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Airfoil with minimum relaxation drag

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THE FOLLOWING problem is discussed: what is the shape of a two-dimensional airfoil which, at a given length, area and lift, makes the relaxation drag (i.e. the drag caused by thermodynamic relaxation) a minimum. The discussion is confined to the cases of subsonic flows and slender airfoils. The optimal shape is actually derived for the case of near equilibrium flow. Thus the problem is reduced to minimizing a certain integral under the side condition that the length, area and lift of the airfoil have prescribed values.

Rozważono następujące zagadnienie: jaki powinien być kształt dwuwymiarowego płata (profilu), aby przy danej długości, powierzchni i wyporze opór relaksacyjny (tzn. opór wywołany przez termodynamiczną relaksację) miał wartość minimalną. Dyskusję ograniczono do przypadku przepływów poddźwiękowych i smukłego profilu. Kształt optymalny wyznacza się w istocie dla przepływów bliskich równowagowych. W ten sposób zadanie sprowadza się do minimalizacji pewnej całki z warunkami bocznymi dotyczącymi stałości długości, powierzchni profilu i wartości wyporu.

Рассмотрена следующая задача: какая должна быть форма двумерного крыла (профиля), чтобы при данной длине, поверхности и подъемной силе, релаксационное сопротивление (т. е. сопротивление вызванное термодинамической релаксацией) имело минимальное значение. Обсуждение ограничено случаями дозвуковых течений и тонкого профиля. Оптимальная форма определяется в сущности для течений близких равновесным. Таким образом задача сводится к минимизации некоторого интеграла с граничными условиями, касающимися постоянства длины, поверхности профиля и значения подьемной силы.

1. Introduction

WE CONSIDER a two-dimensional body moving with constant subsonic speed u_0 through a gas which is uniform and at rest far ahead of the body. Equivalently, in a body fixed frame of reference (with the coordinates x, y, see Fig. 1), the gas flow is steady, the flow

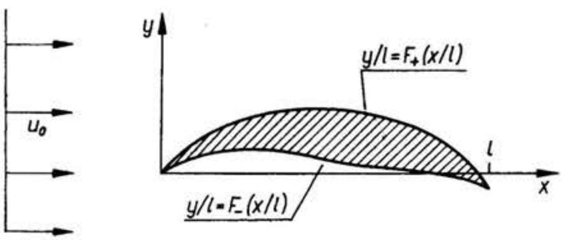

FIG. 1.

speed being u_0 far upstream. The gas is assumed to be a real gas with one internal degree of freedom, which may describe, e.g. a chemical reaction or internal vibrations of molecules. While a fluid particle passes the vicinity of the body, its pressure changes with time.

As a consequence, the particle gets out of its thermodynamic equilibrium state since relaxation processes can only proceed at a finite rate. Thereby entropy is generated which in turn causes a drag force (OSWATITSCH [2], ROMBERG [1], BECKER [3]). We shall call this drag force the "relaxation drag" of the airfoil. It is the aim of this paper to find that shape of the profile which for a given length, area and lift minimizes the relaxation drag.

In order to achieve this, we shall first discuss shortly the thermodynamic properties of the gas, deriving thereby the formula which relates the entropy production rate to the relaxation process. Then we shall simplify this result for flow fields which differ only by a small perturbation from a constant parallel flow. This will lead us to a formula which allows us to compute the entropy production rate if the velocity field is known. Next we shall derive an equation which relates the relaxation drag to the overall entropy production and hence to the velocity field. Then we shall simplify the situation further, assuming sufficiently slow flow, such that the fluid particles are always nearly in their thermodynamic equilibrium states. This being the case, a good approximation to the real velocity field of the relaxing gas will be the velocity field of the non-relaxing equilibrium gas. As a consequence, we can insert this field into our drag formula, thereby obtaining the drag force for slender bodies of arbitrary shape. Finally we shall use this result, which gives the drag as a functional of the body shape, to gain the optimal shape which makes the relaxation drag a minimum.

2. Thermodynamic properties of the gas

The thermodynamic state of the gas is characterized completely by the canonical equation of state

$$h = \hat{h}(p, s, q),$$

h being the specific enthalpy, p — the pressure, s — the specific entropy and q — the internal state variable. By the Gibbs relation (see e.g. [4])

$$dh = \frac{1}{\varrho}\, dp + T ds + \Gamma dq, \tag{2.1}$$

we obtain the density $\varrho = 1/(\partial\hat{h}/\partial p)$, the temperature $T = \partial\hat{h}/\partial s$ and the chemical potential $\Gamma = \partial\hat{h}/\partial q$ as functions of the independent state variables p, s and q. In thermodynamic equilibrium the chemical potential vanishes [5, 4]:

$$\Gamma = \frac{\partial\hat{h}}{\partial q} \quad (p, s, \tilde{q}) = 0.$$

This is an implicit equation for the equilibrium value

$$q = \tilde{q}(p, s)$$

of the internal state variable.

If a gas particle is not in (local) thermodynamic equilibrium, its internal state variable is assumed to change with time according to the "relaxation equation"

$$Dq/Dt = -\frac{\Gamma}{\partial\Gamma/\partial q} \cdot \frac{1}{\tau(p, q, s)} \tag{2.2}$$

(see e.g. [9]). Note that, at least close to a stable equilibrium state, for reasons of thermodynamic stability $\partial\Gamma/\partial q$ is positive. D/Dt denotes the material time derivative and τ is the "relaxation time". The relaxation process, described by Eq. (2.2), is a source of entropy production. This fact follows immediately from the Gibbs relation (2.1), as a consequence of which

$$\frac{Dh}{Dt} = \frac{1}{\varrho}\frac{Dp}{Dt} + T\frac{Ds}{Dt} + \Gamma\frac{Dq}{Dt}. \tag{2.3}$$

Assuming inviscid flow (without heat conduction and diffusion) the energy balance equation may be given by the form

$$\frac{Dh}{Dt} = \frac{1}{\varrho}\frac{Dp}{Dt}.$$

Hence we get from Eqs. (2.2) and (2.3)

$$\frac{Ds}{Dt} = \frac{\Gamma^2}{T\cdot\partial\Gamma/\partial q}\cdot\frac{1}{\tau} =: \frac{\sigma}{\varrho}. \tag{2.4}$$

This equation relates the specific entropy production rate σ to the relaxation process. Since T, $\partial\Gamma/\partial q$ and τ are positive, σ is non-negative.

3. Entropy production rate as a functional of the velocity field for steady small amplitude flow

Next let us simplify Eq. (2.4) assuming that the thermodynamic state of the gas differs only by a small perturbation from a fixed equilibrium state p_0, q_0, s_0. In thermodynamic equilibrium Γ vanishes, hence the Taylor expansion with respect to $p-p_0$, $q-q_0$ and $s-s_0$ leads to

$$\Gamma(p, s, q) = \left(\frac{\partial\Gamma}{\partial p}\right)_0\cdot(p-p_0) + \left(\frac{\partial\Gamma}{\partial q}\right)_0\cdot(q-q_0) + \left(\frac{\partial\Gamma}{\partial s}\right)_0\cdot(s-s_0). \tag{3.1}$$

Note that entropy changes are of second order in the amplitude of the flow field. Therefore, the last term on the right-hand side can be neglected in a first-order approximation for Γ.

The aim of the next steps is to obtain Γ as a functional of the velocity field. To this end we have to express $q-q_0$ and $p-p_0$ in terms of the velocity field. Integration of the linearized version of the relaxation equation

$$u_0\frac{\partial q}{\partial x} = -\frac{1}{\tau_0(\partial\Gamma/\partial q)_0}\cdot\left\{\left(\frac{\partial\Gamma}{\partial p}\right)_0\cdot(p-p_0) + \left(\frac{\partial\Gamma}{\partial q}\right)_0\cdot(q-q_0)\right\} \tag{3.2}$$

leads to

$$q-q_0 = -\frac{(\partial\Gamma/\partial p)_0}{(\partial\Gamma/\partial q)_0}\int_0^\infty e^{-\alpha}\{p(x-u_0\tau_0\alpha, y)-p_0\}d\alpha. \tag{3.3}$$

Integrating the linearized momentum balance

$$\varrho_0 u_0\,\partial u/\partial x = -\partial p/\partial x$$

we get

$$\varrho_0 u_0 (u-u_0) = -(p-p_0). \tag{3.4}$$

Combination of this with Eqs. (3.3) and (3.1) leads finally to the following relation:

$$\Gamma = \left(\frac{\partial \Gamma}{\partial p}\right)_0 \varrho_0 u_0 \int_0^\infty e^{-\alpha}\big(u(x-u_0\tau_0\alpha, y)-u(x, y)\big)d\alpha.$$

Inserting this into the expression (2.4) for the specific entropy production rate, we get

$$\sigma = \varrho_0^3 u_0^2 \frac{1}{T_0\tau_0}\frac{(\partial\Gamma/\partial p)_0^2}{(\partial\Gamma/\partial q)_0}\left[\int_0^\infty e^{-\alpha}\big(u(x-u_0\tau_0\alpha, y)-u(x, y)\big)d\alpha\right]^2.$$

Now, it is known from the literature that the following relation holds between the equilibrium and frozen speeds of sound, a_0 and b_0 (see e.g. [6]):

$$\frac{1}{a_0^2}-\frac{1}{b_0^2} = \varrho_0^2\cdot\frac{(\partial\Gamma/\partial p)_0^2}{(\partial\Gamma/\partial q)_0}.$$

Therefore, we can rewrite our result for the entropy production rate in the form

$$\sigma = \frac{\varrho_0 u_0^2}{T_0\tau_0}\left(\frac{1}{a_0^2}-\frac{1}{b_0^2}\right)\cdot[I\{u(x, y)\}]^2, \tag{3.5}$$

where the functional $I\{u\}$ is given by

$$I\{u(x, y)\} = \int_0^\infty e^{-\alpha}\{u(x-u_0\tau_0\alpha, y)-u(x, y)\}d\alpha. \tag{3.6}$$

By this the entropy production rate is given in terms of the square of $I\{u\}$ where I is a linear functional of the velocity field. Of course, this result can be exploited only if the velocity field is known at least approximately. Fortunately, there are situations where the velocity field nearly coincides with that of an associated non-relaxing gas flow, such that it can be computed by standard methods of classical aerodynamics. This is for instance true if $\partial\hat{h}/\partial q$ is small as compared to $(\partial\hat{h}/\partial p)\cdot(p_0/q_0)$ that is, if the enthalpy function depends only weakly on the internal state variable. In that case one may take the frozen gas as an associated non-relaxing gas. Similarly, it is also true if the relaxation time τ_0 is small or large as compared to the characteristic flow time l/u_0, where l is the length of the profile (see Fig. 1). In the first case, $\tau_0 \ll l/u_0$, the equilibrium flow is the associated non-relaxing gas flow, in the second case, $\tau_0 \gg l/u_0$, it is the completely frozen flow.

Incidentally, for near equilibrium flow the integral $I\{u\}$ can be simplified considerably: Since $u_0\tau_0 \ll 1$ in this case, we have, by the Taylor expansion,

$$e^{-\alpha}[u(x-u_0\tau_0\alpha, y)-u(x, y)] \approx -u_0\tau_0 e^{-\alpha}\cdot\alpha\frac{\partial u}{\partial x}(x, y).$$

Therefore, the functional $I\{u\}$ can be reduced to

$$I\{u\} = -u_0\tau_0\frac{\partial u}{\partial x}(x, y) \quad \text{for} \quad u_0\tau_0 \ll 1. \tag{3.7}$$

Similarly, we obtain for nearly frozen flow, that is, in the limit $u_0\tau_0/l \to \infty$

$$I\{u\} = u_0-u(x, y) \quad \text{for} \quad u_0\tau_0 \gg 1. \tag{3.8}$$

4. Relaxation drag

As we can see from Eqs. (3.5) and (3.6), entropy is produced only in the vicinity of the airfoil, where $I\{u\}$ is not negligible. Therefore, an entropy wake emerges, the width of which is determined by the dimensions of the airfoil. Now, the generalized Crocco's Theorem (see e.g. [4])

$$-\mathbf{v}\times\operatorname{curl}\mathbf{v} = T\nabla s - \Gamma\nabla q \tag{4.1}$$

reduces to

$$-\mathbf{v}\times\operatorname{curl}\mathbf{v} = T\nabla s$$

far away from the body since the gas approaches thermodynamic equilibrium at a great distance, and therefore Γ vanishes. As a consequence, together with the entropy wake, we also find a momentum wake. The appearance of this indicates that due to relaxation a drag force D is produced. The magnitude of D follows immediately from the overall momentum balance, as a consequence of which

$$D = \lim_{x\to\infty}\int_{-\infty}^{\infty} \varrho u(u_0-u)\,dy. \tag{4.2}$$

The integral is to be taken over a line $x = \text{const}$ far downstream (such that the pressure can be assumed to be p_0 again along that line). Linearizing Eqs (4.1) and (4.2) for nearly parallel flow, we arrive at the following pair of equations:

$$u_0\,\partial u/\partial y = -T_0\,\partial s/\partial y, \tag{4.3}$$

$$D = \varrho_0 u_0 \lim_{x\to\infty}\int_{-\infty}^{\infty}(u_0-u)\,dy. \tag{4.4}$$

Integrating Eq. (4.2), we obtain

$$u_0(u-u_0) = -T_0(s-s_0).$$

Inserting this into Eq. (4.4), we get the following drag formula:

$$D = T_0\varrho_0 \lim_{x\to\infty}\int_{-\infty}^{\infty}(s-s_0)\,dy. \tag{4.5}$$

Now, $\varrho_0 u_0(s-s_0)$ is, in linear approximation, the surplus flux of entropy. This flux must be equal to the entropy produced along the stream-line per unit time. Hence, in linear approximation, we get

$$\varrho_0 u_0(s-s_0) = \int_{-\infty}^{x}\sigma\,dx.$$

Combination of this with Eq. (4.4) finally leads to the following result for the relaxation drag:

$$D = \frac{T_0}{u_0}\iint \sigma\,dx\,dy.$$

The integral is to be taken over the whole plane outside the airfoil.

Though this result has been derived here assuming the slenderness of the body, it is of quite general validity (ROMBERG [1], OSWATITSCH [2], BECKER [3]). Defining the drag coefficient C_D in the usual way as

$$C_D = D/\left(\frac{1}{2}\varrho_0 u_0^2 l\right)$$

and inserting the expression (3.5) for σ, we obtain the final result

$$C_D = \frac{2}{u_0 \tau_0 l}\left(\frac{1}{a_0^2} - \frac{1}{b_0^2}\right) \cdot \iint [I\{u\}]^2 dx\, dy \tag{4.6}$$

with

$$I\{u\} = \int_0^\infty e^{-\alpha}\{u(x-u_0\tau_0\alpha, y) - u(x, y)\} d\alpha.$$

For near equilibrium flow we may use the simplified expression (3.7) for $I\{u\}$. This leads to the drag formula

$$C_D = \frac{2u_0\tau_0}{l}\left(\frac{1}{a_0^2} - \frac{1}{b_0^2}\right)\iint \left(\frac{\partial u}{\partial x}\right)^2 dx\, dy \tag{4.7}$$

which, essentially, was first found by ROMBERG [1] under the more restrictive assumption that both the frozen and the equilibrium gas are ideal gases.

Similarly, for nearly frozen flow we may use Eq. (3.8) for $I\{u\}$, with the result

$$C_D = \frac{2}{u_0 \tau_0 l}\left(\frac{1}{a_0^2} - \frac{1}{b_0^2}\right)\iint (u-u_0)^2 dx\, dy, \tag{4.8}$$

which was first found by E. BECKER [7]. Unfortunately, the integral in Eq. (4.8) does not exist if the airfoil has a finite lift. On the other hand, the real physical relaxation drag of course remains finite. That this holds true can be seen from the following arguments: At a great distance from the body (of the order $u_0\tau_0$) the relaxation time τ_0 becomes comparable with the time scale of pressure changes. Therefore, the expression (3.8) for $I\{u\}$ is not valid there. For still larger distances $I\{u\}$ must even be approximated by the near equilibrium formula (3.7); that is, $I\{u\}$ is proportional to $\partial u/\partial x$ for $r \gg u_0\tau_0$. Now, the integral over $(\partial u/\partial x)^2$ does exist at infinite distances. Hence the real physical relaxation drag always remains finite. Since the far field is that of a single vortex with circulation $(1/2)\cdot C_L u_0 l$, where C_L is the lift coefficient of the airfoil, probably the divergent integral in Eq. (4.8) has to be replaced by a term of the order

$$\text{const}\cdot(C_L u_0 l)^2 \cdot \ln\frac{u_0\tau_0}{l}.$$

Indeed this expression becomes infinite for $u_0\tau_0 \to \infty$. This is the reason why the integral in Eq. (4.8) diverges if $C_L \neq 0$. Inserting this expression into Eq. (4.8) instead of the integral, we obtain a lift coefficient which does even vanish in the limit $u_0\tau_0 \to \infty$. This result could be expected from physical arguments.

Although Eq. (4.8) is of no importance in cases of finite lift of the airfoil, it nevertheless holds true if the velocity field decays at least like a dipole field at infinity. This is the case if the lift coefficient of the airfoil is zero.

5. The flow field

In order to evaluate the integral $I\{u\}$ we must know the velocity field at least approximately. As has been explained above, it suffices in many cases to insert the velocity field of the associated non-relaxing gas flow. Standard methods of the classical slender body theory [8] lead to the following result, whereby M_0 denotes the Mach number of the associated non-relaxing gas far ahead of the body:

$$u = \frac{u_0}{\sqrt{1-M_0^2}} \frac{\partial}{\partial \bar{x}} \varphi(\bar{x}, \bar{y}), \tag{5.1}$$

with $\bar{x} = x/l$, $\bar{y} = (y/l) \cdot \sqrt{1-M_0^2}$.

The potential φ is given by

$$\varphi = \varphi_q(\bar{x}, \bar{y}) + \varphi_\gamma(\bar{x}, \bar{y}), \tag{5.2}$$

$$\varphi_q = \frac{1}{2\pi} \int_0^1 q(\xi) \ln \sqrt{(\bar{x}-\xi)^2 + \bar{y}^2}\, d\xi, \tag{5.3}$$

$$\varphi_\gamma = \frac{1}{2\pi} \int_0^1 \gamma(\xi) \arctan \frac{\bar{y}}{\bar{x}-\xi}\, d\xi. \tag{5.4}$$

The dimensionless vortex and source distributions, γ and q, are determined by the shape of the airfoil through the following relations:

$$-\frac{1}{2\pi}\, {}_{\mathrm{C}}\!\!\int \frac{\gamma(\xi)}{\bar{x}-\xi}\, d\xi = \frac{1}{2}\left(F'_+(\bar{x}) + F'_-(\bar{x})\right) =: g'(\bar{x}), \tag{5.5}$$

$$q(\bar{x}) = F'_+(\bar{x}) - F'_-(\bar{x}) =: 2f'(\bar{x}), \tag{5.6}$$

whereby $\bar{y} = F_+(\bar{x})$ and $\bar{y} = F_-(\bar{x})$ are the upper and lower boundary of the airfoil. The symbol ${}_{\mathrm{C}}\!\int$ denotes the Cauchy principal value of the integral. From Eq. (5.6) we see that the source distribution q has to satisfy the "closing-condition"

$$\int_0^1 q(\xi)\, d\xi = 0. \tag{5.7}$$

The area A of the airfoil is

$$A = -l^2 \int_0^1 \xi \cdot q(\xi)\, d\xi. \tag{5.8}$$

The lift coefficient C_L of the body is given by

$$C_L = \frac{2}{\sqrt{1-M_0^2}} \int_0^1 \gamma(\xi)\, d\xi.$$

Finally, the Kutta-Joukowski condition is expressed by

$$\gamma(1) = 0. \tag{5.9}$$

6. Relation between the drag coefficient and the vortex and source distributions for a flow near equilibrium

For near equilibrium flow the drag coefficient is given by Eq. (4.7). Combination of this equation with Eq. (5.1) leads to

$$C_D = C\iint \varphi_{\bar{x}\bar{x}}^2 \, d\bar{x}\, d\bar{y} \tag{6.1}$$

with

$$C = \frac{2u_0^3 \tau_0}{l}\left(\frac{1}{a_0^2}-\frac{1}{b_0^2}\right)(1-M_0^2)^{-3/2}. \tag{6.2}$$

Because $\varphi = \varphi_q + \varphi_\gamma$, we can write, instead of Eqs. (6.1)

$$C_D = C\iint\left\{\left(\frac{\partial^2\varphi_q}{\partial\bar{x}^2}\right)^2+\left(\frac{\partial^2\varphi_\gamma}{\partial\bar{x}^2}\right)^2\right\}d\bar{x}\,d\bar{y}+2C\iint\frac{\partial^2\varphi_q}{\partial\bar{x}^2}\frac{\partial^2\varphi_\gamma}{\partial\bar{x}^2}\,d\bar{x}\,d\bar{y}.$$

Now, a mirror reflexion of the airfoil transforms (F_+, F_-) into $(-F_+, -F_-)$. Thereby φ_q remains unaffected, and φ_γ changes sign. Since the drag of the airfoil obviously does not change sign under mirror reflexion, the last term in this equation must vanish. This leads to the result

$$C_D = C\iint\left\{\left(\frac{\partial^2\varphi_q}{\partial\bar{x}^2}\right)^2+\left(\frac{\partial^2\varphi_\gamma}{\partial\bar{x}^2}\right)^2\right\}d\bar{x}\,d\bar{y}. \tag{6.3}$$

This equation shows that the influences of the vortex and source distributions on the drag are decoupled. Inserting Eqs. (5.3) and (5.4) into Eq. (6.3), one obtains, after some mathematics,

$$C_D = -\frac{C}{4\pi}\left\{\int_0^1 q(\eta)\frac{d}{d\eta}\,\mathrm{C}\!\!\int_0^1\frac{q(\xi)d\xi}{\xi-\eta}\,d\eta+\int_0^1\gamma(\eta)\frac{d}{d\eta}\,\mathrm{C}\!\!\int_0^1\frac{\gamma(\xi)d\xi}{\xi-\eta}\,d\eta\right\}. \tag{6.4}$$

The mathematics which leads from Eq. (6.3) to Eq. (6.4) is rather complicated. It is explained in the Appendix. Equation (6.4) expresses the dependence of the relaxation drag coefficient of a slender airfoil in near equilibrium flow in terms of the source and vortex distributions, $q(\bar{x})$ and $\gamma(\bar{x})$.

7. Airfoil with minimum drag

The problem of finding that shape of a airfoil which for a given area and lift makes the relaxation drag a minimum has now been reduced to the following mathematical problem: find $q(\bar{x})$ and $\gamma(\bar{x})$ such that the right-hand side of Eq. (6.4) becomes a minimum under the side conditions (5.7), (5.8), (5.9). Introducing the Lagrangean multipliers λ, μ, ν in order to satisfy the side conditions, we have to solve the following variational problems:

$$\delta\int_0^1 q(\eta)\left\{\frac{d}{d\eta}\,\mathrm{C}\!\!\int_0^1\frac{q(\xi)}{\xi-\eta}\,d\xi-\lambda\eta-\mu\right\}d\eta = 0, \tag{7.1}$$

$$\delta\int_0^1 \gamma(\eta)\left\{\frac{d}{d\eta}\,\mathrm{C}\!\!\int_0^1\frac{\gamma(\xi)}{\xi-\eta}\,d\xi-\nu\right\}d\eta = 0. \tag{7.2}$$

We show in the appendix that this leads to the following singular integral equations which have to be satisfied by $q(\xi)$ and $\gamma(\xi)$ if, in addition to the side conditions, we have $q(0) = q(1) = 0$ and $\gamma(0) = 0$:

$$\frac{d}{d\eta}\,\mathrm{C}\!\!\int_0^1 \frac{q(\xi)}{\xi-\eta}\,d\xi = \frac{1}{2}\{\lambda\eta+\mu\}, \tag{7.3}$$

$$\frac{d}{d\eta}\,\mathrm{C}\!\!\int_0^1 \frac{\gamma(\xi)}{\xi-\eta}\,d\xi = \frac{1}{2}\nu. \tag{7.4}$$

As a consequence of the conditions $q(0) = q(1) = 0$, the airfoil has a sharp leading and trailing edge. It seems plausible, for physical reasons, that the optimal airfoil should have sharp edges since then stagnation points can be avoided. The appearance of a stagnation point would cause divergence of the integral on the right-hand side of Eq. (4.6). This certainly means that in reality the drag would increase very much if a stagnation point were produced. Similarly, due to $\gamma(0) = 0$, there is no flow around the leading edge. That this condition is satisfied by the optimal airfoil is plausible, since otherwise the drag integral would diverge again.

The solutions of Eqs. (7.3) and (7.4) are well known from classical aerodynamics (see e.g. [8]). Taking into account the side conditions, we can write the result as

$$q\left(\frac{x}{l}\right) = \frac{32}{\pi}\frac{A}{l^2}\left(1-\frac{2x}{l}\right)\sqrt{\frac{x}{l}\left(1-\frac{x}{l}\right)}, \tag{7.5}$$

$$\gamma\left(\frac{x}{l}\right) = \frac{4}{\pi}\,C_L\sqrt{1-M_0^2}\,\sqrt{\frac{x}{l}\left(1-\frac{x}{l}\right)}. \tag{7.6}$$

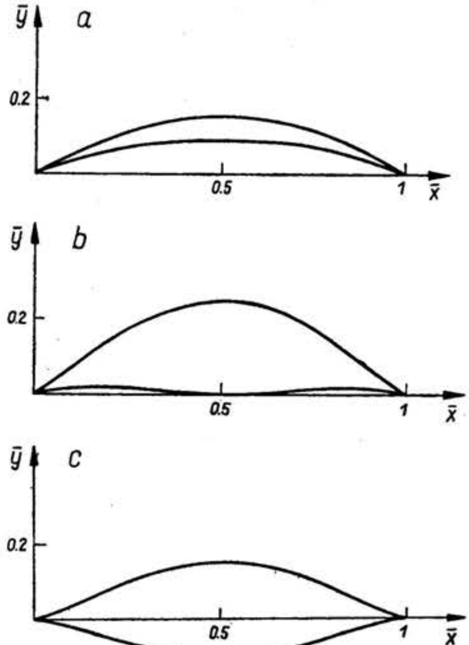

FIG. 2.

The corresponding shape of the airfoil can be determined from Eqs. (5.5) and (5.6), the result being

$$\frac{1}{2}(F_{+}+F_{-}) = g\left(\frac{x}{l}\right) = \frac{64}{3\pi}\frac{A}{l^2}\left\{\frac{x}{l}\left(1-\frac{x}{l}\right)\right\}^{3/2}, \tag{7.7}$$

$$\frac{1}{2}(F_{+}-F_{-}) = f\left(\frac{x}{l}\right) = \frac{1}{\pi}C_L\sqrt{1-M_0^2}\,\frac{x}{l}\left(1-\frac{x}{l}\right). \tag{7.8}$$

In these results $g(\overline{x})$ is the shape of the centerline and $2f(\overline{x})$ is the thickness distribution of the airfoil. The resulting optimal shape is sketched in Fig. 2. It is symmetrical about $x/l = 1/2$, and has a sharp leading and trailing edge. Further, there is no flow around the edges. As a consequence, the perturbation of the constant parallel flow remains small, and therefore our result is consistent with the basic assumption of nearly parallel flow.

From Eq. (6.4) we finally can derive the following expression for the minimum relaxation drag coefficient:

$$C_D = \frac{16}{\pi}\cdot C\left(\frac{A}{l^2}\right)^2 + \frac{1}{2}C\cdot C_L^2(1-M_0^2),$$

where C is given by Eq. (6.2).

Appendix

The calculation of the drag coefficient (6.3) requires an integration over the total flow field. For slender airfoils integration is done over the whole plane, whereby integration over a generating singularity does not exist (in the usual sense). Since the flow around an edge with a finite jump of slope already leads to a divergent integral, it is not to be expected that a reasonable limit process can always be found. It is easy to show that the drag coefficient of a dipole or a source does not exist, hence it follows that for rounded noses this simplification is not allowed. If the airfoil, however, can be generated by source and vortex lines whose strength fulfills a Hölder-condition, then the integrals exist in the Cauchy sense. The optimum is then only sought within the class of the airfoils which can be thus presented. That is to say, there may well be solutions even more optimal in other airfoil classes.

The variational problem can be solved analytically using the following ideas:

a) One optimizes the source and vortex distribution and determines the shape from it.

b) One regards the square in the integrand of Eq. (6.1) $\varphi^2_{\overline{x}\overline{x}}$ as a product of two different fields.

c) One changes the integration sequence, such that integration over the location of the singularities is carried out as the last step.

d) One understands the integration over the y coordinate as the limit of the integration over the following symmetric interval $\lim\limits_{\varepsilon\to 0}\left(\varepsilon \leqslant |y| \leqslant \frac{1}{\varepsilon}\right)$: Equations (6.3), (5.3) and (5.4) lead, under these assumptions, to the representation

$$\text{(A.1)}\quad \frac{C_D}{C} = \frac{1}{4\pi^2}\int_0^1\int_0^1 q(\xi)q(\eta)\,d\xi\,d\eta\,\frac{\partial}{\partial\xi}\frac{\partial}{\partial\eta}\,{}_{\in}\!\int d\bar{y}\int_{-\infty}^{\infty}\frac{(\bar{x}-\xi)}{(\bar{x}-\xi)^2+\bar{y}^2}\cdot\frac{(\bar{x}-\eta)}{(\bar{x}-\eta)^2+\bar{y}^2}\,d\bar{x}$$
$$+\frac{1}{4\pi^2}\int_0^1\int_0^1 \gamma(\xi)\gamma(\eta)\,d\xi\,d\eta\,\frac{\partial}{\partial\xi}\frac{\partial}{\partial\eta}\,{}_{\in}\!\int d\bar{y}\int_{-\infty}^{\infty}\frac{\bar{y}}{(\bar{x}-\xi)^2+\bar{y}^2}\,\frac{\bar{y}}{(\bar{x}-\eta)^2+\bar{y}^2}\,d\bar{x}.$$

The symbol ${}_{\in}\!\int$ has the following meaning:

$$\text{(A.2)}\qquad {}_{\in}\!\int dy = \int_{-\frac{1}{\varepsilon}}^{-\varepsilon} + \int_{\varepsilon}^{\frac{1}{\varepsilon}} dy.$$

In the limit $\varepsilon \to \infty$ this leads to the main part of the Cauchy integral for $y \to 0$ and $y \to \infty$. We then simply carry out this limit, but only if the integrals exist, and begin by calculating with the finite interval. With the aid of the residue theorem (u.h. = upper half plane)

$$\text{(A.3)}\qquad \int_{-\infty}^{\infty} f(\bar{x})\,d\bar{x} = 2\pi i\sum_{\text{u.h.}}\operatorname{Res}(f(\bar{x}))$$

the last integration in each case can be carried out simply; the second integration over y leads to the somewhat surprising result that both kernels are identical

$$\text{(A.4)}\qquad I_1 = I_2 = \frac{\pi}{2}\left[\ln\left(1+\frac{\varepsilon^2}{4}(\xi-\eta)^2\right)-\ln\varepsilon^2-\ln\left(\varepsilon^2+\frac{1}{4}(\xi-\eta)^2\right)\right].$$

Upon a single differentiation $\frac{\partial}{\partial\xi} I_1$ we go to the limit which now exists in the Cauchy sense (symbol ${}_{C}\!\int$)

$$\text{(A.5)}\quad -\frac{C_D}{C} = -\frac{1}{4\pi}\int_0^1 q(\eta)\,d\eta\,\frac{d}{d\eta}\,{}_{C}\!\int_0^1\frac{q(\xi)\,d\xi}{\xi-\eta} - \frac{1}{4\pi}\int_0^1\gamma(\eta)\,d\eta\,\frac{d}{d\eta}\,{}_{C}\!\int_0^1\frac{\gamma(\xi)\,d\xi}{\xi-\eta}$$

in so far as $q, \gamma \in C^\alpha(0, 1)$ fulfill a Hölder condition and vanish at the ends of the interval: $q(0) = q(1) = \gamma(0) = \gamma(1) = 0$.

Should the $q, \gamma \in C^1(0, 1)$ even be differentiable (this will later be the case), then (A.5) can be simplified by partial integration to

$$\text{(A.6)}\quad \frac{C_D}{C} = -\frac{1}{4\pi}\int_0^1\int_0^1 q'(\xi)q'(\eta)\ln|\xi-\eta|\,d\xi\,d\eta - \frac{1}{4\pi}\int_0^1\int_0^1\gamma'(\xi)\gamma'(\eta)\ln|\xi-\eta|\,d\xi\,d\eta.$$

In this form the symmetry of the integration with respect to ξ and η is evident, which is not immediately recognizable (but also valid) from Eq. (A.5) at first glance.

The optimization with the given side conditions (5.7) and (5.8) leads to the following variational problem $(Q(\eta) = \mu+\lambda\eta)$:

$$\text{(A.7)}\qquad \delta\int_0^1 q(\eta)\left[\frac{d}{d\eta}\,{}_{C}\!\int_0^1\frac{q(\xi)\,d\xi}{\xi-\eta} - Q(\eta)\right]d\eta = 0.$$

Due to the symmetry of the integration this requirement is equivalent to

$$\int_0^1 \delta q(\eta)\left[2\frac{d}{d\eta}\,\mathrm{C}\!\!\int_0^1 \frac{q(\xi)d\xi}{\xi-\eta}-Q(\eta)\right]d\eta = 0. \tag{A.8}$$

If this identity holds for all permissible test functions, then the integral equation

$$\frac{d}{d\eta}\,\mathrm{C}\!\!\int_0^1 \frac{q(\xi)d\xi}{\xi-\eta} = \frac{1}{2}Q(\eta) \tag{A.9}$$

follows from the fundamental lemma of variational calculus.

One could invert this integral equation by applying the Betz inversion formula. It is, however, preferable to utilize the fact that $Q(\eta)$ is a finite polynomial. With the slightly modified Glauert formulae

$$\begin{aligned}
&\mathrm{C}\!\!\int_0^1 U_n(1-2\xi)\frac{d\xi}{\xi-\eta} = \pi T_n(1-2\eta),\\
&\mathrm{C}\!\!\int_0^1 \frac{T_n(1-2\xi)}{\sqrt{\xi(1-\xi)}}\frac{d\xi}{\xi-\eta} = -\pi\frac{U_n(1-2\eta)}{\sqrt{\eta(1-\eta)}},
\end{aligned} \tag{A.10}$$

whereby T_n and U_n are the Tschebyscheff functions of the first and second kind,

$$T_n(x) = \cos n(\arccos x), \qquad U_n(x) = \sin n(\arccos x) \tag{A.11}$$

one quickly arrives at the explicit solutions using an obvious ansatz.

From the orthogonality of the Tschebyscheff polynomials a further relation of orthogonality

$$\int_0^1\int_0^1 \frac{T_n(1-2\xi)}{\sqrt{\xi(1-\xi)}}\frac{T_m(1-2\eta)}{\sqrt{\eta(1-\eta)}}\ln(\xi-\eta)d\xi d\eta = \begin{cases} -\dfrac{\pi^2}{2n}\delta_{n,m}, & n\neq 0,\\ -2\pi^2\ln 2\delta_{n,m}, & n=0\end{cases} \tag{A.12}$$

follows using Eq. (A.10) which, finally, by virtue of Eq. (A.6) allows the explicit calculation of the drag coefficient of the optimized airfoils.

For numerical purposes one should note that it is not necessary to calculate the relaxation drag of the linear theory (6.1) by integration over the whole flow field. With the aid of the Gauss and Stokes theorem of the plane (by utilizing the Laplace equation $\Delta\varphi = 0$ after having carried out a Prandtl-Glauert transformation), one can reduce the plane integral (6.1) to a contour integral

$$4\int_A\int u_{\bar{x}}^2 dA = 2\oint_{\partial A} uu_n ds + \oint_{\partial A}(u_{\bar{x}}^2-u_{\bar{y}}^2)(\vec{r}\cdot\vec{n})ds - 2\oint_{\partial A}(u_{\bar{x}}u_{\bar{y}})(\vec{r}\cdot\vec{t})\,ds, \tag{A.13}$$

$$\vec{r} = \bar{x}\vec{i}+\bar{y}\vec{j}, \qquad u_n = (\vec{n}\cdot\nabla)u.$$

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