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# BRIEF NOTES

## Thermal stress in a layered anisotropic elastic half-space

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THE PROBLEM of determining the thermal stress in an inhomogeneous anisotropic half-space is considered. The elastic and thermal parameters for the half-space are assumed to vary with one Cartesian coordinate. An extension of the Bergman series method is used to determine the effect of a prescribed temperature distribution and to solve a particular boundary-value problem.

### 1. Introduction

IN THE INVESTIGATION of real soil strata deformation under loading, the factor of inhomogeneity becomes significant. Moreover, under certain circumstrances soils exhibit anisotropic behaviour. Thus, for example, GIBSON and KALSI [1] have recently investigated the loading of an incompressible cross-anisotropic elastic half-space with rigidity modulus increasing linearly with depth. This has prompted the present treatment of the problem of stress distribution in an anisotropic elastic half-space in which the elastic moduli vary with depth. The effects of a prescribed temperature distribution are included and the constrained boundary problem solved.

### 2. The governing equations

The equilibrium equations in anisotropic thermoelasticity are

$$\frac{\partial}{\partial x_j}\left[c_{ijkl}\frac{\partial u_k}{\partial x_l}-\beta_{ij}\theta\right]=0, \tag{2.1}$$

where the repeated suffix summation convention (summing from 1 to 3) is used for Latin suffixes only and the temperature $\theta$ satisfies the heat conduction equation

$$\left[\frac{\partial}{\partial x_i}k_{ij}\frac{\partial\theta}{\partial x_j}\right]=0. \tag{2.2}$$

In Eq. (2.1) $u_k$ represents the components of displacement, $c_{ijkl}$ are the elastic moduli while $\beta_{ij}$ and $k_{ij}$ denote the thermal moduli and heat conduction coefficients respectively. These material parameters are assumed to satisfy the usual symmetry conditions (see CLEMENTS [2]) and are also assumed to depend on the $x_2$ coordinate.

## 3. The temperature field

Consider the following representation for the temperature $\theta$.

(3.1) $$\theta = \sum_{n=0}^{\infty} T_n(x_2) E_n(S(x_2)+x_1), \quad T_0 \neq 0,$$

where the $E_n$ satisfy the recurrence relations

(3.2) $$E_n' = E_{n-1} \quad \text{for} \quad n = 1, 2, \ldots .$$

By substituting, it is readily seen that the $\theta$ given by Eq. (3.1) will satisfy Eq. (2.2) if the functions $T_n(x_2)$ and $S(x_2)$ are given by

(3.3) $$T_0 = \alpha/[k_{12}+k_{22}S']^{1/2} \quad (\alpha \text{ arbitary constant}),$$

(3.4) $$T_n = -\frac{1}{2}(k_{12}+k_{22}S')^{-1/2} \int \frac{\{k_{22}T_{n-1}''+k_{22}'T_{n-1}'\}dx_2}{\{k_{12}+k_{22}S'\}^{1/2}}, \quad n = 1, 2, \ldots,$$

(3.5) $$S' = [-k_{12} \pm (k_{12}^2 - k_{11}k_{22})^{1/2}]/k_{22}.$$

Since physical considerations require

(3.6) $$k_{12}^2 - k_{11}k_{22} < 0$$

it follows that Eq. (3.5) yields a complex conjugate pair which will be denoted by $\tau(x_2)$ and $\bar{\tau}(x_2)$ where $\tau(x_2)$ is obtained from Eq. (3.5) by taking the positive sign. Hence

(3.7) $$\theta = \sum_{n=0}^{\infty} T_n(x_2)\{E_n(z')+\overline{E}_n(\bar{z}')\},$$

where $z' = x_1 + \tau x_2$. A suitable form for the $E_n$ for our present purposes is

(3.8) $$E_n = \frac{1}{2\pi} \int_0^{\infty} A_n(p) \exp(ipz')\, dp,$$

where $A_n(p) = A(p)\,(ip)^{-n}$ with $A_0(p) \equiv A(p)$. Hence, from Eqs. (3.7) and (3.8)

(3.9) $$\theta(x_1, x_2) = \frac{1}{\pi} R \int_0^{\infty} \left[A(p) \exp(ipz') \sum_{n=0}^{\infty} \frac{T_n(x_2)}{(ip)^n}\right] dp,$$

where $R$ denotes the real part of a complex number and the term $A(p)$ is determined from the boundary conditions.

On the boundary $x_2 = 0$ the temperature is prescribed so that

(3.10) $$\theta(x_1, 0) = f(x_1),$$

where $f(x_1)$ is specified. It is assumed that this temperature distribution may be written as the Fourier integral

(3.11) $$\theta(x_1, 0) = \frac{1}{\pi} R \int_0^{\infty} dp \int_{-\infty}^{\infty} \{f(\xi) \exp[-ip(\xi - x_1)]\}\, d\xi.$$

Comparison of Eqs. (3.9) and (3.11) yields

(3.12) $$A(p) = \left[\int_0^{\infty} f(\xi) \exp(-ip\xi)\, d\xi\right] \Big/ \left\{\sum_{n=0}^{\infty} \frac{T_n(0)}{(ip)n}\right\}.$$

## 4. The stress field

A particular solution of Eq. (2.1) is sought in the form

$$u_k = \sum_{n=0}^{\infty} g_{kn}(x_2) F_n\big(S(x_2)+x_1\big), \tag{4.1}$$

where

$$F_n' = F_{n-1} = E_n. \tag{4.2}$$

Substitution of Eq. (4.1) into Eq. (2.1) yields the recurrence relations

$$\begin{aligned} &\{c_{i2k2}g_{kn}'' + c_{i2k2}' S_{kn}'\} + \{(c_{i1k2}+c_{i2k1}+2S'c_{i2k2})g_{k,n+1}' + (S''c_{i2k2} \\ &\quad + S'c_{i2k2}' + c_{i2k1}')g_{k,n+1} - \beta_{i2}' T_{n+1} - \beta_{i2} T_{n+1}'\} + \{[c_{i1k1} + S'(c_{i1k2}+c_{i2k1}) \\ &\quad + S'c_{i2k2}]q_{k,n+2} - \beta_{i1} T_{n+2} - \beta_{i2} T_{n+2} S') = 0, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &(c_{i1k2}+c_{i2k1}+2S'c_{i2k2})g_{k0}' + (S''c_{i2k2} + S'c_{i2k2}' + c_{i2k1}')S_{k0} \\ &\quad - \beta_{i2}' T_0 - \beta_{i2} T_0' + [(c_{i1k1} + S'(c_{i1k2}+c_{i2k1}) + S'^2 c_{i2k2})g_{k1} \\ &\quad - \beta_{il} T_l - \beta_{i2} T_1 S'] = 0, \end{aligned} \tag{4.4}$$

$$D_{ik} g_{k0} = [\beta_{il} + \beta_{i2} S'] T_0, \tag{4.5}$$

where

$$D_{ik} = c_{i1k1} + S'(c_{i1k2}+c_{i2kl}) + S'^2 c_{i2k2}. \tag{4.6}$$

Combination of Eqs. (3.8) and (4.2) shows that

$$F_n = \frac{1}{2\pi} \int_0^{\infty} \frac{A(p)}{(ip)^{n+1}} \exp(ipz')\,dp, \tag{4.7}$$

whence we obtain a particular solution of Eq. (4.1) in the form

$$u_k = \frac{1}{\pi} R \sum_{n=0}^{\infty} g_{kn}(x_2) \left( \int_0^{\infty} \left[ \frac{A(p)}{(ip)^{n+1}} \exp(ipz') \right] dp \right\}. \tag{4.8}$$

In addition to the displacement (4.8), any displacement may be superimposed which is a solution of the equations

$$\frac{\partial}{\partial x_j} \left\{ c_{ijkl} \frac{\partial u_k}{\partial x_l} \right\} = 0. \tag{4.9}$$

Solutions of Eq. (4.9) are sought in the form

$$u_k = \sum_{n=0}^{\infty} h_{kn}(x_2) H_n\big(\Lambda(x_2)+x_1\big), \tag{4.10}$$

where

$$H_n' = H_{n-1}, \quad n = 1, 2, \ldots. \tag{4.11}$$

Substitution of Eq. (4.10) into Eq. (4.9) yields

$$(4.11)\quad c_{i2k2}h''_{kn}+c'_{i2k2}h'_{kn}+(c_{i1k2}+c_{i2k1}+2\Lambda'c_{i2k2})h'_{k,n+1}+(\Lambda''c_{i2k2}+\Lambda'c'_{i2k2}+c'_{i2k1})h_{k,n+1}+[c_{i1k1}+\Lambda'(c_{i1k2}+c_{i2k1})+\Lambda'^2c_{i2k2}]h_{k,n+2}=0,\quad n=0,1,2,\ldots,\quad i=1,2,3;$$

$$(4.12)\quad [c_{i1k2}+c_{i2k1}+2\Lambda'c_{i2k2}]h'_{k0}+[\Lambda''c_{i2k2}+\Lambda'c'_{i2k2}+c'_{i2k1}]h_{k0}+[c_{i1k1}+\Lambda'(c_{i1k2}+c_{i2k1})+\Lambda'^2c_{i2k2}]h_{k1}=0,\quad i=1,2,3;$$

$$(4.13)\quad [c_{i1k1}+\Lambda'(c_{i1k2}+c_{i2k1})+\Lambda'c_{i2k2}]h_{k0}=0,\quad i=1,2,3;$$

where $h_{k,n}\equiv h_{kn}$.

The consistency condition for Eq. (4.13) provides the sextic-equation

$$(4.14)\quad |c_{i1k1}+\Lambda'(c_{i1k2}+c_{i2k1})+\Lambda'^2c_{i2k2}|=0$$

determining $\Lambda$. Equation (4.13) further provides, in general, two algebraic equations for the $h_{k0}$, $k=1,2,3$. Moreover, in view of Eq. (4.14) there exists a linear combination of the rows of the matrix

$$(4.15)\quad S_{ik}=(c_{i1k1}+\Lambda'(c_{i1k2}+c_{i2k1})+\Lambda'^2c_{i2k2})$$

which is zero. This linear combination may be used to eliminate the $h_{kl}$ in Eq. (4.12) to provide an ordinary differential equation relating the $h_{k0}$. Combination with the algebraic equations given by Eq. (4.13) defines each of the $h_{k0}$ via an ordinary differential equation. In a similar manner, in addition to the ordinary differential equation relating the $h_{k0}$, Eq. (4.12) gives two algebraic equations for the $h_{kl}$. Appropriate linear combination of Eq. (4.11) for the case $n=0$ eliminates the $h_{k2}$ and provides an ordinary differential equation for the $h_{kl}$. Thus, the $h_{kl}$ are defined. In general, Eq. (4.11) provides an ordinary differential equation for the $h_{k,n+1}$ together with two algebraic equations for the $h_{k,n+2}$. Thus, the $h_{kn}$ are completely recursively defined by the system of equations: (4.11)–(4.13).

The sextic (4.15) in $\Lambda'$ has only imaginary roots (see ESHELBY *et al.* [5]) which occur in conjugate pairs. The six roots are taken to be $\Lambda'_\alpha$, $\overline{\Lambda}'_\alpha$; $\alpha=1,2,3$ and the associated values of the $h_{kn}$ generated by the recurrence relations (4.11)–(4.13) are denoted by $h_{kn\alpha}$. Hence the solution of Eq. (4.9) in the form (4.10) is given by

$$(4.16)\quad \begin{aligned} u_k&=\sum_\alpha\sum_{n=0}^{\infty}h_{kn\alpha}(x_2)\{H_{n\alpha}(z_\alpha)+\overline{H}_{n\alpha}(\overline{z}_\alpha)\},\\ H'_{n\alpha}&=H_{n-1\alpha},\end{aligned}$$

where the summation over $\alpha$ is from 1 to 3 and $z_\alpha=x_1+\Lambda_\alpha(x_2)$. Thus, the solution of Eq. (2.1) is generated in the form

$$(4.17)\quad \begin{aligned} u_k&=\sum_\alpha\sum_{n=0}^{\infty}h_{kn\alpha}(x_2)\{H_{n\alpha}(z_\alpha)+\overline{H}_{n\alpha}(\overline{z}_\alpha)\}+\sum_{n=0}^{\infty}g_{kn}(x)\{F_n(z')+\overline{F}_n(\overline{z}')\},\\ F'_n&=F_{n-1}.\end{aligned}$$

The stress $\sigma_{ij}$ may now be obtained by substituting into the equations

$$\sigma_{ij} = c_{ijkl}\frac{\partial u_k}{\partial x_l} - \beta_{ij}\theta. \tag{4.18}$$

A suitable form for the $H_{n\alpha}(z)$ which gives zero stress at infinity is

$$H_{n\alpha}(z_\alpha) = \frac{1}{2\pi}\int\limits_0^\infty \frac{B_\alpha(p)}{(ip)^n}\exp(ipz_\alpha)\,dp, \tag{4.19}$$

where the $B_\alpha(p)$ are chosen to satisfy particular boundary conditions on $x_2 = 0$.

## 5. Constrained boundary

If the boundary is fully constrained, then the boundary conditions are

$$u_k(x_1, 0) = 0, \quad k = 1, 2, 3.$$

The arbitrary constants of integration in the expressions for the $h_{kn\alpha}$, $g_{kn\alpha}$, $T_n$ are selected so that

$$h_{kn\alpha}(0) = g_{kn\alpha}(0) = T(0) = 0, \quad n > 0,$$
$$h_{k0\alpha}(0) = A_{k\alpha} = \text{constant}, \quad g_{k0}(0) = \text{constant} = C_k iK,$$
$$T_0(0) = 1$$

whence, from Eq. (4.17)

$$\sum_\alpha A_{k\alpha}\{\psi_\alpha(x_1)+\overline{\psi}_\alpha(x_1)\} + C_k iK\{F_0(x_1)+F_0(x_1)\} = 0,$$
$$H_{0\alpha} \equiv \psi_\alpha,$$

so that

$$\sum_\alpha A_{k\alpha}B_\alpha(p) + \frac{1}{p}C_k KA(p) = 0.$$

Thus, if the matrix $[A_{k\alpha}]$ is non-singular, then

$$B_\alpha(p) = -\frac{1}{p}R_{\alpha j}C_j kA(p),$$

where

$$\sum_\alpha A_{i\alpha}R_{\alpha j} = \delta_{ij}.$$

The stress in the half-space with constrained boundary and a specified distribution of heat on the surface may now be readily calculated from Eq. (4.18).

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