the formulae (2.14)—(2.17), space and time-derivatives of cyclic functions must be calculated. After simple transformations we obtain

(4.11)
$$\nabla_p\Omega_{(3)} = -\frac{1}{4\pi}\oint_{D(t)}\epsilon_{psl}\nabla_s\left(\frac{1}{r}\right)d\zeta_l,$$
$$\partial_t\Omega_{(3)} = \frac{1}{4\pi}\oint_{D(t)}\epsilon_{kjp}\dot{\zeta}_j\nabla_p\left(\frac{1}{r}\right)d\zeta_k;$$

(4.12)
$$\nabla_p\Omega_{(4)} = -\frac{1}{4\pi}\oint_{-\infty}^{t}d\tau\oint_{D(\tau)}\epsilon_{psl}\left(\nabla_s+\frac{1}{c_2^2}\dot{\zeta}_t\partial_t\right)\frac{\delta(\theta-r/c_2)}{r}d\zeta_l,$$
$$\partial_t\Omega_{(4)} = \frac{1}{4\pi}\oint_{-\infty}^{t}d\tau\oint_{D(\tau)}\epsilon_{kjp}\dot{\zeta}_j\nabla_p\frac{\delta(\theta-r/c_2)}{r}d\zeta_k.$$

Introduction of the tensor $K_{ij}(\mathbf{x}-\boldsymbol{\xi},t-\tau)$ such that ([6])

$$\nabla^2K_{ij}(\mathbf{x}-\boldsymbol{\xi},t-\tau) = -G_{ij}(\mathbf{x}-\boldsymbol{\xi},t-\tau) \tag{4.13}$$

which in the case of isotropic materials has the form [6]

$$K_{ij} = \frac{H(\theta)}{4\pi\varrho}\left\{\frac{\delta_{ij}}{c_2}\frac{r-c_2\theta}{r}H(r-c_2\theta)+\frac{1}{6}\nabla_i\nabla_j\frac{1}{r}\left[\frac{(r-c_1\theta)^3}{c_1}H(r-c_1\theta) - \frac{(r-c_2\theta)^3}{c_2}H(r-c_2\theta\right]\right\} \tag{4.14}$$

makes it possible to integrate Eq. (2.17) by parts and obtain

$$\mathring{u}_m = -\int_{-\infty}^{t}d\tau\int_{V_\infty}\nabla^2K_{mj}(\mathbf{x}-\boldsymbol{\xi},t-\tau)A_j(\boldsymbol{\xi},\tau)dV(\boldsymbol{\xi}) = -\int_{-\infty}^{t}d\tau\int_{V_\infty}K_{mj}\overset{(\xi)}{\nabla^2}A_j(\boldsymbol{\xi},\tau)dV(\boldsymbol{\xi}). \tag{4.15}$$

The vector A_j corresponding to the cyclic function Ω_3 is found by substituting Eqs. (4.11) in Eq. (2.16)

(4.16)
$$A_j^{(3)}(\boldsymbol{\xi},\tau) = -\frac{1}{4\pi}b_l\epsilon_{ksp}\nabla_s\left[C_{ijkl}\nabla_i\oint_{D(\tau)}\frac{1}{r}d\zeta_p+\varrho\delta_{jl}\frac{d}{d\tau}\oint_{D(\tau)}\dot{\zeta}_k\frac{1}{r}d\zeta_p\right],$$
$$r = \boldsymbol{\xi}-\boldsymbol{\zeta}(\tau).$$

Inserting Eq. (4.16) into Eq. (4.15) we obtain

$$(4.17)\quad \mathring{u}_m(\mathbf{x},t)=\frac{1}{4\pi}\int_{-\infty}^{t}d\tau\int K_{mj}(\mathbf{x}-\boldsymbol{\xi},t-\tau)\in_{ksp}b_l\nabla_s\Big[C_{ijkl}\nabla_i\oint_{D(\tau)}\overset{(\xi)}{\nabla}{}^2\Big(\frac{1}{r}\Big)d\zeta_p$$

$$+\varrho\delta_{jl}\frac{d}{d\tau}\oint_{D(\tau)}\dot{\zeta}_k\overset{(\xi)}{\nabla}{}^2\Big(\frac{1}{r}\Big)d\zeta_p\Big]dV(\boldsymbol{\xi})$$

$$=-\int_{-\infty}^{t}d\tau\int_{V_\infty}K_{mj}(\mathbf{x}-\boldsymbol{\xi},t-\tau)\in_{ksp}b_l\nabla_s\Big[C_{ijkl}\nabla_i\oint_{D(\tau)}\delta(\boldsymbol{\xi}-\boldsymbol{\zeta}(\tau))d\zeta_p$$

$$+\varrho\delta_{jl}\frac{d}{d\tau}\nabla_s\oint_{D(\tau)}\dot{\zeta}_k\delta(\boldsymbol{\xi}-\boldsymbol{\zeta}(\tau))d\zeta_p\Big]dV(\boldsymbol{\xi})$$

$$=-\int_{-\infty}^{t}d\tau\oint_{D(\tau)}^{t}b_l\in_{ksp}C_{ijkl}\nabla_i\nabla_sK_{mj}(\mathbf{x}-\boldsymbol{\zeta}(\tau),t-\tau)d\zeta_p$$

$$-\varrho b_l\delta_{jl}\in_{ksp}\int_{-\infty}d\tau\int_{V_\infty}\Big[(\nabla_sK_{mj})\frac{d}{d\tau}\oint_{D(\tau)}\dot{\zeta}_k\delta(\boldsymbol{\xi}-\boldsymbol{\zeta})d\zeta_p\Big]dV(\boldsymbol{\xi})$$

$$=-\int_{-\infty}^{t}d\tau\oint_{D(\tau)}b_l\in_{ksp}C_{ijkl}\nabla_i\nabla_sK_{mj}d\zeta_p$$

$$-\varrho\delta_{jl}b_l\in_{ksp}\overset{(x)}{\nabla}_s\int_{\infty}^{t}d\tau\int_{V_\infty}\Big\{\frac{d}{d\tau}\Big[K_{mj}\oint_{D(\tau)}\dot{\zeta}_k\delta(\boldsymbol{\xi}-\boldsymbol{\zeta})d\zeta_p\Big]$$

$$-\Big[\frac{d}{d\tau}K_{mj}(\mathbf{x}-\boldsymbol{\xi},t-\tau)\Big]\oint_{D(\tau)}\dot{\zeta}_k\delta(\boldsymbol{\xi}-\boldsymbol{\zeta})d\zeta_p\Big\}dV(\boldsymbol{\xi})$$

$$=-\int_{-\infty}^{t}d\tau\oint_{D(\tau)}b_l\in_{ksp}C_{ijkl}\nabla_i\nabla_sK_{mj}d\zeta_p-\int_{-\infty}^{t}d\tau\oint_{D(\tau)}b_l\in_{ksp}\varrho\delta_{jl}\dot{\zeta}_k\partial_t\nabla_sK_{mj}d\zeta_p$$

$$-\varrho b_j\in_{ksp}\oint_{D(\tau)}\nabla_sK_{mj}(\mathbf{x}-\boldsymbol{\zeta}(\tau),t-\tau)\dot{\zeta}_kd\zeta_p\Big|_{-\infty}^{t}$$

$$=-\int_{-\infty}^{t}d\tau\oint_{D(\tau)}d\zeta_pb_l\in_{ksp}(C_{ijkl}\nabla_i+\varrho\delta_{jl}\dot{\zeta}_k\partial_t)\nabla_sK_{mj}(\mathbf{x}-\boldsymbol{\zeta}(\tau),t-\tau).$$

Hence the displacement field corresponding to the cyclic function $\Omega_{(3)}(\mathbf{x},t)$ has the form

$$(4.18)\quad u_m(\mathbf{x},t)=\frac{b_m}{4\pi}\oint_{D(t)}\frac{(\mathbf{k}\times\mathbf{r})\cdot d\boldsymbol{\zeta}}{r(r-\mathbf{r}\cdot\mathbf{k})}-\int_{-\infty}^{t}d\tau\oint_{D(\tau)}d\zeta_pb_l\in_{ksp}(C_{ijkl}\nabla_i+\varrho\delta_{jl}\dot{\zeta}_k\partial_t)\nabla_sK_{mj}.$$

This formula is analogous to the solution derived by E. Kossecka in the case of a surface model of a dislocation. The difference occurs in the first term only, [6].

The vector A_j corresponding to $\Omega_{(4)}$ may be written as

$$(4.19)\qquad A_j^{(4)}(\boldsymbol{\xi},\tau) = -\frac{1}{4\pi}b_l\Bigg[C_{ljkl}\overset{(\xi)}{\nabla}_i\int\limits_{-\infty}^{\tau}dT\oint\limits_{D(T)}d\zeta_q(T)\epsilon_{kqp}\left(\overset{(\xi)}{\nabla}_p+\frac{1}{c_2^2}\dot{\zeta}_p\partial_\tau\right)$$

$$\cdot\frac{\delta(\tau-T-r(\boldsymbol{\xi},\boldsymbol{\zeta})/c_2)}{r}+\varrho\delta_{jl}\partial_\tau\int\limits_{-\infty}^{\tau}dT\oint\limits_{D(T)}d\zeta_q(T)\epsilon_{iqp}\dot{\zeta}_i(T)\overset{(\xi)}{\nabla}_p\frac{\delta(\tau-T-r/c_2}{r}\Bigg].$$

Introducing the tensor M_{ij} defined by the formula

$$(4.20)\qquad \left(\nabla^2-\frac{1}{c_2^2}\partial_t^2\right)M_{jl} = G_{jl}$$

which, in the case of isotropy, has the form

$$(4.21)\qquad M_{ij} = \frac{1}{4\pi\varrho}\Bigg\{-\frac{\delta_{ij}}{2c_2}H(\theta-r/c_2)+\nabla_i\nabla_j\Bigg[\frac{c_1^2c_2^2}{6(c_1^2-c_2^2)}\left(\frac{(\theta-r/c_1)^3}{r}H(\theta-r/c_1)\right.$$

$$\left.-\frac{(\theta-r/c_2)^3}{r}H(\theta-r/c_2)\right)+\frac{c_2}{4}(\theta-r/c_2)^2H(\theta-r/c_2)\Bigg]\Bigg\},$$

we can write $\hat{\mathbf{u}}$,

$$(4.22)\qquad \hat{u}_m = -\int\limits_{-\infty}^{t}d\tau\int\limits_{V_\infty}G_{mj}(\mathbf{x}-\boldsymbol{\xi},t-\tau)A_j(\boldsymbol{\xi},\tau)dV(\boldsymbol{\xi})$$

$$= -\int\limits_{-\infty}^{t}d\tau\int\limits_{V_\infty}\left(\overset{(\xi)}{\nabla^2}-\frac{1}{c_2^2}\partial_\tau^2\right)M_{mj}(\mathbf{x}-\boldsymbol{\xi},t-\tau)A_j(\boldsymbol{\xi},\tau)dV(\boldsymbol{\xi}).$$

The expression for A_j in Eq. (4.19) is now substituted into Eq. (4.22) and integration by parts with respect to space and time variables is performed. The result is as follows:

$$(4.23)\qquad \hat{u}_m(\mathbf{x},t) = -\int\limits_{-\infty}^{t}d\tau\int\limits_{D(\tau)}b_l\epsilon_{kqp}\left[C_{ljkl}\nabla_i\left(\nabla_p+\frac{1}{c_2^2}\dot{\zeta}_p\partial_t\right)+\varrho\delta_{jl}\dot{\zeta}_k\nabla_p\partial_t\right]$$

$$\times M_{mj}(\mathbf{x}-\boldsymbol{\zeta},t-\tau)d\zeta_q.$$

The displacement corresponding to the cyclic function has the form

$$(4.24)\qquad \mathbf{u} = \mathbf{b}\Omega_{(4)}+\hat{\mathbf{u}}.$$

Here $\hat{\mathbf{u}}$ is given by Eq. (4.23); this is the most general solution of the problem of dislocation lines moving in a linear elastic medium.

Two formulae for the displacement $\mathbf{u}$ having the form of integrals taken along the dislocation lines are known so far in the literature. These are the solutions given in [7, 8]. The solutions belonging to the class of distributions differ from each other essentially. One of them contains a jump at the time-like surface [7], the other one — at the space-like surface [8]; also different are the terms corresponding to $\hat{\mathbf{u}}$. Consideration of the solutions belonging to the class of cyclic functions enables us to prove the solutions [7] and [8] to be identical since they may be considered as the solutions corresponding to two representants of the same cyclic function $\Omega_{(4)}$ and following from Eqs. (4.7) and (4.8). After tedious transformations also the corresponding terms of $\hat{\mathbf{u}}$ can be shown to be identical.

From the considerations of Sect. 2 it is seen that the introduction of cyclic functions not only makes it possible to find the relations between the known solutions [6, 7, 8], but also — what is probably more important — enables us to formulate exactly the displacement equations of the medium containing dislocation lines, without using the fields of distortion and velocities of the points of the medium or the surface model.

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