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Magnetohydrodynamic flow in a rectangular duct under a uniform transverse magnetic field at high Hartmann number

II. The volumetric flow-rate in a duct having non-conducting walls (*)

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THIS PAPER is an extension of an earlier publication by the author [1] on the fully developed, laminar, unidirectional flow of a uniformly conducting, incompressible fluid through a duct having uniform rectangular cross-section, the walls of which are all non-conducting. Here the leading terms of the high-M series form for the volumetric flow-rate are derived from the series expansion for the velocity field obtained in [1] and checked with a closed-form estimate from the same source. The results match exactly with those obtained by previous authors using different approaches.

Praca jest uogólnieniem wcześniejszej publikacji autora [1] dotyczącej w pełni rozwiniętego, laminarnego, jednokierunkowego przepływu jednorodnie przewodzącej nieściśliwej cieczy przez przewód o przekroju prostokątnym, którego ścianki są nieprzewodzące. Główny człon szeregu dla dużych M opisuje wydatek. Wyraz ten otrzymuje się za pomocą rozkładu w szereg pola prędkości otrzymanego w [1] oraz porównania z oszacowaniem podanym w tej samej pracy. Otrzymane wyniki wykazują zgodność z rezultatami innych autorów otrzymanymi na innej drodze.

Работа является обобщением более ранней публикации автора [1], касающейся вполне развернутого однонаправленного течения однородно проводящей несжимаемой жидкости через канал с прямоугольным сечением, стенки которого непроводящие. Главный член в ряде для больших M описывает расход; этот член получается при помощи разложения в ряд поля скорости полученного в [1] и сравнен он с оценкой приведенной в этой же самой работе. Полученные результаты указывают на согласие с результатами других авторов, полученными по другому пути.

Introduction

IN AN EARLIER paper [1], the author considered the fully developed, laminar, unidirectional flow of a uniformly conducting, incompressible fluid through a rectangular duct of uniform cross-section, the walls of which were all non-conducting. For values of the Hartmann number $M \gg 1$, classical asymptotic analysis revealed the leading terms in the expansions of the induced velocity and magnetic fields in all key regions, with the exception of certain boundary layers near the corners of the duct. As was promised in Sect. 4 of [1] we will, in the current paper, estimate the leading terms in the series form for the volumetric flow-rate in powers of $M^{-1/2}$ A closed-form estimate for the flow-rate will also be derived and the results will be compared with those obtained by earlier researchers.

In the following sections, references of the forms (2.1) to (6.31) relate to key expressions and results which were featured in [1].

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1. The series form for the volumetric flow-rate

Since v is even in y and b is odd in y, the volumetric flow-rate down the duct may be expressed in the form (see the definitions (3.1) in [1])

$$F = 2\int_{y=-1}^{1}\int_{x=-l}^{0}\{u-M^{-1}(1+y)\}dx\,dy. \tag{1.1I}$$

One must integrate u_I, u_H, $(u_{lc})_f$, $(u_c)_r$ and $(u_{lc})_r$ over the entire rectangle. Although the results for u_s are not defined in the $(ic)_f$ layers, and integration of u_s over such regions may thus seem to introduce an error into the eventual expression for F, it was shown in Sect. 6A of [1] that by integrating $\{(u_{lc})_f - u_s\} = \hat{u}$ over the entire cross-section, the error is effectively cancelled. This is due to the fact that u_s is, in fact, the outer expansion of $(u_{lc})_f$ as one moves out from the $(ic)_f$ layer into the (s) layer. The reader's attention is drawn to this and other salient comments in Sect. 5 of [1].

The contribution to F from the (I) and (H) regions is (see the result (4.3)):

$$F_{I+H} \sim 2lM^{-1}\int_{-1}^{1}(1-y-2e^{-M(1+y)})dy = 4lM^{-1}(1-M^{-1}), \tag{1.2I}$$

correct to asymptotically small terms in M.

The side layers on $x = \pm l$ contribute a term

$$F_s = 2M^{-1/2}\int_{y=-1}^{1}\int_{X=0}^{\infty} u_s(X,y)\,dX\,dy,$$

where (see the results (4.11′), (4.18) and (4.20))

$$u_s = -M^{-1}\int_{0}^{1-y}\operatorname{erfc}(X/2\theta^{1/2})d\theta + M^{-2}(1-y)\frac{\partial}{\partial y}(\operatorname{erfc}\{X/2(1-y)^{1/2}\}) + M^{-3}(1-y)\left\{-2\frac{\partial^2}{\partial y^2}+\frac{1}{2}(1-y)\frac{\partial^3}{\partial y^3}\right\}(\operatorname{erfc}\{X/2(1-y)^{1/2}\}) + 0(M^{-4}). \tag{1.3I}$$

Since

$$\int_{X=0}^{\infty}\operatorname{erfc}(X/2a^{1/2})dX = [X\operatorname{erfc}(X/2a^{1/2})]_0^{\infty} + (\pi a)^{1/2}\int_0^{\infty} Xe^{-X^2/4a}dX = 2(a/\pi)^{1/2}, \tag{1.4I}$$

hence

$$F_s = \frac{4M^{-3/2}}{\pi^{1/2}}\int_{y=-1}^{1}\left\{-\int_0^{1-y}\theta^{1/2}d\theta + M^{-1}(1-y)\frac{d}{dy}(1-y)^{1/2} - M^{2}(1-y)\left\{2\frac{d^2}{dy^2} - \frac{1}{2}(1-y)\frac{d^3}{dy^3}\right\}(1-y)^{1/2}\right\}dy + 0(M^{-9/2}) = -\frac{64\sqrt{2}M^{-3/2}}{15\pi^{1/2}} - \frac{8\sqrt{2}M^{-5/2}}{3\pi^{1/2}} + \frac{5\sqrt{2}M^{-7/2}}{2\pi^{1/2}}0(M^{9/2}). \tag{1.5I}$$

It is worth noting that although $u_s^{(3)}$ and subsequent $u_s^{(n)}$, $n \geqslant 4$, are unbounded at the corner $X = 0 = 1-y$ (see the comments following the result (4.25)), their respective contributions to F are all finite.

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The $(c)_r$ layers on $y = -1$ contribute (see the results (4.40), (4.43) and (4.46))

$$(1.6\mathrm{I})\quad F_{(c)_r} = 2M^{-3/2} \int\limits_{X=0}^{\infty} \int\limits_{Y^\circ=0}^{\infty} u_c(X, Y^\circ)\, dY^\circ dX = \frac{2M^{-5/2}}{\pi^{1/2}} \Bigg\{ \int\limits_0^2 2\theta^{1/2} d\theta$$

$$+M^{-1} \int\limits_{X=0}^{\infty} \left\{ \frac{Xe^{-X^2/8}}{2^{3/2}} + \operatorname{erfc}\left(\frac{X}{2^{3/2}}\right) \right\} dX + \frac{M^{-2}}{2^{7/2}} \int\limits_{X=0}^{\infty} \Bigg\{ \frac{X}{64} (X^4 - 8X^2 - 144) e^{-X^2/8}$$

$$+ \frac{X}{4}(X^2 - 12) e^{-X^2/8} - X \int\limits_0^1 (X^2 s^2 - 12) s^2 e^{-X^2 s^2/8} ds \Bigg\} dX \Bigg\} + 0(M^{-11/2}) = \frac{16\sqrt{2}M^{-5/2}}{3\pi^{1/2}}$$

$$+ \frac{6\sqrt{2}M^{-7/2}}{\pi^{1/2}} + \frac{7\sqrt{2}M^{-9/2}}{8\pi^{1/2}} + 0(M^{-11/2}).$$

The $(ic)_r$ layers contribute (see the results (4.51) and (4.56))

$$(1.7\mathrm{I})\quad F_{(ic)_r} = 2M^{-2} \int\limits_{\chi=0}^{\infty} \int\limits_{Y=0}^{\infty} u_{ic}(\chi, Y^\circ)\, dY^\circ d\chi = \frac{4M^{-4}}{\pi} \int\limits_0^{\infty} k^{-4}(1 - k^2 - \beta^{-1})\, dk + 0(M^{-5})$$

$$= \frac{4M^{-4}}{\pi} \int\limits_0^{l} (\omega^4 - \omega^2 - 2)\, d\omega + 0(M^{-5}),$$

on setting $k = \omega(1-\omega^2)^{-1}$, $dk = (1+\omega^2)(1-\omega^2)^{-2} d\omega$.

That is,

$$(1.7\mathrm{I}')\quad F_{(ic)_r} = \frac{-128M^{-4}}{15\pi} + 0(M^{-5}).$$

Summarising, the results (1.2I), (1.5I), (1.6I) and (1.7I′) yield

$$(1.8\mathrm{I})\quad F \sim 4lM^{-1} - \frac{64\sqrt{2}M^{-3/2}}{15\pi^{1/2}} - 4lM^{-2} + \frac{8\sqrt{2}M^{-5/2}}{3\pi^{1/2}} + \frac{17\sqrt{2}M^{-7/2}}{2\pi^{1/2}} + 0(M^{-4}).$$

The full $0(M^{-4})$ contribution cannot be obtained without first deriving the complete solution for u in the $(ic)_f$ layers (see earlier comments); this is not available by means of the classical approach.

2. A closed-form estimate for the flow-rate

Performing a double integration, over the entire rectangular cross-section, of the closed-form solution (6.6) yields the full contribution to the flow rate from the $(ic)_f$ and (s) layers. Integration of the result (6.21) likewise yields those from the $(c)_r$ and $(ic)_r$ layers. From the result (6.6),

$$(2.1\mathrm{I})\quad F_{(ic)_f+s} = 2M^{-2} \int\limits_{\chi=0}^{\infty} \int\limits_{Y=0}^{2M} \{u_{ic}(\chi, Y)\}_f\, d\chi\, dY = \frac{4M^{-4}}{\pi} \int\limits_{k=0}^{\infty} \int\limits_{\chi=0}^{\infty} \int\limits_{Y=0}^{2M} k^{-3} \{-k^2 Y$$

$$+ (1 - e^{-\alpha Y})\} \sin(k\chi)\, dY\, d\chi\, dk = \frac{4M^{-4}}{\pi} \int\limits_{k=0}^{\infty} \{-2M^2 k^{-2} + 2Mk^{-4} + (\alpha k^4)^{-1}(e^{-2\alpha M} - 1)\}\, dk.$$

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From the expression (6.5′)

$$(2.2\text{I})\quad (\alpha k^4)^{-1}(e^{-2\alpha M}-1) \sim -\frac{2(k^2-k^4+2k^6M)+4(k^2-k^4)^2M^2}{k^4(k^2-k^4)} \sim -2Mk^{-4} +2M^2k^{-2}+0(k^{\circ}),\quad \text{near } k=0,$$

and hence the integral of the result (2.1I) does exist.

In the Appendix A1, the expression (2.1I) is expanded as a power series in $M^{-1/2}$; the result is

$$(2.3\text{I})\quad F_{(ic)_r+s} = -\frac{64\sqrt{2}M^{-3/2}}{15\pi^{1/2}} - \frac{8\sqrt{2}M^{-5/2}}{3\pi^{1/2}} + \frac{5\sqrt{2}M^{-7/2}}{2\pi^{1/2}} - \frac{128M^{-4}}{15\pi} + \frac{16}{\pi}\sum_{n=2}^{\infty}\frac{(2n-1)\cdot\{(-5/2)(-7/2)(-9/2)\ldots(-3/2-n)\}\Gamma(n-3/2)\cdot(2M)^{-5/2-n}}{(n+1)!},$$

in which the three leading terms clearly match with the expression (1.5I).

The full contribution from the (r) layers is (see the result (6.21)):

$$(2.4\text{I})\quad F_r = 2M^{-2}\int_{Y^{\circ}=0}^{\infty}\int_{\chi=0}^{\infty}u_r(\chi, Y^{\circ})\,d\chi\,dY^{\circ} = \frac{4M^{-4}}{\pi}\int_{k=0}^{\infty}\int_{\chi=0}^{\infty}\int_{Y^{\circ}=0}^{\infty}k^{-3}\{2k^2Me^{-Y^{\circ}} -(1-e^{-2\alpha M})e^{-\beta Y^{\circ}}\}\sin(k\chi)\,d\chi = \frac{4M^{-4}}{\pi}\int_{k=0}^{\infty}k^{-4}\{2k^2M-\beta^{-1}(1-e^{-2\alpha M})\}\,dk.$$

In the Appendix A2, the latter expression is expanded out as a power series in $M^{-1/2}$, yielding

$$(2.5\text{I})\quad F_r = \frac{16\sqrt{2}M^{-5/2}}{3\pi^{1/2}} + \frac{6\sqrt{2}M^{-7/2}}{\pi^{1/2}} - \frac{128M^{-4}}{15\pi^{1/2}} + \frac{16}{\pi}\sum_{n=2}^{\infty}\frac{(2n-5)}{n!}\{(-7/2)(-9/2)\ldots(-3/2-n)\}\Gamma(n-3/2)\cdot(2M)^{-5/2-n},$$

the third term being the sole contribution from the $(ic)_r$ layers. The leading four terms check with the results (1.6I) and (1.7I′).

Combining results (2.3I), (2.5I) with the closed form (1.2I)

$$(2.6\text{I})\quad F = F_{I+H}+F_{s+(ic)_r}+F_r = 4lM^{-1} - \frac{64\sqrt{2}M^{-3/2}}{15\pi^{1/2}} + 4lM^{-2} + \frac{8\sqrt{2}M^{-5/2}}{3\pi^{1/2}} + \frac{17\sqrt{2}M^{-7/2}}{2\pi^{1/2}} - \frac{256M^{-4}}{15\pi} + \frac{8}{\pi}\sum_{n=2}^{p}\frac{(4n^2-16n-5)}{(n+1)!}\{(-7/2)(-9/2)\ldots(-3/2-n)\}\Gamma(n-3/2)(2M)^{-5/2-n}+0(M^{-p-7/2}).$$

One cannot actually proceed to the $p=\infty$ limit because the infinite series is not convergent (see ERDÉLYI [2]). SHERCLIFF [3] obtained the first three terms of Eq. (2.6I), and WILLIAMS [4] obtained the first four terms explicitly but did not continue his expansion so as to yield the remaining terms of the series. We have (see Appendix A3) extended his result for V_0

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([4], p. 265, result 22) — in which the numerator of the third term in { } was incorrectly given as 32a, rather than 64a — using asymptotic expansions of the Bessel functions $K_2(M)$, $K_3(M)$ for $M \gg 1$; the series so obtained matches exactly with Eq. (2.6I) above. One may also note that addition of the results (2.1I) and (2.4I) yields the single closed-form expression obtained by TODD [5].

Conclusion

In a future publication we shall consider a duct having non-conducting walls parallel to, and walls of arbitrary conductivity perpendicular to, the imposed magnetic field. The boundary conditions on the induced fields do not decouple in such a configuration, unlike the situation considered in [1] and the current paper.

Finally, the author wishes to express his sincere gratitude to Professor L. TODD of the Laurentian University, Sudbury, Ontario, Canada, for his collaboration on this project.

Appendix A1. Expansion of the expression (2.1I) as a power series in $M^{-1/2}$

Introducing $s = M\alpha(k)$ into the expression (6.5′) yields

(A1.1) $$\alpha+\frac{1}{2} = sM^{-1}+\frac{1}{2} = \left(k^2+\frac{1}{4}\right)^{1/2}, \quad k^2 = sM^{-1}(1+sM^{-1}),$$

$2k\,dk = M^{-1}(1+2sM^{-1})ds$, and hence Eq. (2.1I) may be re-written in the form

(A1.2) $$F_{z+(lc)_f} = \frac{2M^{-3/2}}{\pi}\int_0^\infty \left\{\frac{e^{-2s}-1+2s(1-s-s^2M^{-1})}{(1+sM^{-1})^{5/2}}\right\}(1+2sM^{-1})s^{-7/2}ds.$$

The highest-order contribution is

(A1.3) $$\frac{2M^{-3/2}}{\pi}\int_0^\infty (e^{-2s}-1+2s-2s^2)s^{-7/2}ds = \frac{4M^{-3/2}}{5\pi}\int_0^\infty (2-4s-2e^{-2s})s^{-5/2}ds$$

$$= \frac{32M^{-3/2}}{15\pi}\int_0^\infty (e^{-2s}-1)s^{-3/2}ds = -\frac{128M^{-3/2}}{15\pi}\int_0^\infty s^{-1/2}e^{-2s}ds = -\frac{64\sqrt{2}M^{-3/2}}{15\pi^{1/2}}.$$

There remains

(A1.4) $$\frac{2M^{-3/2}}{\pi}\int_0^\infty (e^{-2s}-1+2s-2s^2)\left\{\frac{1+2sM^{-1}}{(1+sM^{-1})^{5/2}}-1\right\}s^{-7/2}ds$$

$$-\frac{4M^{-5/2}}{\pi}\int_0^\infty \frac{(1+2sM^{-1})s^{-1/2}}{(1+sM^{-1})^{5/2}}ds,$$

the leading $(0(M^{-2}))$ contribution here being (setting $s = \alpha M$)

(A1.5) $$-\frac{4M^{-2}}{\pi}\int_0^\infty \{(1+2\alpha)(1+\alpha)^{-3/2}-1\}\alpha^{-3/2}d\alpha.$$

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Setting $\alpha = \tan^2\theta$, $d\alpha = 2\sec^2\theta\tan\theta\, d\theta$, the latter integral equals

$$\text{(A1.5')} \qquad \int_0^{\pi/2} \{2\cos\theta - 2(1-\cos\theta)\operatorname{cosec}^2\theta\} d\theta = 2 - \int_0^{\pi/2} \sec^2\left(\frac{1}{2}\theta\right) d\theta = 0.$$

Thus the expression (A1.4) reduces to the form

$$\text{(A1.4')} \qquad \frac{2M^{-3/2}}{\pi} \int_0^\infty (e^{-2s} - 1 + 2s)\left\{\frac{(1+2sM^{-1})}{(1+sM^{-1})^{5/2}} - 1\right\} s^{-7/2} ds$$

$$= \frac{2M^{-3/2}}{\pi} \int_0^\infty \frac{(e^{-2s}-1+2s)}{(1+sM^{-1})^{5/2}} \left\{\frac{-sM^{-1}+0(s^2M^{-2})}{1+2sM^{-1}+(1+sM^{-1})^{5/2}}\right\} s^{-7/2} ds,$$

in which the highest contribution is

$$\text{(A1.6)} \qquad \frac{M^{-5/2}}{\pi} \int_0^\infty (1-2s-e^{-2s}) s^{-5/2} ds = -\frac{8\sqrt{2}M^{-5/2}}{3\pi^{1/2}},$$

(see integration-by-parts in line (A1.3)) and the next is (setting $sM^{-1} = \alpha = \tan^2\theta$)

$$\text{(A1.7)} \qquad \frac{4M^{-3}}{\pi} \int_0^\infty \left\{(1+2\alpha)(1+\alpha)^{-5/2} - 1 + \frac{1}{2}\alpha\right\} \alpha^{-5/2} d\alpha = \frac{40M^{-3}}{3\pi} \int_0^{\pi/2} (6\cos^5\theta - 7\cos^7\theta) d\theta = 0.$$

This leaves

$$\text{(A1.8)} \qquad \frac{2M^{-3/2}}{\pi} \int_0^\infty (e^{-2s}-1)\left\{(1+2sM^{-1})(1+sM^{-1})^{-5/2} - 1 + \frac{1}{2}sM^{-1}\right\} ds,$$

in which, for $M \gg 1$,

$$\text{(A1.8')} \qquad \{\ \} \sim (1+2sM^{-1})\left(1 - \frac{5}{2}sM^{-1} + \frac{35}{8}s^2M^{-2} + \ldots\right) - 1 + \frac{1}{2}sM^{-1} = -\frac{5}{8}s^2M^{-2} + 0(s^3M^{-3}).$$

The leading term in (A1.8) is thus

$$\text{(A1.9)} \qquad \frac{5M^{-7/2}}{4\pi} \int_0^\infty (1-e^{-2s}) s^{-3/2} ds = \frac{5\sqrt{2}}{2\pi^{1/2}} M^{-7/2},$$

the next (setting $sM^{-1} = \alpha = \tan^2\theta$) being

$$\text{(A1.10)} \qquad \frac{2M^{-4}}{\pi} \int_0^\infty \left\{-2(1+\alpha)^{-3/2} + (1+\alpha)^{-5/2} + 1 - \frac{1}{2}\alpha - \frac{5}{8}\alpha^2\right\} \alpha^{-7/2} d\alpha$$

$$= -\frac{M^{-4}}{\pi} \int_0^{\pi/2} \{8\cos^3\theta + 8\cos\theta + 7(1+\cos\theta)^{-2} - 2(1+\cos\theta)^{-3}\} d\theta$$

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(A1.10) [cont.]
$$= -\frac{M^{-4}}{\pi}\left\{\frac{20}{3}+\frac{1}{4}\int_0^1 (1+\sigma^2)(6-\sigma^2)d\sigma\right\}, \quad \text{where } \sigma = \tan\left(\frac{1}{2}\theta\right),$$
$$= -\frac{128M^{-4}}{15\pi}.$$

This is the leading contribution from the $(ic)_f$ layers, which can clearly only generate flow-rate terms involving integral powers of M^{-1}. Furthermore, it is the only such contribution; this becomes clear on noting that the only term in (A1.2) not yet explicitly considered is

$$\text{(A1.11)} \quad \frac{2M^{-3/2}}{\pi}\int_0^\infty e^{-2s}\left\{(1+2sM^{-1})(1+sM^{-1})^{-5/2}-1+\frac{1}{2}sM^{-1}+\frac{5}{8}s^2M^{-2}\right\}s^{-7/2}ds,$$

where

$$(1+2sM^{-1})(1+sM^{-1})^{-5/2} \equiv 2(1+sM^{-1})^{-3/2}-(1+sM^{-1})^{-5/2}$$
$$= 1+\sum_{i=0}^{\infty}\frac{2\{(-3/2)(-5/2)\dots(-3/2-i)\}}{(i+1)!}\left(\frac{s}{M}\right)^{i+1}$$
$$-\sum_{j=0}^{\infty}\frac{\{(-5/2)(-7/2)\dots(-5/2-j)\}}{(j+1)!}\left(\frac{s}{M}\right)^{j+1}$$
$$= 1-\frac{1}{2}sM^{-1}-\frac{5}{8}s^2M^{-2}+\sum_{n=2}^{\infty}\frac{(2n-1)\{(-5/2)(-7/2)\dots(-3/2-n)\}}{2(n+1)!}\left(\frac{s}{M}\right)^{n+1}.$$

Since, for all $n \geqslant 2$,

$$\text{(A1.12)} \quad \int_0^\infty s^{n-5/2}e^{-2s}ds = 2^{3/2-n}\Gamma(n-3/2),$$

therefore (A1.11) reduces to the series form

$$\text{(A1.13)} \quad \frac{16}{\pi}\sum_{n=2}^{\infty}\frac{(2n-1)\{(-5/2)(-7/2)\dots(-3/2-n)\}\Gamma(n-3/2)}{(n+1)!}(2M)^{-5/2-n},$$

this being the flow-rate contribution due to $\sum_{n=4}^{\infty} u_s^{(n)}M^{-n}$.

Finally. $F_{s+(ic)_f}$ is obtained by combining the contributions (A1.3), (A1.6), (A1.9), (A1.10) and (A1.13).

Appendix A2. Expansion of the expression (2.4I) as a power series in $M^{-1/2}$

Combining the expression (6.20) for $\beta(k)$ with line (A1.1) and substituting into the result (2.4I) yields

$$\text{(A2.1)} \quad F_r = \frac{2M^{-5/2}}{\pi}\int_0^\infty \{e^{-2s}-1+2s(1+sM^{-1})^2\}(1\mp 2sM^{-1})(1+sM^{-1})^{-7/2}s^{-5/2}ds,$$

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in which the leading contribution (due to $u_c^{(1)}$) is

$$\text{(A2.2)} \qquad \frac{2M^{-5/2}}{\pi}\int_0^\infty (e^{-2s}-1+2s)s^{-5/2}ds = \frac{16\sqrt{2}M^{-5/2}}{3\pi^{1/2}}.$$

The $0(M^{-3})$ contribution is readily seen to be zero, while $u_c^{(2)}$ contributes a term

$$\text{(A2.3)} \qquad -\frac{3M^{-7/2}}{\pi}\int_0^\infty (e^{-2s}-1)s^{-3/2}ds = \frac{6\sqrt{2}M^{-7/2}}{\pi^{1/2}}.$$

There remains

$$\text{(A2.4)} \quad \frac{2M^{-5/2}}{\pi}\int_0^\infty \left\{(e^{-2s}-1)(1+2sM^{-1})(1+sM^{-1})^{-7/2}-1+\frac{3}{2}sM^{-1}\right\}s^{-5/2}ds,$$

the leading term from which (on re-introducing $\alpha = sM^{-1}$) equals

$$\text{(A2.5)} \qquad -\frac{2M^{-4}}{\pi}\int_0^\infty \left\{(1+2\alpha)(1+\alpha)^{-7/2}-1+\frac{3}{2}\alpha\right\}\alpha^{-5/2}d\alpha,$$

which reduces, after three integrations by parts, substituting $\alpha = \tan^2\theta$ and the use of Wallis' formula, to the form

$$\text{(A2.5')} \qquad \frac{2M^{-4}}{3\pi}\int_0^{\pi/2} (630\cos^7\theta-1323\cos^9\theta+693\cos^{11}\theta)d\theta = -\frac{128M^{-4}}{15\pi}.$$

The other term in (A2.4) is

$$\text{(A2.6)} \quad \frac{2M^{-5/2}}{\pi}\int_0^\infty e^{-2s}\left\{2(1+sM^{-1})^{-5/2}-(1+sM^{-1})^{-7/2}-1+\frac{3}{2}sM^{-1}\right\}s^{-5/2}ds,$$

in which

$$\{\ \} = \sum_{n=2}^{\infty} \frac{(n-5/2)\{(-7/2)(-9/2)\ldots(-3/2-n)\}}{n!}\left(\frac{s}{M}\right)^n.$$

Thus, using the result (A1.12), we may express the contribution (A2.6) in the series form

$$\text{(A2.6')} \quad \frac{16}{\pi}\sum_{n=2}^{\infty} \frac{(2n-5)\{(-7/2)(-9/2)\ldots(-3/2-n)\}}{n!}\,\Gamma\left(n-\frac{3}{2}\right)\cdot(2M)^{-5/2-n}.$$

The $(ic)_r$ layers contribute only the term (A2.5′) which exactly matches the contribution (A1.10) from the $(ic)_f$ regions.

Finally, F_r is obtained by combining the contributions (A2.2), (A2.3), (A2.5′) and (A2.6′).

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Appendix A3. Completion of Williams' series expansion for the mean velocity

WILLIAMS ([4], p. 265, result 22) obtained the mean velocity v_0 in the (corrected) closed form

$$\text{(A3.1)} \qquad v_0 \sim \frac{ka^2}{M}\left\{1-\frac{1}{M}-\frac{64}{15\pi l M^3}-\frac{32e^M}{15\pi l}K_3(M)+\frac{32e^M}{3\pi l M}K_2(M)\right\}.$$

The two last terms in this result can be expanded, when $M \gg 1$, in the series forms (see WATSON [6], p. 202)

$$\text{(A3.2)} \qquad e^M K_3(M) = \left(\frac{\pi}{2M}\right)^{1/2}\cdot\left\{1+\sum_{k=1}^{\infty}\frac{35\cdot 27\cdot 11\ \dots\ \{36-(2k-1)^2\}}{k!(8M)^k}\right\}$$

and

$$\text{(A3.3)} \qquad e^M K_2(M) = \left(\frac{\pi}{2M}\right)^{1/2}\cdot\left\{1+\sum_{j=1}^{\infty}\frac{15\cdot 7\cdot(-9)\ \dots\ \{16-(2j-1)^2\}}{j!(8M)^j}\right\}.$$

Thus the flow rate is given by

$$\text{(A3.4)} \qquad F=\frac{4lv_0}{ka^2}=\frac{4l}{M}-\frac{64\sqrt{2}}{15\pi^{1/2}M^{3/2}}-\frac{4l}{M^2}+\frac{8\sqrt{2}}{3\pi^{1/2}M^{5/2}}+\frac{17\sqrt{2}}{2\pi^{1/2}M^{7/2}}-\frac{256}{15\pi M^4}+\frac{64\sqrt{2}}{15\pi^{1/2}}\sum_{s=2}^{\infty}\frac{C(s)}{(s+1)!8^{s+1}M^{s+5/2}},$$

where

$$\text{(A3.5)} \qquad C(s) = 40(s+1)[15\cdot 7\cdot(-9)\ \dots\ \{16-(2s-1)^2\}]-35\cdot 27\cdot 11\ \dots\ \{36-(2s+1)^2\}.$$

Factorisation of each term involving the difference of two squares yields

$$\text{(A3.5')} \qquad C(s) = 40(s+1)\{(5)(7)(9)\ \dots\ (2s+1)(2s+3)\}\{(3)(1)(-1)(-3)\ \dots\ (7-2s)(5-2s)\}$$
$$-\{(7)(9)(11)\ \dots\ (5+2s)(7+2s)\}\{(5)(3)(1)\ \dots\ (7-2s)(5-2s)\}$$
$$=\frac{15}{8}\left\{\left(-\frac{7}{2}\right)\left(-\frac{9}{2}\right)\left(-\frac{11}{2}\right)\dots\left(-s-\frac{1}{2}\right)\left(-s-\frac{3}{2}\right)\right\}\cdot\left\{\left(s-\frac{5}{2}\right)\left(s-\frac{7}{2}\right)\right.$$
$$\left.\dots\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\right\}\cdot 2^{2s}\cdot(4s^2-16s-5)=\frac{15}{8\pi^{1/2}}(4s^2-16s-5)$$
$$\cdot 2^{2s}\cdot\left\{\left(-\frac{7}{2}\right)\left(-\frac{9}{2}\right)\dots\left(-s-\frac{3}{2}\right)\right\}\Gamma\left(s-\frac{3}{2}\right).$$

Thus, for any $s \geq 2$, the coefficient of $M^{-s-5/2}$ in the series expansion of F is

$$\text{(A3.6)} \qquad \frac{(4s^2-16s-5)}{\pi(s+1)!2^{s-1/2}}\cdot\left\{\left(-\frac{7}{2}\right)\left(-\frac{9}{2}\right)\left(-\frac{11}{2}\right)\dots\left(-s-\frac{3}{2}\right)\right\}\cdot\Gamma\left(s-\frac{3}{2}\right),$$

and hence our expansion (2.6I) matches exactly with (A3.4), the full series expansion of Williams' solution.

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