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# Perturbation solution for rigid-viscoplastic spherical container

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A PERTURBATION technique around a rigid-perfectly plastic solution is proposed for analysing the plastic response of strain-rate sensitive metal structures subjected to high intensity dynamic loads. The proposed method is explained on an example of an impulsively loaded thick-walled, rigid-viscoplastic spherical container. An exact solution involving a non-dimensional parameter $\alpha$ is obtained and shown to tend to the rigid-perfectly plastic solution when $\alpha \to 0$. Using the Linstedt-Poincaré procedure a perturbation solution is found and then modified with the help of Shank's transformation so as to extend the admissible range of the small parameter $\alpha$. By comparing the perturbation and exact solutions, practical approximate formulae are proposed.

Zaproponowano metodę perturbacyjną analizy plastycznego zachowania się dynamicznie obciążonych konstrukcji metalowych z uwzględnieniem wpływu prędkości odkształcenia. Metodę tę wyjaśniono na przykładzie obciążonego impulsem prędkości grubościennego zbiornika kulistego z materiału sztywno-lepkoplastycznego. Uzyskano rozwiązanie zamknięte, które zależy od bezwymiarowego parametru $\alpha$ i przechodzi w rozwiązanie sztywno-idealnie plastyczne, gdy $\alpha \to 0$. Stosując metodę Linstedta-Poincaré znaleziono rozwiązanie perturbacyjne wokoło rozwiązania sztywno-idealnie plastycznego. Zastosowano transformację Shanka dla rozszerzenia dopuszczalnego zakresu małego parametru $\alpha$. Przez porównanie rozwiązania perturbacyjnego z rozwiązaniem zamkniętym zaproponowano praktyczne wzory przybliżone.

Предложен пертурбационный метод анализа пластического поведения динамически нагруженных металлических конструкций с учетом влияния скорости деформации. Этот метод выяснен на примере нагруженного импульсом скорости толстостенного сферического резервуара из жестко-вязкопластического материала. Получено замкнутое решение, которое зависит от безразмерного параметра $\alpha$ и переходит в жестко-идеально пластическое решение, когда $\alpha \to 0$. Применяя метод Линстэдта-Пуанкаре найдено пертурбационное решение в окрестности жестко-идеально пластического решения. Применено преобразование Шанка для расширения допустимого интервала малого параметра $\alpha$. Путем сравнения пертурбационного решения с замкнутым решением предложены практические приближенные формулы.

## 1. Introduction

THEORETICAL analysis of plastic response of strain-rate sensitive metal structures subjected to high intensity dynamic loads leads as a rule to a nonlinear initial boundary value problem. Consequently, exact solutions were obtained only for the simplest cases of structures with a high degree of symmetry, for example the solution for a circular ring with a power type of stress-strain rate law, published in [1] or the solution for a thick-walled spherical container with a linear excess stress function, reported in [8](1). To analyse the behaviour of more complicated structures, such as beams, plates and shells, various simplifying assumptions and approximations were introduced. Review of relevant literature can be found in Ref. [3].

(1) In this case the corresponding initial boundary value problem is linear.

According to the approximation technique proposed by PERRONE in [6], the solution for the rigid-perfectly viscoplastic structure is approximated by the rigid-perfectly plastic one with a correction for strain-rate sensitivity in the form of an empirical modification of the static yield stress correspondingly to initial strain rate magnitude. Hence, the method is particularly proficient in solving impulsive loading problems when solutions for rigid-perfectly plastic structures are available. Since most metals exhibit a rapid increase of the yield stress over a relatively narrow range of strain rates, Perrone's approach can not describe correctly the stress field during the deformation process.

In the present paper an alternative approach is proposed which is also applicable whenever the exact rigid-perfectly plastic solution is known and free of the disadvantage of Perrone's simplifying assumption. This is a perturbation technique around the rigid-perfectly plastic solution.

Rather than seek for generality, we shall develop the proposed perturbation method directly on an example of an impulsively loaded thick-walled, rigid-perfectly viscoplastic spherical container. First, an exact solution of the stated problem is obtained in a parametric form. This solution, involving a non-dimensional parameter $\alpha$, is shown to tend to the rigid-perfectly plastic solution when $\alpha \to 0$. Next, using the Linstedt-Poincaré technique a perturbation solution around the limiting rigid-perfectly plastic one is developed and then modified with the help of the Shank's transformation so as to extend the admissible range of the small parameter $\alpha$. Finally, by comparing the exact and perturbation solutions practical approximate formulae are proposed.

## 2. Formulation of the problem

Consider a thick-walled spherical container made of incompressible, homogeneous rigid-perfectly viscoplastic material obeying the constitutive relation formulated by Perzyna [5]. Let $a$ and $b$ denote the inner and outer radii. Our task is to describe a motion of the container, initiated at a time $t = 0$ by a radial velocity field $v_0(r) > 0$, $a \leqslant r \leqslant b$, under the assumptions that the internal and external pressures and the initial radial displacements are equal to zero and deformations are small.

Because of the symmetry the normal stress $\sigma_r$ and the hoop stresses $\sigma_\theta = \sigma_\varphi$ are principal stresses. Moreover, in view of the incompressibility condition the radial velocity distribution at arbitrary time $t$ must be of the form

$$v(r, t) = \left(\frac{a}{r}\right)^2 v_a(t), \tag{2.1}$$

in which $v_a(t)$ denotes the radial velocity of points on the internal surface of the container. For $t = 0$ Eq. (2.1) gives the relation

$$v_0(r) = \left(\frac{a}{r}\right)^2 v_{a0}, \quad v_{a0} \equiv v_a(0) > 0 \tag{2.2}$$

for the initial radial velocity distribution. Integrating Eq. (2.1) with respect to time we obtain the formula for the radial displacement field

$$u(r, t) = \left(\frac{a}{r}\right)^2 u_a(t), \tag{2.3}$$

where

$$u_a(t) = \int_0^t v_a(t)dt \tag{2.4}$$

is the displacement of the internal surface.

Let us introduce the non-dimensional independent variables

$$r^* = \frac{r}{a}, \quad t^* = \frac{2\sqrt{3}k\ln\eta^{-1}}{\varrho a v_{a0}(1-\eta)} t \tag{2.5}$$

and the non-dimensional dependent ones(2)

$$v^* = \frac{v}{v_{a0}}, \quad \sigma_r^* = \frac{\sigma_r}{\sqrt{3}k}, \quad \sigma_\varphi^* = \frac{\sigma_\varphi}{\sqrt{3}k}, \quad u^* = \frac{2\sqrt{3}k\ln\eta^{-1}}{\varrho a v_{a0}^2(1-\eta)} u, \tag{2.6}$$

where $k$ denotes the static yield stress in pure shear, $\varrho$ is the mass density and the parameter $\eta = a/b$ characterizes the geometry of the container. Then a full set of governing equations of the problem stated above, expressed in these non-dimensional variables, consists of the equation of motion

$$\frac{\partial \sigma_r^*}{\partial r^*} + 2\frac{\sigma_r^* - \sigma_\varphi^*}{r^*} = \frac{2\ln\eta^{-1}}{1-\eta}\frac{\partial v^*}{\partial t^*}, \tag{2.7}$$

the constitutive relation

$$\frac{v^*}{r^*} = \frac{a\gamma}{\sqrt{3}v_{a0}}\Phi(\sigma_\varphi^* - \sigma_r^* - 1), \tag{2.8}$$

and Eq. (2.2)

$$v^* = \frac{v_a^*(t^*)}{r^{*2}}. \tag{2.9}$$

for the radial velocity distribution, in which $\gamma$ stands for the viscosity constant and the function $\Phi(\cdot)$ is chosen to best represent results of experiments on the dynamic behaviour of materials. Equations (2.7)–(2.9) together with the initial condition

$$v_a^*(0) = 1 \tag{2.10}$$

for the radial velocity and the boundary conditions

$$\sigma_r^*(1, t^*) = \sigma_r^*(\eta^{-1}, t^*) = 0 \tag{2.11}$$

for the radial stress furnish an initial boundary value problem for the sought functions $v^*(r^*, t^*)$, $\sigma_r^*(r^*, t^*)$ and $\sigma_\varphi^*(r^*, t^*)$.

To complete our formulation let us note that the non-dimensional form of Eq. (2.4) is read as

$$u_a^*(t^*) = \int_0^{t^*} v_a^*(\tau)d\tau. \tag{2.12}$$

(2) From Eq. $(2.6)_{1,4}$ it follows that

$$v_a^* = \frac{v_a}{v_{a0}}, \quad u_a^* = \frac{2\sqrt{3}k\ln\eta^{-1}}{\varrho a v_{a0}^2(1-\eta)} u_a.$$

### 3. Solution for the power function $\Phi(\cdot)$

In the case of the power function $\Phi(\cdot) = (\cdot)^n$, where $n$ is a natural number, the constitutive equation (2.8) takes the form

$$\frac{v^*}{r^*} = \frac{a\gamma}{\sqrt{3}v_{a0}}(\sigma_\varphi^* - \sigma_r^* - 1)^n. \tag{3.1}$$

With a view of Eq. (2.9), Eq. (3.1) yields the relation

$$\sigma_\varphi^* - \sigma_r^* = 1 + \alpha\frac{3}{n}\frac{\ln\eta^{-1}}{1-\eta^{\frac{3}{n}}}\left(\frac{v_a^*}{r^{*3}}\right)^{\frac{1}{n}}, \tag{3.2}$$

where the non-dimensional parameter

$$\alpha = 3^{\frac{1-2n}{2n}} n \frac{1-\eta^{\frac{3}{n}}}{\ln\eta^{-1}}\left(\frac{v_{a0}}{a\gamma}\right)^{\frac{1}{n}} \tag{3.3}$$

is seen to be non-negative.

Let us substitute Eqs. (2.9) and (3.2) into the equation of motion (2.7) and integrate with respect to $r^*$ from 1 to $r^*$. Making use of the initial and boundary conditions (2.10) and (2.11), we infer that $v_a^*(t^*)$ must be a solution to the nonlinear (for $n \neq 1$) initial value problem

$$\frac{dv_a^*}{dt^*} + \alpha v_a^{*\frac{1}{n}} = -1, \quad v_a^*(0) = 1 \tag{3.4}$$

for the ordinary differential equation. The radial stress is related to this solution by

$$\sigma_r^*(r^*, t^*; \eta, n, \alpha) = A(r^*; \eta) - \alpha B(r^*; \eta, n) v_a^{*\frac{1}{n}}, \tag{3.5}$$

where the terms

$$\frac{1}{2}A(r^*; \eta) = \ln r^*\eta + \frac{r^{*-1}-\eta}{1-\eta}\ln\eta^{-1}, \tag{3.6}$$

$$\frac{1}{2}B(r^*; \eta, n) = \frac{\ln\eta^{-1}}{1-\eta^{\frac{3}{n}}}\left[r^{*-\frac{3}{n}} - \eta^{\frac{3}{n}} - \frac{(r^{*-1}-\eta)(1-\eta^{\frac{3}{n}})}{1-\eta}\right] \tag{3.7}$$

are independent of the parameter $\alpha$.

Finally, combining Eqs. (3.5) and (3.2) we obtain the relation between the hoop stress and the radial velocity

$$\sigma_\varphi^*(r^*, t^*; \eta, n, \alpha) = 1 + A(r^*; \eta) - \alpha\left[B(r^*; \eta, n) - \frac{3}{n}\frac{\ln\eta^{-1}}{1-\eta^{\frac{3}{n}}} r^{*-\frac{3}{n}}\right] v_a^{*\frac{1}{n}}. \tag{3.8}$$

It can be proved by direct substitution that the solution to the problem (3.4) can be represented in the parametric form

$$\bar{v}_a(\zeta; \alpha) = \left(-\frac{\zeta}{\alpha}\right)^n, \tag{3.9}$$

$$(3.10)\quad t^*(\zeta;\alpha) = (-1)^{n-1}\frac{n}{\alpha^n}\int_{-\alpha}^{\zeta}\frac{\xi^{2-1}}{1-\xi}d\xi$$

$$= (-1)^{n-1}\frac{n}{\alpha^n}\left\{\sum_{i=1}^{n-1}\frac{1}{n-i}[(-\alpha)^{n-i}-\zeta^{n-i}]-\ln\frac{1-\zeta}{1+\alpha}\right\},$$

where $\bar{v}_a^*(\zeta;\alpha) \equiv v_a^*\big(t^*(\zeta;\alpha);\alpha\big)$ and the parameter $\zeta$ changes within the range $-\alpha \leqslant \zeta \leqslant 0$. Now, with the help of Eqs. (3.9) and (3.10) we can change the variable under the integral sign in Eq. (2.12) and obtain a parametric solution for the displacement

$$(3.11)\quad \bar{u}_a^*(\zeta;\alpha) = -\frac{n}{\alpha^{2n}}\int_{-\alpha}^{\zeta}\frac{\xi^{2n-1}}{1-\xi}d\xi = \frac{n}{\alpha^{2n}}\left\{\sum_{i=1}^{2n-1}\frac{1}{2n-i}[\zeta^{2n-i}-(-\alpha)^{2n-i}]+\ln\frac{1-\zeta}{1+\alpha}\right\},$$

where $\bar{u}_a^*(\zeta;\alpha) \equiv u_a^*\big(t^*(\zeta;\alpha);\alpha\big)$.

According to Eq. (3.9) the motion of the container ends when $\zeta = 0$. Hence, by setting $\zeta = 0$ in Eqs. (3.10) and (3.11) we arrive at the formulae for the response time

$$(3.12)\qquad t_k^* \equiv t^*(0;\alpha) = (-1)^{n-1}\frac{n}{\alpha^n}\left[\sum_{i=1}^{n-1}\frac{(-1)^{n-i}}{n-i}\alpha^{n-i}+\ln(1+\alpha)\right],$$

and for the corresponding final displacement

$$(3.13)\qquad u_{ak}^* \equiv u_a^*(t_k^*;\alpha) = \frac{n}{\alpha^{2n}}\left[\sum_{i=1}^{2n-1}\frac{(-1)^{2n-i+1}}{2n-i}\alpha^{2n-i}-\ln(1+\alpha)\right].$$

When $n = 1$ the initial value problem becomes linear and has an explicit form solution. Inverting Eq. (3.10) for $n = 1$ we have

$$(3.14)\qquad \zeta(t^*;\alpha) = 1-(1+\alpha)e^{-\alpha t^*},$$

which, when substituted into Eq. (3.10) with $n = 1$, yields

$$(3.15)\qquad v_a^*(t^*;\alpha) = \frac{1}{\alpha}[(1+\alpha)e^{-\alpha t^*}-1].$$

Further substitution of Eq. (3.14) into Eqs. (3.11)–(3.13) gives the expressions for the remaining quantities:

$$(3.16)\qquad u_a^*(t^*;\alpha) = \frac{1}{\alpha^2}[(1+\alpha)(1-e^{-\alpha t^*})-\alpha t^*],$$

$$(3.17)\qquad t_k^* = \frac{1}{\alpha}\ln(1+\alpha),$$

$$(3.18)\qquad u_{ak}^* = \frac{1}{\alpha^2}[\alpha-\ln(1+\alpha)].$$

When $n \neq 1$, the relation (3.10) can be inverted only in an approximate way using, for example, an iterative procedure which for relatively large $n$ could be troublesome. Therefore, to obtain an explicit approximate solution to the problem (3.4) a perturbation technique which appears to be more simple in the considered case is proposed.

## 4. Limiting case of rigid-perfectly plastic motion

It was shown in [5] that by letting $\gamma \to \infty$ the constitutive equation for rigid-perfectly viscoplastic material reduces to that describing a rigid-perfectly plastic material. It will be proved now that, with $\alpha \to 0$, the solution (3.5), (3.8)–(3.11) indeed tends to that for the rigid-perfectly plastic container.

Expanding $\ln(1+\alpha)$ and $\ln(1-\zeta)$ into power series, Eqs. (3.10)–(3.12) for $\alpha < 1$ and $-\zeta < 1$ can be written in the form

$$t^*(\zeta;\alpha) = 1-\left(-\frac{\zeta}{\alpha}\right)^n\left(1+n\sum_{i=1}^{\infty}\frac{\zeta^i}{i+n}\right)+n\sum_{i=1}^{\infty}\frac{(-1)^i}{i+n}\alpha^i, \tag{4.1}$$

$$t_k^* = 1+n\sum_{i=1}^{\infty}\frac{(-1)^i}{i+n}\alpha^i, \tag{4.2}$$

$$\bar{u}_a^*(\zeta;\alpha) = \frac{1}{2}-\left(-\frac{\zeta}{\alpha}\right)^{2n}\left(\frac{1}{2}+n\sum_{i=1}^{\infty}\frac{\zeta^i}{i+2n}\right)-n\sum_{i=1}^{\infty}\frac{(-1)^{i+2n+1}}{i+2n}\alpha^i. \tag{4.3}$$

With the parameters $v_{a0}, a, \eta, n$ held fixed the condition $\gamma \to \infty$ implies $\alpha \to 0$, and since $-\alpha \leqslant \zeta \leqslant 0$ also $-\zeta \to 0+$. Consequently, the infinite series in powers of $\alpha$ and $\zeta$ disappear from the right hand side of the relations (4.1)–(4.3). The quantity $\left(-\frac{\zeta}{\alpha}\right)^n$ tends to a limit which, as can be concluded from Eq. (3.9), becomes the limiting velocity $v_{ap}^*$. By virtue of Eq. (4.1) this limiting velocity is found to be described by the relation

$$v_{ap}^* = 1-t^*, \quad \alpha \to 0, \tag{4.4}$$

which is the solution to the reduced initial value problem

$$\frac{dv_{ap}^*}{dt^*} = -1, \quad v_{ap}^*(0) = 1, \tag{4.5}$$

resulting from Eq. (3.4) upon setting $\alpha = 0$.

In the limit the relation (4.2) gives $t_k^* = 1$ and Eqs. (4.3) and (4.4) yield

$$u_{ap}^* = \frac{1}{2}[1-(1-t^*)^2], \quad \alpha \to 0. \tag{4.6}$$

Finally, bearing in mind Eq. (4.4) we also infer from Eqs. (3.5) and (3.8) that

$$\sigma_{rp}^*(r^*;\eta) = A, \tag{4.7}$$

$$\sigma_{\varphi p}^*(r^*;\eta) = 1+A, \quad \alpha \to 0. \tag{4.8}$$

From Eqs. (4.7) and (4.8) it can be readily seen that

$$\sigma_{\varphi p}^*-\sigma_{rp}^* = 1, \quad \alpha \to 0, \tag{4.9}$$

which expresses the static Huber-Mises yield condition([3]).

([3]) In the dimensional quantities (see $(2.6)_{2,3}$) the condition (4.9) is read as

$$\sigma_{\varphi p}-\sigma_{rp} = \sqrt{3}k, \quad \alpha \to 0.$$

Let us denote by $f$ an arbitrary response function from the set $v_a^*, u_a^*, \sigma_r^*, \sigma_\varphi^*$ and by $f_p$ its limit from the set $v_{ap}^*, u_{ap}^*, \sigma_{rp}^*, \sigma_{\varphi p}^*$. Then we can define "the limiting rigid-perfectly plastic motion" of the container as

$$f_p = \lim f \quad \text{when} \quad \alpha \to 0, \quad r^*, t^*; \eta, n \text{ fixed,} \tag{4.10}$$

and by performing a perturbation around $f_p$ derive an approximate solution to the problem formulated in Sect. 2.

## 5. Perturbation solution

Let us think of the parameter $\alpha$ as being small and assume the perturbation solution to the problem (3.4) to have a straight-forward expansion in powers of $\alpha$. Then, making use of the regular parameter perturbation technique (see for example [4]), the first order expansion for the sought solution can be found to have the form

$$v_a^*(t^*; \alpha) = 1-t^*-\alpha \frac{n}{1+n} [1-(1-t^*)^{\frac{1}{n}+1}]+0(\alpha^2), \quad \alpha \to 0. \tag{5.1}$$

According to Eq. (4.4) the motion of the rigid-perfectly plastic container takes place within the time interval $0 \leqslant t^* \leqslant 1$. When the container is made of the rigid-perfectly viscoplastic material, the corresponding time interval can be expected not to be substantially smaller if the inequality $\alpha \ll 1$ holds. It seems thus reasonable to require for the expansion (5.1) to be uniformly valid in the sense of Lighthill over the time interval $0 \leqslant t^* \leqslant 1$ (see [4])([4]). However, it follows from the relations (4.4) and (5.1) that $|v_{a_1}^*/v_{ap}^*| \to \infty$ when $t^* \to 1-0$, which means that the first order expansion (5.1) violates the assumed requirement.

To remove the above mentioned singularity from the relation (5.1) we shall apply the Linstedt-Poincaré procedure (see for example [4]).

Introduce a new variable

$$t^* = \tau\omega \tag{5.2}$$

and

$$\omega = 1+\alpha\omega_1+\alpha^2\omega_2+ \ldots, \tag{5.3}$$

where $\omega_1, \omega_2, \ldots$ are constants, so that the initial value problem (3.4) becomes

$$\frac{dv_a^*}{d\tau} = -(1+\alpha\omega_1+\alpha\omega_2+ \ldots)(\alpha v_a^{*\frac{1}{n}}+1), \quad v_a^*(0) = 1. \tag{5.4}$$

Then we let

$$v_a^* = v_{ap}^*(\tau)+\alpha v_{a_1}^*(\tau)+\alpha^2 v_{a_2}^*(\tau)+ \ldots, \tag{5.5}$$

in Eq. (5.4) and find $v_{ap}^*(\tau)$, $v_{a_1}^*(\tau)$, $v_{a_2}^*(\tau)$, ... choosing $\omega_1, \omega_2, \ldots$ in such a way to make the power expansion (5.5) uniformly valid within $0 \leqslant \tau \leqslant 1$.

([4]) Here, following Lighthill we demand for the first perturbation in Eq. (5.1) not to be more singular than the term given by Eq. (4.4), that is, for the ratio $v_{a_1}^*/v_{ap}^*$ to be limited within $0 \leqslant t^* \leqslant 1$.

With this purpose we first set $\alpha = 0$ in Eq. (5.4) to get

$$\frac{dv^*_{ap}}{d\tau} = -1, \quad v^*_{ap}(0) = 1, \tag{5.6}$$

which yields the zero-th approximation

$$v^*_{ap} = 1-\tau. \tag{5.7}$$

As a next step we differentiate Eq. (5.4) with respect to $\alpha$, then set $\alpha = 0$ arriving at

$$\frac{dv^*_{a_1}}{d\tau} = -\omega_1 - v^{*\frac{1}{n}}_{ap}, \quad v^*_{a_1}(0) = 0. \tag{5.8}$$

Bearing in mind Eq. (5.7), the solution to Eq. (5.8) is

$$v^*_{a_1} = -\frac{n}{1+n}\left[\left(1+\frac{1+n}{n}\omega_1\tau\right) - (1-\tau)^{\frac{1}{n}}\right]. \tag{5.9}$$

Now, it results from both Eqs. (5.7) and (5.9) that in order to make $v^*_{a_1}/v^*_{ap}$ limited within $0 \leqslant \tau \leqslant 1$ it is enough to require

$$\omega_1 = -\frac{n}{1+n}, \tag{5.10}$$

and then Eq. (5.9) is transformed into the form

$$v^*_{a_1} = -\frac{n}{1+n}(1-\tau)\left[1-(1-\tau)^{\frac{1}{n}}\right]. \tag{5.11}$$

A double differentiation of Eq. (5.4) with respect to $\alpha$, followed by setting $\alpha = 0$, yields the problem

$$\frac{dv^*_{a_2}}{d\tau} = -\omega_2 - \left(\omega_1 + \frac{1}{n}\frac{v^*_{a_1}}{v^*_{ap}}\right)v^{*\frac{1}{n}}_{ap}, \quad v^*_{a_2}(0) = 0, \tag{5.12}$$

whose solution, obtained with the help of the relations (5.7), (5.10) and (5.11), has the form

$$v^*_{a_2} = \frac{n}{1+n}\left\{\frac{1+n}{2+n}\left(1-\frac{2+n}{n}\omega_2\tau\right) - (1-\tau)^{\frac{1}{n}+1}\left[1-\frac{1}{2+n}(1-\tau)^{\frac{1}{n}}\right]\right\}. \tag{5.13}$$

Since the expression $v^*_{a_1}/v^*_{ap}$ has already been limited, the ratio $\frac{v^*_{a_2}}{v^*_{a_1}} = \frac{v^*_{a_2}}{v^*_{ap}}\frac{v^*_{ap}}{v^*_{a_1}}$ will be finite if only $v^*_{a_2}/v^*_{ap}$ is limited. It turns out from both Eqs. (5.7) and (5.13) that this requirement is satisfied if we assume

$$\omega_2 = \frac{n}{2+n}. \tag{5.14}$$

Using Eq. (5.14), Eq. (5.13) can be written in the final form

$$v^*_{a_2} = -\frac{n}{1+n}(1-\tau)\left\{(1-\tau)^{\frac{1}{n}}\left[1-\frac{1}{2+n}(1-\tau)^{\frac{1}{n}}\right] - \frac{1+n}{2+n}\right\}. \tag{5.15}$$

Introducing Eqs. (5.7), (5.11) and (5.15) into Eq. (5.5) and substituting Eqs. (5.10) and (5.14) respectively for $\omega_1$ and $\omega_2$ in Eq. (5.3), we have the second order expansion which is now uniformly valid within $0 \leqslant \tau \leqslant 1$ as follows:

$$(5.16) \quad v_a^{*\frac{1}{n}}(\tau;\alpha) = 1-\tau-\alpha\frac{n}{1+n}(1-\tau)\left[1-(1-\tau)^{\frac{1}{n}}\right] + \alpha^2\frac{n}{1+n}(1-\tau)\left\{(1-\tau)^{\frac{1}{n}}\left[1-\frac{1}{2+n}(1-\tau)^{\frac{1}{n}}\right]-\frac{1+n}{2+n}\right\}+0(\alpha^3),$$

$$(5.17) \quad \omega = 1-\alpha\frac{n}{1+n}+\alpha^2\frac{n}{2+n}+0(\alpha^3), \quad \alpha\to 0,$$

where, as it can be seen from Eq. (5.2),

$$(5.18) \quad \tau = \frac{t^*}{\omega}.$$

Finally, on the basis of Eq. (5.16) we get the two-term asymptotic expansion

$$(5.19) \quad v_a^{*\frac{1}{n}}(\tau;\alpha) = (1-\tau)^{\frac{1}{n}}+\alpha\frac{1}{1+n}(1-\tau)^{\frac{1}{n}}\left[1-(1-\tau)^{\frac{1}{n}}\right]+0(\alpha^2), \quad \alpha\to 0.$$

Using this result the second order expansions for the stresses $\sigma_r^*$ and $\sigma_\varphi^*$ can be readily found.

It follows from any of the first three approximations to the expansion (5.16) that the motion of the container ceases when $\tau = 1$, which by virtue of Eq. (5.18) corresponds to $t_k^* = \omega$. Thus, the sum ${}_3\omega$ of the first three terms in the expansion (5.17) defines the response time up to a small term of the order $0(\alpha^3)$.

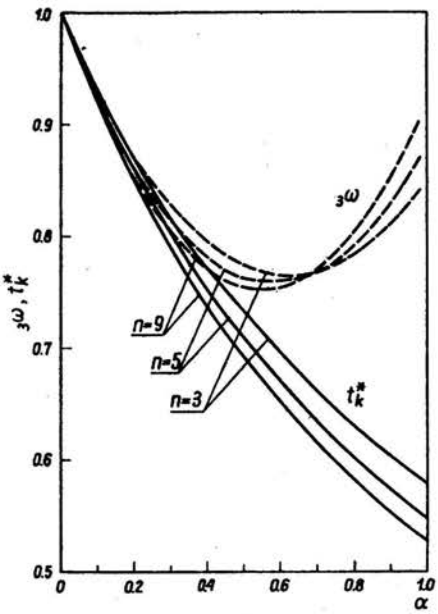


FIG. 1.

The sum ${}_3\omega$ versus $\alpha$ for $n = 3, 5, 9$ is represented in Fig. 1 by the dashed lines, whereas the solid lines show plots of the exact response time $t_k$ obtained from the relation (3.12). This figure shows that when taking ${}_3\omega$ as the response time we restrict the percentage relative error $100\%\ ({}_3\omega - t_k^*)/t_k^*$ so as not to exceed 2%, the admissible magnitudes of $\alpha$ must be confined to the range $0 \leqslant \alpha \leqslant 0.3$. Any direct extension of the range would cause a substantial increase of the error.

Note from Eq. (3.3) that for fixed magnitudes of $n, \gamma, \eta$ and $a$, the parameter $\alpha$ is proportional to $v_{a_0}^{\frac{1}{n}}$. Thus any additional assumption about the magnitudes of $\alpha$ imposes a restriction on an admissible range of initial velocities, which in turn diminishes the practical value of the expansion (5.16). This deficiency, however, can be overcome by introducing the nonlinear Shank's transformation(5) [7] performed on the sum ${}_3\omega$. As a result of the transformation we obtain a simple rational fraction

$$\omega_k = \frac{n+2+\dfrac{\alpha}{1+n}}{n+2+(1+n)\alpha}. \tag{5.20}$$

Equation (5.20) gives an approximation to the exact response time with strikingly high accuracy within the range $0 \leqslant \alpha < 1$ which is much larger than before. Since on the scale of Fig. 1 the curves representing $\omega_k$ and $t_k^*$ versus $\alpha$, for $n = 3, 5, 9$ are undistinguishable, these results are replotted in Fig. 2, where a distribution of the percentage relative

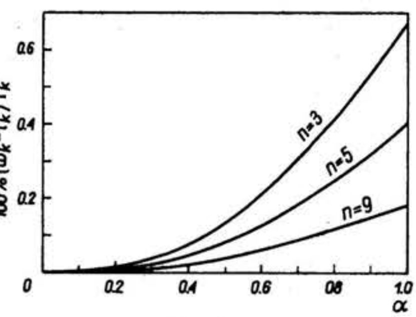


FIG. 2.

error $100\%\ (\omega_k - t_k^*)/t_k^*$ is shown. We observe that for each $\alpha$ the approximate response time obtained from Eq. (5.20) is larger than the exact one and becomes more accurate as $n$ increases.

Note that the first three terms in the expansions (5.17) and (4.2) are the same. Therefore, the rational fraction (5.20) can also be treated as an approximation to the infinite

(5) Applying Shank's transformation to the first three terms of a power series $1+a\alpha+b\alpha^2+\ldots$ results in a simple rational fraction

$$\frac{a+(a^2-b)\alpha}{a-b\alpha},$$

which often gives a more accurate approximation to the sum of the series than the sum of the original three terms does. The fraction can be seen to yield the exact sum when the original series is a geometric one, whether convergent or divergent [2].

series in Eq. (4.2). Since for $\alpha$ close to unity the infinite series turns out to be slowly convergent([6]), the rational fraction gives much better approximation to its sum than ${}_3\omega$ does. Moreover, for larger $n$ the infinite series becomes almost geometric, which reflects in the increasing accuracy of $\omega_k$.

Taking advantage of the fact that $\omega_k$ is a good approximation to $t_k^*$ within the extended range $0 \leqslant \alpha < 1$, we first substitute $\omega$ appearing in Eq. (5.18) by Eq. (5.20), then introduce the result into Eqs. (5.16) and (5.19) to get the modified expansions

$$v_a^*(t^*;\alpha) = 1-\frac{t^*}{\omega_k}-\alpha\frac{n}{1+n}\left(1-\frac{t^*}{\omega_k}\right)\left[1-\left(1-\frac{t^*}{\omega_k}\right)^{\frac{1}{n}}\right]+0(\alpha^2), \tag{5.21}$$

$$v_a^{*\frac{1}{n}}(t^*;\alpha) = \left(1-\frac{t^*}{\omega_k}\right)^{\frac{1}{n}}+\alpha\frac{1}{1+n}\left(1-\frac{t^*}{\omega_k}\right)^{\frac{1}{n}}\left[1-\left(1-\frac{t^*}{\omega_k}\right)^{\frac{1}{n}}\right]+0(\alpha^2). \tag{5.22}$$

By inserting Eq. (5.21) into Eq. (2.12), the expansion for the radial displacement is obtained:

$$u_a^*(t^*;\alpha) = \frac{1}{2}\,\omega_k\left[1-\left(1-\frac{t^*}{\omega_k}\right)^2\right] + \alpha\frac{\omega_k}{2}\frac{n}{1+n}\left\{\left(1-\frac{t^*}{\omega_k}\right)^2\left[1-\frac{2n}{1+2n}\left(1-\frac{t^*}{\omega_k}\right)^{\frac{1}{n}}\right]-\frac{1}{1+2n}\right\}+0(\alpha^2). \tag{5.23}$$

Finally, setting $t^* = \omega_k$ in Eq. (5.23), we have the two-term expansion

$$u_{ak}^* = \frac{1}{2}\,\omega_k\left[1-\alpha\frac{n}{(1+n)(1+2n)}\right]+0(\alpha^2) \tag{5.24}$$

for the final displacement.

The diagrams of the final displacement versus $\alpha$, respectively for $n = 3, 5, 9$, are shown in Figs. 3 to 5. The dashed lines represent the one-term expansion in Eq. (5.24)

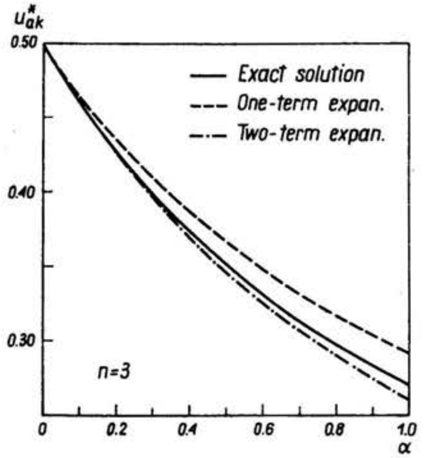


FIG. 3.

([6]) For example, when $\alpha = 0.96887$ and $n = 5$ we have $\omega_k = 0.5594$, whereas to get the same magnitude from Eq. (4.2) the sum of the thirty nine terms must be taken.

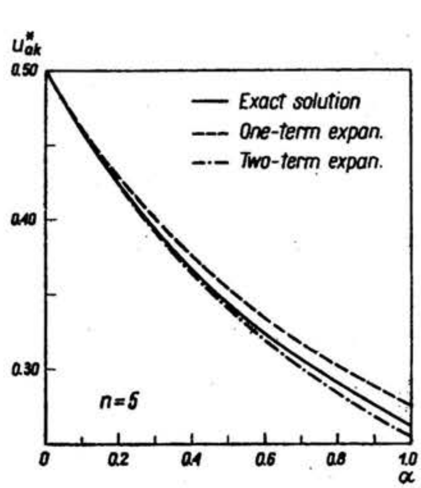


FIG. 4.

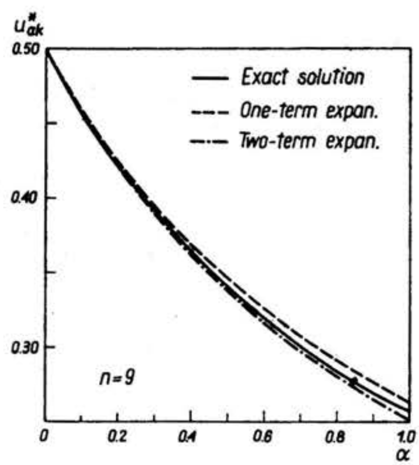


FIG. 5.

(i.e. when only the first term is retained) and the dash-dot curves correspond to the two-term approximation. The continuous lines plot the exact final displacement obtained from Eq. (3.13). It can be seen that for each $\alpha$ the one-term expansion gives the final displacement larger than the exact one, the percentage relative error not exceeding 7.8%, 5% and 2.9% respectively for $n = 3, 5, 9$. The final displacement resulting from the two-term expansion is smaller, the corresponding relative error being less than 3.8%, 3% and 2%. We can also infer that the accuracy of the two approximations becomes higher for larger $n$.

## 6. Example

To demonstrate the practical applicability of the present solution consider a spherical container made of mild steel and assume the following numerical values: $k =$

$= 147.15$ MPa, $\gamma = 40.4\ \mathrm{s}^{-1}$, $n = 5$, $\varrho = 7.8\ \mathrm{kg dcm}^{-3}$, $a = 1.68$ m, $b = 2.4$ m ($\eta = 0.7$), $v_{a0} = 56\ \mathrm{ms}^{-1}$(7). A corresponding value of the small parameter equals to $\alpha = 0.96687$. According to Eq. (3.12) the exact response time is $t_k^* = 0.55732$, whereas the approximate value computed from the Eq (5.20) equals $\omega_k = 0.55941$. The exact final displacement is $u_{ak}^* = 0.26244$, while the one-term and the two-term expansions in Eq. (5.24) give the approximate values 0.27971 and 0.25922 respectively. In view of Eq. $(2.6)_4$ the dimensional final displacements are 1.77954 cm, 1.89664 cm and 1.75771 cm respectively. These magnitudes give the maximum final strain intensities(8) equal to 3.17%, 3.387% and 3.139%.

The diagrams of the radial velocity $v_a^*$ and displacement $u_a^*$ versus time are shown in Fig. 6, the continuous lines representing the results obtained from the exact solution (3.9)–(3.11). The dashed curves correspond to the one-term approximations to the expansions (5.21) and (5.23) while the dash-dot ones illustrate the results given by the two-

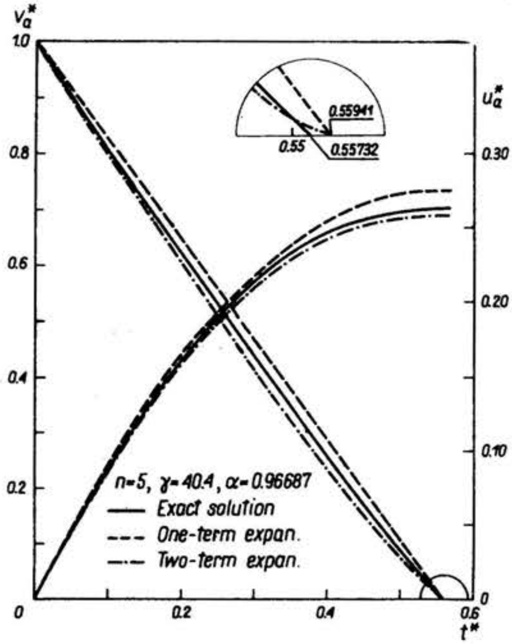


FIG. 6.

(7) The initial radial velocity $v_{a0} = 56\ \mathrm{m\,s}^{-1}$ gives the maximum strain rate intensity $\dot{\varepsilon}_a = 100\ \mathrm{s}^{-1}$ resulting from

$$\dot{\varepsilon} = \frac{1}{\sqrt{2}}\left[(\dot{\varepsilon}_r-\dot{\varepsilon}_\varphi)^2+(\dot{\varepsilon}_\varphi-\dot{\varepsilon}_\theta)^2+(\dot{\varepsilon}_\theta-\dot{\varepsilon}_r)^2\right]^{1/2} = 3\,\frac{a^2}{r^3}\,v_a(t)$$

for $r = a$ and $t = 0$.

(8) The maximum final strain intensity $\varepsilon_a$ is given by the expression

$$\varepsilon = \frac{1}{\sqrt{2}}\left[(\varepsilon_r-\varepsilon_\varphi)^2+(\varepsilon_\varphi-\varepsilon_\theta)^2+(\varepsilon_\theta-\varepsilon_r)^2\right]^{1/2} = 3\,\frac{a^2}{r^3}\,u_a(t)$$

for $r = a$ and $t = t_k$ or $t = \omega_k T$, where $T$ is defined by Eq. (7.4).

term approximations. It can be inferred from Fig. 6 and the obtained numerical results that the one-term approximations to the expansions (5.21) and (5.23) describe the radial velocity and displacement with sufficiently good accuracy.

Figure 7 shows the radial (Eq. (3.5) (and hoop (Eq. (3.8)) stresses for fixed $r^* = 1{,}2$. The continuous lines correspond to the case when $v_a^{*\frac{1}{n}}$ is calculated from the

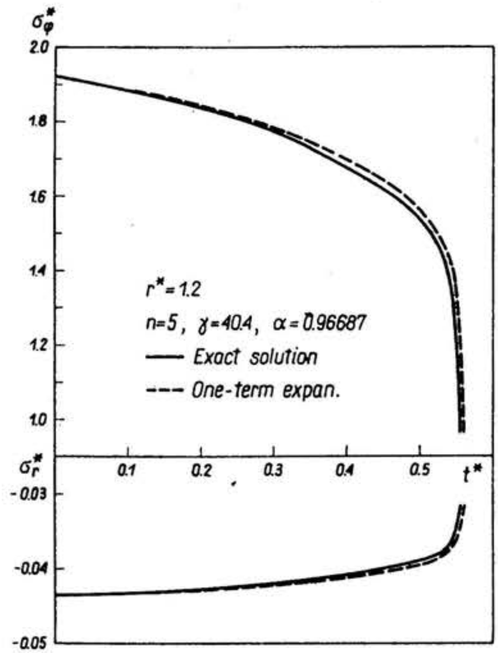


Fig. 7.

exact solution (3.9) and (3.10), whereas the dashed ones represent the results obtained with the help of the one-term approximation. Again, the approximate result can be seen to be in good agreement with the exact solution.

## 7. Conclusion

Analysis of the final results obtained in Sect. 5 and the numerical example have shown that just the one-term approximations to the expansions (5.21), (5.23) and (5.24) give a remarkable good accuracy for a relatively large range $0 \leqslant \alpha < 1$ of the "small parameter". In view of the relations (2.2), (2.3), (2.5) and $(2.6)_{1,4}$ the approximate solution in physical quantities takes the form

$$v \simeq v_{a0}\left(\frac{a}{r}\right)^2\left(1-\frac{t^*}{T\omega_k}\right), \tag{7.1}$$

$$u \simeq \frac{1}{2} v_{a0} T\omega_k\left(\frac{a}{r}\right)^2\left[1-\left(1-\frac{t}{T\omega_k}\right)^2\right], \tag{7.2}$$

$$u_k \simeq \frac{1}{2} v_{a0} T\omega_k \left(\frac{a}{r}\right)^2, \tag{7.3}$$

where

$$T = \frac{\varrho a v_{a0}(1-\eta)}{2\sqrt{3}k \ln \eta^{-1}}. \tag{7.4}$$

To get the approximate dimensional formulae for the radial and circumferential stresses we retain only the first term in the expansion (5.19) and substitute it for $v_a^{*\frac{1}{n}}$ in Eqs. (3.5) and (3.8). Bearing in mind Eqs. (2.5) and $(2.6)_{1,2,3}$, we finally arrive at the expressions

$$\sigma_r \simeq \sqrt{3}k\left[A - \alpha B\left(1 - \frac{t}{T\omega_k}\right)^{\frac{1}{n}}\right], \tag{7.5}$$

$$\sigma_\varphi \simeq \sqrt{3}k\left\{1 + A - \alpha\left[B - \frac{3}{n}\frac{\ln\eta^{-1}}{1-\eta^{\frac{3}{n}}}\left(\frac{a}{r}\right)^{\frac{3}{n}}\right]\left(1 - \frac{t}{T\omega_k}\right)^{\frac{1}{n}}\right\} \tag{7.6}$$

in which

$$\frac{1}{2}A = \ln\frac{r}{b} + \frac{\frac{a}{r} - \eta}{1-\eta}\ln\eta^{-1}, \tag{7.8}$$

$$\frac{1}{2}B = \frac{\ln\eta^{-1}}{1-\eta^{\frac{3}{n}}}\left[\left(\frac{a}{r}\right)^{\frac{3}{n}} - \eta^{\frac{3}{n}} - \frac{\left(\frac{a}{r} - \eta\right)(1-\eta^{\frac{3}{n}})}{1-\eta}\right]. \tag{7.7}$$

It has already been pointed out that the larger the exponent $n$, the more accurate the expressions (7.1)–(7.3) become, their simple forms remaining unchanged. On the other hand, the increase of $n$ makes the exact formulae (3.9)–(3.13) more difficult to work with. Thus, for larger $n$ the approximate formulae developed above seem to be more convenient in practical applications.

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