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Reflection and refraction of an acceleration wave at boundary between two nonlinear elastic materials

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THE PAPER considers the propagation of acceleration waves in nonlinear elastic materials. The propagation condition of the acceleration wave is derived. Then the definition of the slowness surface for the acceleration waves is introduced. The geometrical properties of the slowness surface and the relation between the condition of strict hyperbolicity and the condition of strong ellipticity are discussed. With the aid of the slowness surface ray curves are introduced and it is shown that the material energy flux-density vector is tangent to the ray curve. In the second part of the paper two laws governing reflection and refraction of the acceleration wave are derived. Using the derived laws and the slowness surface, all the parameters of reflected acceleration waves are determined.

W niniejszej pracy rozważa się propagację fali przyspieszenia w materiałach nieliniowo sprężystych. Na wstępie podano warunek propagacji fali przyspieszenia. Dalej zdefiniowano powierzchnię opóźnienia dla fal przyspieszenia i przedyskutowano jej własności geometryczne. Omówiony został również związek pomiędzy warunkiem ścisłej hiperboliczności i warunkiem silnej eliptyczności. Wykorzystując pojęcie powierzchni opóźnienia wprowadzono promienie i udowodniono, że materialny wektor strumienia gęstości energii jest styczny do promienia. W drugiej części pracy wyprowadzono dwa prawa rządzące odbiciem i załamaniem fali przyspieszenia. Wykorzystując wyprowadzone prawa odbicia i załamania a także powierzchnię opóźnienia wyznaczono wszystkie parametry odbitych i załamanych fal przyspieszenia.

В настоящей работе рассматривается распространение волны ускорения в нелинейно упругих материалах. Во введении дается условие распространения волны ускорения. Далее определена поверхность замедления для волн ускорения и обсуждены ее геометрические свойства. Обсуждено тоже соотношение между условием точной гиперболичности и условием точной эллиптичности. Используя понятие поверхности замедления, введены радиусы и доказано, что материальный вектор потока энергии касателен к радиусу. Во второй части работы выведены два закона управляющие отражением и преломлением волны ускорения. Используя выведенные законы отражения и преломления, а также поверхность замедления, определены все параметры отраженных и преломленных волн ускорения.

Introduction

PROPAGATION of acceleration waves in unbounded nonlinear elastic materials was investigated in many works, see, for example, the papers [1] to [4] and the references cited there.

In this paper we investigate the problem of reflection and refraction of an acceleration wave at an infinite boundary on both sides of which lie unbounded nonlinear elastic media differing in mass density and elastic properties.

In Sect. 1 we recall the basic facts concerning the acceleration wave propagation, that is we derive the propagation condition, we introduce the definition of the slowness surface and then we consider the ray curves along which the mechanical energy carried by the acceleration wave propagate.

In Sect. 2 we derive the first and the second law of reflection and refraction of the acceleration wave. These two laws are the counterparts of the well-known Snell's law governing reflection and refraction of an electromagnetic wave at an interface between two anisotropic media [8]. Then, with the aid of the derived laws and the slowness surface we calculate all parameters of reflected and refracted acceleration waves.

1. An acceleration wave in an unbounded nonlinear elastic material

Let the motion of an elastic body be given by [6]

$$x^i = x^i(X^\alpha, t), \quad i, \alpha = 1, 2, 3, \tag{1.1}$$

where X^α are the Cartesian coordinates of a particle in a natural (unstressed) configuration $\mathcal{B}_R$, and x^i are the Cartesian coordinates of the same particle at time t.

The balance of linear momentum of an elastic body in its local form may be expressed by the following differential equation [5]:

$$T_i^\alpha{}_{,\alpha} + \varrho_R b_i = \varrho_R x_{i,tt}, \tag{1.2}$$

where the comma denotes partial differentiation

$$(\cdot)_{,\alpha} \equiv \frac{\partial(\cdot)}{\partial X^\alpha}, \quad (\cdot)_{,t} \equiv \frac{\partial(\cdot)}{\partial t},$$

T_i^α is the Piola-Kirchhoff stress tensor, b_i denotes a body force, $x_{i,tt}$ is the acceleration and ϱ_R is the mass density in the natural configuration $\mathcal{B}_R$.

For simple elastic materials the Piola-Kirchhoff stress tensor T_i^α depends on the deformation gradient tensor $x^i{}_{,\alpha}$ and additionally on X^α for inhomogeneous materials

$$T_i^\alpha = T_i^\alpha(x^k{}_{,\beta}, X^\beta). \tag{1.3}$$

Hence with the aid of Eq. (1.3), Eq. (1.2) takes the following form [5]:

$$A_i{}^{(\alpha}{}_k{}^{\beta)} x^k{}_{,\alpha\beta} + q_i + \varrho_R b_i = \varrho_R x_{i,tt}, \tag{1.4}$$

where

$$A_i{}^{(\alpha}{}_k{}^{\beta)} \equiv \frac{1}{2}\left(A_i{}^\alpha{}_k{}^\beta + A_k{}^\beta{}_i{}^\alpha\right),$$

$$A_i{}^\alpha{}_k{}^\beta = \frac{\partial T_i^\alpha}{\partial x^k{}_{,\beta}}, \tag{1.5}$$

$$q_i \equiv \left(\frac{\partial T_i^\alpha}{\partial X^\alpha}\right)_{x^k{}_{,\beta}=\mathrm{const}}.$$

The tensor $A_i{}^\alpha{}_k{}^\beta$ defined by Eq. $(1.5)_2$ is the elasticity tensor depending on $x^i{}_{,\alpha}$ and X^α. For a hyperelastic material the Piola-Kirchhoff stress tensor is expressed by [5]

$$T_i^\alpha = \frac{\partial}{\partial x^i{}_{,\alpha}}\{\sigma(x^k{}_{,\beta}, X^\beta)\}, \tag{1.6}$$

where $\sigma = \sigma(x^i{}_{,\alpha}, X^\alpha)$ is the strain energy function. Hence, for the hyperelastic materials the elasticity tensor $A_i{}^\alpha{}_k{}^\beta$ fulfills the following symmetry condition:

$$A_i{}^\alpha{}_k{}^\beta = A_k{}^\beta{}_i{}^\alpha.$$

Consider a propagating discontinuity surface (wave) in the natural configuration $\mathscr{B}_R$ [6]

$$\Psi(X^\alpha)-t = 0, \tag{1.7}$$

with a speed of propagation

$$U \equiv (S_\alpha S_\alpha)^{-\frac{1}{2}}. \tag{1.8}$$

The vector

$$S_\alpha \equiv \Psi_{,\alpha}, \tag{1.9}$$

is called the slowness vector of the wave (1.7). The direction of propagation (the unit normal vector) of Eq. (1.7) is expressed as follows:

$$N_\alpha = US_\alpha, \tag{1.10}$$

where U is given by Eq. (1.8) and S_α is defined by the relation (1.9). Note, that the length of the slowness vector S_α equals $1/U$, i.e. the reciprocal value of the speed of propagation.

We assume that the function (1.1) itself and its first derivatives with respect to X^α and t are continuous across Eq. (1.7); then the jumps of the second derivatives of the function (1.1) with respect to X^α and t may be written as [6, 2]

$$\begin{aligned} [\![x^i{}_{,\alpha\beta}]\!] &= sD^iN_\alpha N_\beta, \\ [\![x^i{}_{,\alpha t}]\!] &= [\![x^i{}_{,t\alpha}]\!] = -sUD^iN_\beta, \\ [\![x^i{}_{,tt}]\!] &= sU^2D^i, \end{aligned} \tag{1.11}$$

where U and N_α are given by Eqs. (1.8) and (1.10) respectively, the non-zero unit vector D^i is called the amplitude of the jump and the scalar s denotes the intensity of the jump. Note that on $\Psi(X^\alpha)-t = 0$ the jump in the acceleration differs from zero; thus we call Eq. (1.7) an acceleration wave.

From the above considerations it follows that T_i^α, $A_i{}^\alpha{}_k{}^\beta$ and q_i defined by Eq. $(1.5)_3$ are continuous across Eq. (1.7) as the functions of $x^i{}_{,\alpha}$. Hence, with the aid of Eqs. (1.3), (1.5) and $(1.11)_{1,2}$ the following relations

$$\begin{aligned} [\![T_{i,t}^\alpha]\!] &= A_i{}^\alpha{}_k{}^\beta[\![x^k{}_{,\beta t}]\!] = -sUA_i{}^\alpha{}_k{}^\beta D^kN_\beta, \\ [\![T_{i,\alpha}^\alpha]\!] &= A_i{}^\alpha{}_k{}^\beta[\![x^k{}_{,\alpha\beta}]\!] = sA_i{}^\alpha{}_k{}^\beta D^kN_\alpha N_\beta, \end{aligned} \tag{1.12}$$

hold.

The balance of linear momentum holds in the whole region ahead and behind the acceleration wave; while on the acceleration wave it must be replaced by its jump. Assuming that the body force b_i is continuous across the acceleration wave, making use of Eqs. $(1.11)_3$ and $(1.12)_2$, the jump of Eq. (1.2) on Eq. (1.7) may be written as [2]

$$(A_i{}^\alpha{}_k{}^\beta N_\alpha N_\beta - \varrho_R U^2\delta_{ik})D^k = 0. \tag{1.13}$$

The relation (1.13) is the propagation condition of the acceleration wave in the natural configuration $\mathscr{B}_R$. In accordance with the Fresnel-Hadamard theorem [5], the amplitude D^i of the acceleration wave propagating in the direction N_α must be the eigenvector of the acoustical tensor

$$Q_{ik} \equiv A_i{}^\alpha{}_k{}^\beta N_\alpha N_\beta,$$

and the speed of propagation U of the acceleration wave must be such that $\varrho_R U^2$ is the corresponding eigenvalue of the acoustical tensor Q_{ik}.

The system of equations (1.13) has non-trivial solutions D^i (non-trivial acceleration waves exist) if and only if

$$\det(A_i{}^\alpha{}_k{}^\beta N_\alpha N_\beta - \varrho_R U^2 \delta_{ik}) = 0, \tag{1.14}$$

what, with the aid of Eq. (1.10), may be written in two equivalent forms:

$$\begin{aligned} H(\Psi_{,\alpha}, X^\alpha) &\equiv \det(A_i{}^\alpha{}_k{}^\beta \Psi_{,\alpha}\Psi_{,\beta} - \varrho_R \delta_{ik}) = 0, \\ H(S_\alpha, X^\alpha) &\equiv \det(A_i{}^\alpha{}_k{}^\beta S_\alpha S_\beta - \varrho_R \delta_{ik}) = 0. \end{aligned} \tag{1.15}$$

Equation $(1.15)_1$ is identical with the characteristic condition of Eq. (1.4), and this proves that the acceleration waves are carried by the characteristic surfaces of Eq. (1.4), for this see [7] Chapter VI. This fact enables us to construct the acceleration waves with the aid of the bi-characteristic ray curves which will be introduced later on. Equation $(1.15)_1$ is the counterpart of the Fresnel's differential equation (eiconal equation) occurring in geometrical optics [8].

In a space with the $\{S_\alpha\}$ Cartesian system of coordinates, Eq. $(1.15)_2$ defines the slowness (normal) surface — compare with [7], [9], [10] and [11]. If the S_α's are real, then Eq. $(1.15)_2$ gives us the real surface of the sixth degree symmetrical with respect to the origin of the $\{S_\alpha\}$ system of coordinates. If the acoustical tensor Q_{ik} is positive-definite for every direction of propagation N_α (the $S-E$ condition holds [5]), then, from Eq. (1.14) follows that three real speeds of propagation U are possible. This, with the aid of Eq. (1.10), leads to the conclusion that the slowness surface consists at the most of the three real sheets. Thus every straight line drawn through the origin of the $\{S_\alpha\}$ system of coordinates cuts the slowness surface in six points symmetrical in pairs.

By definition the slowness surface is the locus of the ends of the slowness vectors S_α of all possible acceleration waves emanating from the point X^α at which the origin of the $\{S_\alpha\}$ system of coordinates is located.

From the fact that slowness surface consists of three sheets and it is a sixth degree surface it follows that the inner sheet of Eq. $(1.15)_2$ (having no points in common with the other sheets of Eq. $(1.15)_2$) must be convex. Otherwise, there could exist a straight line intersecting the inner sheet of Eq. $(1.15)_2$ in four or more points and the remaining sheets at least in four further points; what contradicts the fact that Eq. $(1.15)_2$ is of the sixth degree.

In this paper we consider the case when the slowness surface consists of three real and separate sheets, that is Eq. (1.4) is strictly hyperbolic — compare with [7], Chapter VI. With the aid of Eq. (1.10) this implies that for every direction of propagation three real and different speeds of propagation are possible. Then, by Eq. (1.13) a uniquely determined real orthogonal triad of amplitudes exists.

Hence the condition of strict hyperbolicity implies the condition of strong ellipticity (the $S-E$ condition [5]), but the opposite implication does not hold. With the aid of the slowness surface we may define ray curves in the following way [7, 8]:

$$\begin{aligned} \frac{dX^\alpha}{d\varkappa} &= \frac{\partial}{\partial S_\alpha}\{H(S_\alpha, X^\alpha)\}, \\ \frac{dS_\alpha}{d\varkappa} &= -\frac{\partial}{\partial X^\alpha}\{H(S_\alpha, X^\alpha)\}, \end{aligned} \tag{1.16}$$

where $H(S_\alpha, X^\alpha) = 0$ is given by Eq. $(1.15)_2$ and $\varkappa$ denotes a parameter along the ray curve. From Eq. $(1.16)_1$ it follows that the tangent vector to the ray curve is parallel to the suitable normal vector to the slowness surface — see Fig. 1.

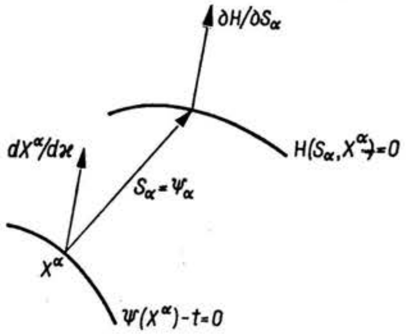

FIG. 1.

The right hand side of Eq. $(1.16)_1$ may be written as [12] page 597

$$\frac{\partial}{\partial S_\alpha}\{H(S_\alpha, X^\alpha)\} = \frac{\partial H_{ik}(S_\alpha, X^\alpha)}{\partial S_\alpha}\{\operatorname{cof} H_{ik}(S_\alpha, X_\alpha)\}, \tag{1.17}$$

where

$$H_{ik}(S_\alpha, X^\alpha) \equiv A_i{}^\alpha{}_k{}^\beta S_\alpha S_\beta - \varrho_R \delta_{ik}, \tag{1.18}$$

and $\operatorname{cof} H_{ik}(S_\alpha, X^\alpha)$ denotes the cofactor of $H_{ik}(S_\alpha, X^\alpha)$ satisfying the following relation:

$$\{\operatorname{cof} H_{ik}\} H_{ij} = H\delta_{kj} = 0, \tag{1.19}$$

where $H = 0$ denotes the slowness surface (the arguments S_α and X^α of H_{ik} and H for simplicity are omitted). With the aid of Eqs. (1.10) and (1.18), Eq. (1.13) takes the form

$$H_{ik} D^k = 0. \tag{1.20}$$

Comparing Eq. (1.19) with Eq. (1.20) we conclude that

$$\operatorname{cof} H_{ik} \propto D^i D^k, \tag{1.21}$$

where $\propto$ denotes proportionality. Making use of Eq. (1.18) we obtain

$$\frac{\partial H_{ik}}{\partial S_\alpha} = 2A_i{}^\alpha{}_k{}^\beta S_\beta.$$

This with the aid of the relation (1.21) allows us to write Eq. (1.17) in the following way:

$$\frac{\partial}{\partial S_\alpha}\{H(S_\alpha, X^\alpha)\} = mA_i{}^\alpha{}_k{}^\beta D^i D^k S_\beta, \tag{1.22}$$

where m is a constant.

In the case when the acceleration wave propagates into the region where $x^i{}_{,t} = 0$ and $T_i^\alpha = 0$, it is possible to show that the material energy flux-density vector P^α may be written as [4]

$$P^\alpha = -[\![T_i^\alpha{}_{,t}]\!][\![x^i{}_{,tt}]\!]. \tag{1.23}$$

Using Eqs. $(1.11)_3$ and $(1.12)_1$ this becomes

(1.24) $$P^\alpha = (sU^2)^2 A_i{}^\alpha{}_k{}^\beta D^i D^k S_\beta.$$

Comparing Eq. (1.24) with Eq. $(1.16)_1$ and making use of Eq. (1.22), we conclude that the material energy flux-density vector is the tangent vector to the ray curve. This proves that the mechanical energy carried by the acceleration wave propagates along the ray curve.

2. Reflected and refracted acceleration waves

In the following section we consider propagation of the acceleration waves in bounded nonlinear elastic media.

Let two nonlinear elastic bodies differing in elastic properties and having different mass densities be given:

(2.1) $$\overset{\mathrm{I}}{A}{}_i{}^\alpha{}_k{}^\beta \neq \overset{\mathrm{II}}{A}{}_i{}^\alpha{}_k{}^\beta, \qquad \overset{\mathrm{I}}{\varrho}_R \neq \overset{\mathrm{II}}{\varrho}_R,$$

where $\overset{\mathrm{I}}{A}{}_i{}^\alpha{}_k{}^\beta(\overset{\mathrm{II}}{A}{}_i{}^\alpha{}_k{}^\beta)$ denotes the elasticity tensor for the first (second) medium and $\overset{\mathrm{I}}{\varrho}_R$ $(\overset{\mathrm{II}}{\varrho}_R)$ is the mass density of the first (second) medium in its natural configuration. These two bodies are assumed to be rigidly coupled at the interface. The surface of the interface may be written in the parametric form

(2.2) $$X^\alpha = B^\alpha(\xi, \eta), \quad \alpha = 1, 2, 3,$$

where ξ and η are surface parameters. The unit normal vector to the interface (2.2) is denoted by M_α, thus the following relations

(2.3) $$M_\alpha M_\alpha = 1, \qquad M_\alpha B^\alpha_{,\xi} = M_\alpha B^\alpha_{,\eta} = 0,$$

hold.

We assume that at each point (ξ, η) of the interface (2.2) the unit normal vector M_α is uniquely determined and it is directed from the first into the second medium.

Let the acceleration wave

(2.4) $$\overset{\mathscr{I}}{\Psi}(X^\alpha) - t = 0, \quad \mathscr{I} \text{ fixed},$$

be given propagating in the first medium. The acceleration wave (2.4) is called the incident acceleration wave on the interface (2.2) when it intersects that interface for some time t. We assume that the slowness vector $\overset{\mathscr{I}}{S}{}^\alpha$ and the intensity $\overset{\mathscr{I}}{s}$ of Eq. (2.4) at some point (ξ, η) on the interface (2.2) are known. Then, by Eqs. (1.8), (1.10) and (1.13) we know the speed of propagation $\overset{\mathscr{I}}{U}$, the propagation direction $\overset{\mathscr{I}}{N}_\alpha$ and the amplitude $\overset{\mathscr{I}}{D}{}^i$ of Eq. (2.4).

As Eq. (2.4) is the incident wave for some time t, we have [8]

(2.5) $$\Phi(\xi, \eta) \equiv \overset{\mathscr{I}}{\Psi}\{B^\alpha(\xi, \eta)\}, \quad \mathscr{I} \text{ fixed}.$$

The incident acceleration wave gives rise to a reflected acceleration wave

$$\overset{\mathscr{R}}{\Psi}(X^\alpha)-t = 0, \quad \mathscr{R} \text{ fixed}, \tag{2.6$_1$}$$

propagating back into the first medium, while in the second medium Eq. (2.4) gives rise to a refracted (transmitted) acceleration wave

$$\overset{\mathscr{T}}{\Psi}(X^\alpha)-t = 0, \quad \mathscr{T} \text{ fixed}. \tag{2.6$_2$}$$

The reflected $(2.6)_1$ (refracted $(2.6)_2$) acceleration wave at a point (ξ, η) on the interface (2.2) must have the same value as the incident acceleration wave (2.4). Thus our aim is to find the reflected (refracted) acceleration wave which, on the interface (2.2), satisfies the following condition:

$$\begin{aligned} &\overset{\mathscr{R}}{\Psi}\{B^\alpha(\xi, \eta)\} = \Phi(\xi, \eta), \\ &\overset{\mathscr{T}}{\Psi}\{B^\alpha(\xi, \eta)\} = \Phi(\xi, \eta), \quad \mathscr{R}, \mathscr{T} \text{ fixed}, \end{aligned} \tag{2.7}$$

where $\Phi(\xi, \eta)$ is defined by Eq. (2.5). Note that the equation

$$\Phi(\xi, \eta)-t = 0,$$

in the (ξ, η) Cartesian systems of coordinates represents the curve along which Eqs. (2.4) and $(2.6)_{1,2}$ meet on the interface (2.2), see Fig. 2. We assume that the functions $\overset{\mathscr{J}}{\Psi}$, $\overset{\mathscr{R}}{\Psi}$

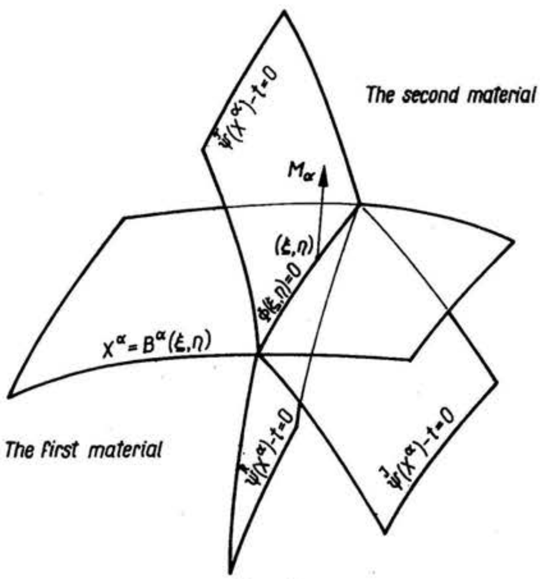

FIG. 2.

and $\overset{\mathscr{T}}{\Psi}$ have continuous derivatives up to and on the interface (2.2). By Eqs. (2.5) and (2.7) at any point (ξ, η) on the interface (2.2) the following relations

$$\begin{aligned} &\Phi_{,\xi} = \overset{\mathscr{J}}{\Psi}_{,\alpha} B^\alpha_{,\xi} = \overset{\mathscr{R}}{\Psi}_{,\alpha} B^\alpha_{,\xi} = \overset{\mathscr{T}}{\Psi}_{,\alpha} B^\alpha_{,\xi}, \\ &\Phi_{,\eta} = \overset{\mathscr{J}}{\Psi}_{,\alpha} B^\alpha_{,\eta} = \overset{\mathscr{R}}{\Psi}_{,\alpha} B^\alpha_{,\eta} = \overset{\mathscr{T}}{\Psi}_{,\alpha} B^\alpha_{,\eta}, \quad \mathscr{J}, \mathscr{R}, \mathscr{T} \text{ fixed}, \end{aligned} \tag{2.8}$$

hold, where $\overset{\mathcal{I}}{\Psi}_{,\alpha}$, $\overset{\mathcal{R}}{\Psi}_{,\alpha}$ and $\overset{\mathcal{T}}{\Psi}_{,\alpha}$ are the derivatives on one side of the interface (2.2). Remembering that $S_\alpha \equiv \Psi_{,\alpha}$, where S_α is the slowness vector, Eq. (2.8) yields

$$\begin{aligned} &(\overset{\mathcal{R}}{S}_\alpha - \overset{\mathcal{I}}{S}_\alpha) B^\alpha_{.\xi} = 0, \\ &(\overset{\mathcal{R}}{S}_\alpha - \overset{\mathcal{I}}{S}_\alpha) B^\alpha_{.\eta} = 0, \quad \mathcal{I}, \mathcal{R}, \mathcal{T} \text{ fixed}, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &(\overset{\mathcal{T}}{S}_\alpha - \overset{\mathcal{I}}{S}_\alpha) B^\alpha_{.\xi} = \theta, \\ &(\overset{\mathcal{T}}{S}_\alpha - \overset{\mathcal{I}}{S}_\alpha) B^\alpha_{.\eta} = 0, \quad \mathcal{I}, \mathcal{R}, \mathcal{T} \text{ fixed}, \end{aligned} \tag{2.10}$$

where $\overset{\mathcal{I}}{S}_\alpha$, $\overset{\mathcal{R}}{S}_\alpha$ and $\overset{\mathcal{T}}{S}_\alpha$ are the slowness vectors of the incident, reflected and refracted acceleration waves. Form Eqs. (2.9) and (2.3) it follows that $\overset{\mathcal{R}}{S}_\alpha - \overset{\mathcal{I}}{S}_\alpha$ is parallel to M_α at a common point (ξ, η) on the interface (2.2). Equations (2.10) and (2.3) tell us that $\overset{\mathcal{T}}{S}_\alpha - \overset{\mathcal{I}}{S}_\alpha$ is parallel to M_α at a common point (ξ, η) on the interface (2.2). This allows us to write the following relations:

$$\begin{aligned} &\overset{\mathcal{R}}{S}_\alpha = \overset{\mathcal{I}}{S}_\alpha + \overset{\mathcal{R}}{\lambda} M_\alpha, \\ &\overset{\mathcal{T}}{S}_\alpha = \overset{\mathcal{I}}{S}_\alpha + \overset{\mathcal{T}}{\lambda} M_\alpha, \quad \mathcal{I}, \mathcal{R}, \mathcal{T} \text{ fixed}, \end{aligned} \tag{2.11}$$

where $\overset{\mathcal{R}}{\lambda}$ and $\overset{\mathcal{T}}{\lambda}$ are real fixed constants. Employing the absolute notation it follows, from Eq. (2.11),

$$\begin{aligned} &\overset{\mathcal{I}}{\mathbf{S}} \times \mathbf{M} = \overset{\mathcal{R}}{\mathbf{S}} \times \mathbf{M}, \\ &\overset{\mathcal{I}}{\mathbf{S}} \times \mathbf{M} = \overset{\mathcal{T}}{\mathbf{S}} \times \mathbf{M}, \quad \mathcal{I}, \mathcal{R}, \mathcal{T} \text{ fixed}, \end{aligned} \tag{2.12}$$

where $\times$ denotes the vector product. The relations (2.12) express the first law of reflection and refraction of the acceleration wave:

The slowness vector $\overset{\mathcal{I}}{S}_\alpha$ of the incident acceleration wave, the slowness vector $\overset{\mathcal{R}}{S}_\alpha$ of the reflected acceleration wave, the slowness vector $\overset{\mathcal{T}}{S}_\alpha$ of the refracted acceleration wave and the unit normal vector M_α to the interface are co-planar. This common plane determined by $\overset{\mathcal{I}}{S}_\alpha$ and M_α is called the plane of incidence [8, 11].

Taking into account the fact that the length of the slowness vector is equal to the reciprocal value of the speed of propagation, then, if we make use of Eq. $(2.3)_1$ and the definition of the vector product, Eqs. (2.12) yield

$$\frac{\overset{\mathcal{I}}{U}}{\sin\overset{\mathcal{I}}{\theta}} = \frac{\overset{\mathcal{R}}{U}}{\sin\overset{\mathcal{R}}{\theta}} = \frac{\overset{\mathcal{T}}{U}}{\sin\overset{\mathcal{T}}{\theta}}, \quad \mathcal{I}, \mathcal{R}, \mathcal{T} \text{ fixed}, \tag{2.13}$$

where $\overset{\mathcal{I}}{U}$, $\overset{\mathcal{R}}{U}$ and $\overset{\mathcal{T}}{U}$ are the speeds of propagation of the incident, reflected and refracted acceleration waves. The angles $\overset{\mathcal{I}}{\theta}$, $\overset{\mathcal{R}}{\theta}$ and $\overset{\mathcal{T}}{\theta}$ measured from M_α are the angles of incidence,

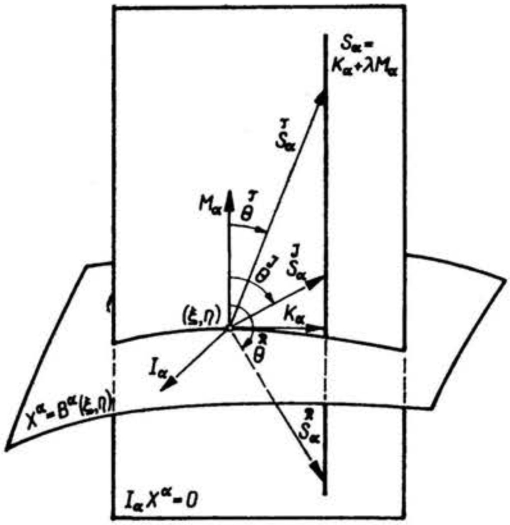

FIG. 3.

reflection and refraction, for this see Fig. 3. The relation (2.13) expresses the second law of reflection and refraction of the acceleration wave:

The speeds of propagation of the incident, reflected and refracted acceleration waves are proportional to the sines of the angles of incidence, reflection and refraction.

The relations (2.12) and (2.13) are the counterparts of the well-known Snell's law of reflection and refraction [8, 11].

The normal vector to the plane of incidence may be defined as (see Eq. (2.12))

$$\mathbf{I} \equiv \overset{\mathscr{J}}{\mathbf{S}} \times \mathbf{M}, \quad \mathscr{J} \text{ fixed}, \tag{2.14}$$

hence the equation of the plane of incidence is

$$I_\alpha X_\alpha = 0, \tag{2.15}$$

where I_α is defined by Eq. (2.14). We define the following vector

$$\mathbf{K} \equiv \mathbf{I} \times \mathbf{M}, \tag{2.16}$$

where $\mathbf{I}$ is given by Eq. (2.14). Using Eq. (2.14) and the definition of the vector triple product, Eq. (2.16) yields

$$\mathbf{K} = \overset{\mathscr{J}}{\mathbf{S}} - (\mathbf{M} \cdot \overset{\mathscr{J}}{\mathbf{S}})\mathbf{M}, \quad \mathscr{J} \text{ fixed}, \tag{2.17}$$

where a dot denotes the scalar product. Making use of Eq. (2.12) we conclude that $\mathbf{K}$ may be expressed in the following way as well:

$$\mathbf{K} = \overset{\mathscr{R}}{\mathbf{S}} - (\mathbf{M} \cdot \overset{\mathscr{R}}{\mathbf{S}})\mathbf{M} = \overset{\mathscr{T}}{\mathbf{S}} - (\mathbf{M} \cdot \overset{\mathscr{T}}{\mathbf{S}})\mathbf{M}. \tag{2.18}$$

From Eq. (2.16) it follows that $\mathbf{K}$ lies on the plane of incidence (2.15) and it is perpendicular to $\mathbf{M}$, see Fig. 3. Making use of Eqs. (2.17) and (2.18) it is easy to show

$$|\mathbf{K}| = \frac{\sin\overset{\mathscr{J}}{\theta}}{\overset{\mathscr{J}}{U}} = \frac{\sin\overset{\mathscr{R}}{\theta}}{\overset{\mathscr{R}}{U}} = \frac{\sin\overset{\mathscr{T}}{\theta}}{\overset{\mathscr{T}}{U}}, \tag{2.19}$$

where $|\mathbf{K}|$ is the length of $\mathbf{K}$. The relations (2.17), (2.18) and (2.19) express the fact that the incident, reflected and refracted acceleration waves at any point (ξ, η) on the interface (2.2) have their vector $\mathbf{K}$ vectorially equal.

Now we proceed to the computation $\overset{\mathcal{R}}{S}_\alpha(\overset{\mathcal{T}}{S}_\alpha)$ explicitly and we determine the other parameters of the reflected (refracted) acceleration waves. From Eqs. (2.11) it follows that the ends of $\overset{\mathcal{R}}{S}_\alpha$ and $\overset{\mathcal{T}}{S}_\alpha$ must lie on a straight line parallel to M_α

$$S_\alpha = K_\alpha + \lambda M_\alpha, \tag{2.20}$$

where K_α is defined by Eq. (2.17) and λ is parameter along the line.

On the other hand we know that the slowness surface is the locus of the ends of the slowness vectors. Thus the end of $\overset{\mathcal{R}}{S}_\alpha$ and $\overset{\mathcal{T}}{S}_\alpha$ must lie on the line (2.20) and on one of three sheets of the slowness surface. In order to find the slowness vectors $\overset{\mathcal{T}}{S}_\alpha$ and $\overset{\mathcal{R}}{S}_\alpha$ satisfying Eqs. (2.12) and (2.13) we proceed as follows:

(i). We take the equations of the slowness surfaces for the first and the second medium at a point (ξ, η) on the interface (2.2)

$$\begin{aligned} \overset{\mathrm{I}}{H}\{\overset{\mathrm{I}}{S}_\alpha, B^\alpha(\xi, \eta)\} &\equiv \det(\overset{\mathrm{I}}{A}{}_i{}^\alpha{}_k{}^\beta \overset{\mathrm{I}}{S}_\alpha \overset{\mathrm{I}}{S}_\beta - \overset{\mathrm{I}}{\varrho}_R\, \delta_{ik}) = 0, \\ \overset{\mathrm{II}}{H}\{\overset{\mathrm{II}}{S}_\alpha, B^\alpha(\xi, \eta)\} &\equiv \det(\overset{\mathrm{II}}{A}{}_i{}^\alpha{}_k{}^\beta \overset{\mathrm{II}}{S}_\alpha \overset{\mathrm{II}}{S}_\beta - \overset{\mathrm{II}}{\varrho}_R\, \delta_{ik}) = 0, \end{aligned} \tag{2.21}$$

where $\overset{\mathrm{I}}{A}{}_i{}^\alpha{}_k{}^\beta(\overset{\mathrm{II}}{A}{}_i{}^\alpha{}_k{}^\beta)$ denotes the elasticity tensor of the first (second) medium, $\overset{\mathrm{I}}{\varrho}_R(\overset{\mathrm{II}}{\varrho}_R)$ is the mass density in the natural configuration of the first (second) medium and $\overset{\mathrm{I}}{S}_\alpha(\overset{\mathrm{II}}{S}_\alpha)$ is the slowness vector of any acceleration wave proceeding from a point (ξ, η) on the interface (2.2) into the first (second) medium.

(ii). We construct the curves of intersection of the slowness surfaces (2.21) by the plane of incidence (2.15).

(iii). The points of intersection of these curves with the line (2.20) are the ends of $\overset{\mathcal{R}}{S}_\alpha$ and $\overset{\mathcal{T}}{S}_\alpha$. On Fig. 4 the above construction is shown for $\overset{\mathcal{R}}{S}_\alpha$ only and the plane of the picture is the plane of incidence.

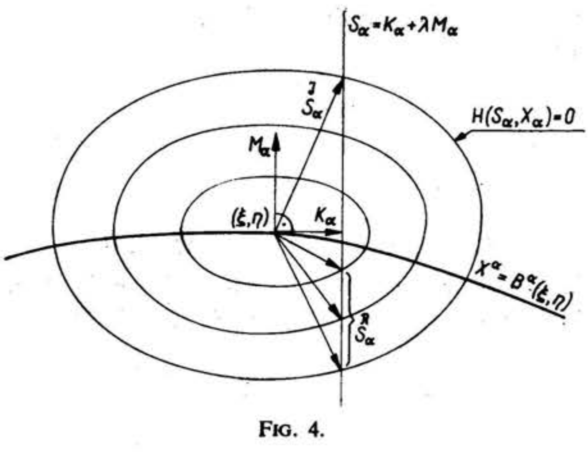

FIG. 4.

To compute $\overset{\mathcal{R}}{S}_\alpha$ and $\overset{\mathcal{T}}{S}_\alpha$ analytically, we substitute Eq. (2.20) into Eq. (2.21)

$$
\begin{aligned}
&\overset{\mathrm{I}}{H}\{K_\alpha+\lambda M_\alpha,\, B^\alpha(\xi,\eta)\} = 0,\\
&\overset{\mathrm{II}}{H}\{K_\alpha+\lambda M_\alpha,\, B^\alpha(\xi,\eta)\} = 0,
\end{aligned}
\tag{2.22}
$$

where K_α and M_α are known and λ is to be determined. The slowness surfaces (2.21) are of the sixth degree in $\overset{\mathrm{I}}{S}_\alpha$ and $\overset{\mathrm{II}}{S}_\alpha$ respectively, hence Eqs. (2.22) are of the sixth degree in λ.

Consider the case when six roots λ of Eqs. (2.22) are real and distinct, what implies that the line (2.20) intersects all three sheets of each of the slowness surfaces (2.21). Introducing into Eq. (2.20) the λ's computed from Eq. (2.22), we obtain the slowness vectors $\overset{\mathcal{R}}{S}_\alpha$ and $\overset{\mathcal{T}}{S}_\alpha$. But not all of them are the slowness vectors of the reflected (refracted) acceleration waves.

We remember that the material energy flux-density vector is parallel to the normal vector to the slowness surface (see Sect. 1). On the basis of this fact we formulate the selection rule for the slowness vector $\overset{\mathcal{R}}{S}_\alpha(\overset{\mathcal{T}}{S}_\alpha)$ corresponding to the reflected (refracted) acceleration waves:

Only these $\overset{\mathcal{R}}{S}_\alpha(\overset{\mathcal{T}}{S}_\alpha)$ are the slowness vectors of the reflected (refracted) acceleration waves for which the corresponding material energy flux-density vector points inside the first (second) medium.

Thus these points of intersection of the line (2.20) with the surface $(2.21)_1$ $\left((2.21)_2\right)$ are the ends of the slowness vectors $\overset{\mathcal{R}}{S}_\alpha(\overset{\mathcal{T}}{S}_\alpha)$ of the reflected (refracted) acceleration waves in which the normal vector to the surface $(2.21)_1$ $\left((2.21)_2\right)$ points inside the first (second) medium.

In any compressible nonlinear elastic material at least three acceleration waves are possible. Thus, taking into account the above selection rule we conclude that in the case when Eqs. (2.22) have six different and real roots three reflected

$$
\overset{\mathcal{R}}{\Psi}(X^\alpha)-t = 0, \quad \mathcal{R} = 1,2,3 \quad \text{(refracted } \overset{\mathcal{T}}{\Psi}(X^\alpha)-t = 0, \quad \mathcal{T} = 1,2,3)
$$

acceleration waves are possible.

The slowness vectors $\overset{\mathcal{R}}{S}_\alpha(\overset{\mathcal{T}}{S}_\alpha)$ give us, with the aid of Eqs. (1.8) and (1.10), the speeds of propagation $\overset{\mathcal{R}}{U}(\overset{\mathcal{T}}{U})$ and the directions of propagation $\overset{\mathcal{R}}{N}_\alpha(\overset{\mathcal{T}}{N}_\alpha)$ of the reflected (refracted) acceleration waves. Then the second law of reflection and refraction expressed by Eq. (2.13) yields the angles of reflection $\overset{\mathcal{R}}{\theta}$ and refraction $\overset{\mathcal{T}}{\theta}$. Finally, from Eq. (1.13) the amplitudes $\overset{\mathcal{R}}{D^i}(\overset{\mathcal{T}}{D^i})$ of the reflected (refracted) acceleration waves may be determined.

The compatibility condition and the sum of the material energy flux-density vectors at a point (ξ, η) on the interface (2.2) in which the incident, reflected and refracted acceleration waves meet each other may be written in the following way:

$$
\begin{gathered}
[\![x^i{}_{,tt}]\!]^{\mathcal{J}}+[\![x^i{}_{,tt}]\!]^{\mathcal{R}}+[\![x^i{}_{,tt}]\!]^{\mathcal{T}}=0,\\
\overset{\mathcal{J}}{P}{}^{\alpha}-\overset{\mathcal{R}}{P}{}^{\alpha}-\overset{\mathcal{T}}{P}{}^{\alpha}=0, \quad \mathcal{J} \text{ fixed}, \quad \mathcal{R}, \mathcal{T}=1,2,3,
\end{gathered} \tag{2.23}
$$

where $[\![\cdot]\!]^{\mathcal{J}}$, $[\![\cdot]\!]^{\mathcal{R}}$ and $[\![\cdot]\!]^{\mathcal{T}}$ denote the jump across the incident, reflected and refracted acceleration waves, while $\overset{\mathcal{J}}{P}{}^{\alpha}$, $\overset{\mathcal{R}}{P}{}^{\alpha}$ and $\overset{\mathcal{T}}{P}{}^{\alpha}$ are material energy flux-density vectors. With the aid of Eqs. $(1.11)_3$ and (1.24), Eq. (2.23) takes the form

$$
\begin{gathered}
\overset{\mathcal{J}}{s}\overset{\mathcal{J}}{U}{}^2\overset{\mathcal{J}}{D}{}^i+\overset{\mathcal{R}}{s}\overset{\mathcal{R}}{U}{}^2\overset{\mathcal{R}}{D}{}^i+\overset{\mathcal{T}}{s}\overset{\mathcal{T}}{U}{}^2\overset{\mathcal{T}}{D}{}^i=0,\\
\overset{\mathrm{I}}{A}{}_i{}^{\alpha}{}_k{}^{\beta}\{(\overset{\mathcal{J}}{s}\overset{\mathcal{J}}{U}{}^2)^2\overset{\mathcal{J}}{D}{}^i\overset{\mathcal{J}}{D}{}^k\overset{\mathcal{J}}{S}_{\beta}-(\overset{\mathcal{R}}{s}\overset{\mathcal{R}}{U}{}^2)^2\overset{\mathcal{R}}{D}{}^i\overset{\mathcal{R}}{D}{}^k\overset{\mathcal{R}}{S}_{\beta}\}-(\overset{\mathcal{T}}{s}\overset{\mathcal{T}}{U}{}^2)^2\overset{\mathrm{II}}{A}{}_i{}^{\alpha}{}_k{}^{\beta}\overset{\mathcal{T}}{D}{}^i\overset{\mathcal{T}}{D}{}^k\overset{\mathcal{T}}{S}_{\beta}=0,\\
\mathcal{J} \text{ fixed}, \quad \mathcal{R}, \mathcal{T}=1,2,3.
\end{gathered} \tag{2.24}
$$

These relations represent the system of six equations from which six intensities $\overset{\mathcal{R}}{s}$ and $\overset{\mathcal{T}}{s}$, $\mathcal{R}, \mathcal{T} = 1, 2, 3$ of the reflected (refracted) acceleration waves may be computed. In this way all the parameters of the reflected (refracted) acceleration waves are determined.

In the case when we consider only reflection of the acceleration wave, the relation $(2.24)_1$ is sufficient for the unique determination of the intensities $\overset{\mathcal{R}}{s}$, $\mathcal{R} = 1, 2, 3$. This is possible provided that the amplitudes $\overset{\mathcal{R}}{D}{}^i$ are linearly independent [13].

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