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Influence of kinematic hardening on plastic flow localization in damaged solids

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THE PAPER aims at the investigation of the influence of the induced anisotropy generated by large plastic deformations and micro-damage process on the shear band localization conditions. The constitutive equations are formulated within the framework of the rate type material structure with internal state variables. To describe the intrinsic micro-damage process the porosity parameter is introduced as a fundamental internal state variable. Particular attention is given to the formulation of the evolution equation for the tensorial kinematic hardening state variable, interpreted as the residual stress tensor. The critical value of the rate hardening modulus for localization of plastic deformations into a shear band and the direction of shear band are investigated. It has been confirmed that two cooperative phenomena, namely the kinematic hardening and the micro-damage process cause that the material is more inclined to instability. The conditions for localization for some particular examples of the state of stress are considered.

Celem pracy jest zbadanie wpływu anizotropii materiałowej wywołanej dużymi plastycznymi deformacjami oraz procesem mikro-uszkadzania materiału na lokalizację odkształceń plastycznych w postaci pasm ścinania. Równania konstytutywne zostały sformułowane w ramach struktury materiałowej typu prędkościowego z parametrami wewnętrznymi. W celu opisu wewnętrznego procesu mikro-uszkadzania wprowadzono parametr porowatości jako podstawową wewnętrzną zmienną stanu. Szczególną uwagę zwrócono na sformułowanie równania ewolucji dla tensorowej zmiennej stanu opisującej kinematyczne wzmocnienie, interpretowanej jako tensor naprężeń resztkowych. Zbadano krytyczną wartość modułu prędkości wzmocnienia dla lokalizacji plastycznych deformacji w postaci pasma ścinania oraz kierunek pasma ścinania. Badania potwierdziły, że dwa współdziałające zjawiska, a mianowicie kinematyczne wzmocnienie i proces mikro-uszkadzania powodują, że materiał jest bardziej wrażliwy na wystąpienie niestabilności. Zanalizowano warunki wystąpienia lokalizacji dla kilku szczególnych przykładów stanu naprężenia.

Целью работы является исследование влияния материальной анизотропии, вызванной большими пластическими деформациями и процессом микроповреждения материала, на локализацию пластических деформаций в виде полос сдвига. Определяющие уравнения сформулированы в рамках материальной структуры скоростного типа с внутренними параметрами. С целью описания внутреннего процесса микроповреждения, введен параметр пористости как основная внутренняя переменная состояния. Особенное внимание обращено на формулировку уравнения эволюции для тензорной переменной состояния, описывающей кинематическое упрочнение, интерпретированной как тензор остаточных напряжений. Исследовано критическое значение модуля скорости упрочнения для локализации пластических деформаций в виде полосы сдвига и направление полосы сдвига. Исследования подтвердили, что два взаимодействующих явления, именно кинематическое упрочнение и процесс микроповреждения, приводят к явлению, что материал более чувствителен на выступание явления нестабильности. Анализируются условия выступания локализации для нескольких частных случаев напряженного состояния.

1. Introduction

IN THE PREVIOUS paper [3] of the authors the investigation of shear band localization conditions for finite elastic-plastic rate independent deformations of damaged solids was presented. The influence of two main effects, namely the induced anisotropy and the micro-damage process, on the shear band localization phenomenon was discussed.

The present paper aims at investigating the influence of the induced anisotropy generated by large plastic deformations on the shear band localization conditions. In comparison with the paper [3], the evolution equation for the residual stress $\boldsymbol{\alpha}$ is postulated in a different form. Moreover, the alternative constitutive formulation is presented and discussed. The particular cases of the state of stress have also been considered.

In Sect. 2, the formulation of the constitutive relations for elastic-plastic solids when isotropic and kinematic hardening effects and the micro-damage process are taken into consideration is given.

The constitutive equations are formulated within the framework of the rate type material structure with internal state variables.

The kinematic hardening effect is generated by finite plastic deformations. The intrinsic micro-damage process is treated as a sequence of nucleation, growth and coalescence of microvoids. Both effects are described by means of the internal state variable method.

Particular attention is given to the formulation of the evolution equation for the tensorial kinematic hardening state variable, interpreted as the residual stress tensor.

The yield criterion is assumed to depend on the first two invariants of the deviator of the stress difference between the loading point and the center of the yield surface as well as on the porosity parameter.

To describe the intrinsic micro-damage process, the porosity parameter is introduced as a fundamental internal state variable. In the evolution equation for the porosity parameter, the first two terms are responsible for the description of the nucleation process while the third relates to the growth mechanism of microvoids.

As a result of the effects considered, the fundamental matrix $\mathbf{L}^p$ which describes the linear relationship between the plastic strain rate and the flux of the Kirchhoff stress is not symmetric. Consequently the normality does not apply, so the plastic flow direction and the effective normal to the actual yield surface are not coincident.

In Sect. 3 the alternative constitutive formulation is presented provided the yield criterion has a different form.

In Sect. 4 the conditions for localization of plastic deformations into a shear band are investigated. These conditions determine the direction of the shear band and the critical value of the hardening modulus rate for localization.

Section 5 presents a discussion and investigation of different effects on the localization phenomenon. Particular attention is focused on kinematic hardening. It has been found that the form of the evolution equation for the residual stress $\boldsymbol{\alpha}$ has a very important influence on the results obtained.

It has been confirmed that two cooperative phenomena, namely kinematic hardening and the micro-damage process account for the fact that the material is more inclined to instability.

In Sect. 6 some particular examples of the state of stress are considered. For these particular cases the critical values of the hardening modulus rate have been investigated.

In the last Section final conclusions are collected.

2. Constitutive relations

The yield criterion is postulated in the form (cf. [7, 8])

$$\tilde{f}(\cdot) = \varkappa. \tag{2.1}$$

The yield function $\tilde{f}(\cdot)$ is assumed as

$$\tilde{f}(\cdot) = \sqrt{\tilde{J}_2} + (n_1 + n_2 \xi)|\tilde{J}_1|, \tag{2.2}$$

where

$$\tilde{J}_1 = \tilde{\tau}_{ii}, \quad \tilde{J}_2 = \frac{1}{2} \tilde{\tau}'_{ij} \tilde{\tau}'_{ij}, \tag{2.3}$$

$$\tilde{\tau}_{ij} = \tau_{ij} - \alpha_{ij}, \tag{2.4}$$

$\boldsymbol{\tau}$ denotes the Kirchhoff stress tensor, $\boldsymbol{\alpha}$ the residual stress tensor which specifies the current center of the yield surface; by the prime the deviatoric part of the tensor is denoted, n_1 and n_2 are the material constants and ξ denotes the void volume fraction or porosity parameter.

The isotropic hardening-softening parameter is as follows(¹):

$$\varkappa = \varkappa_0(1 + k_0 \varepsilon^p)\left(1 - \frac{\xi}{\xi^F}\right), \tag{2.5}$$

where $\varkappa_0$ is the yield stress for the matrix material, k_0 denotes the isotropic hardening constant, ξ^F is the porosity at fracture and ε^p is the equivalent plastic deformation defined by the relation

$$\varepsilon^p = \int_0^t \dot{\varepsilon}^p dt = \int_0^t \left(\frac{2}{3} D^p_{ij} D^p_{ij}\right)^{\frac{1}{2}} dt \tag{2.6}$$

if $\mathbf{D}^p$ is the rate of the plastic deformation tensor.

The material function $\varkappa$ (cf. Eq. (2.5)) satisfies the fracture criterion

$$\varkappa|_{\xi=\xi^F} = 0. \tag{2.7}$$

It is noteworthy that the influence of the micro-damage process on localization within the shear band is of great importance practically in its initial stage only.

To investigate this conjecture in depth, let us consider the typical variation of tensile stress with porosity, Fig. 1. The trajectory tensile stress-porosity represents the real deformation process for a mild steel tensile specimen subjected to a constant strain rate. The process starts at the initial porosity ξ_0 and when the tensile stress reaches the threshold value for nucleation, the nucleation process begins. The process goes on, the tensile stress peaks up at the value of porosity ξ^m and dramatically breaks down to attain at $\xi = \xi^c$ the point at which coalescence of microcracks begins. The segment of the trajectory marked by the broken line represents the mechanism of final fracture.

The expected localization within shear bands can take place in the segment of the

(¹) For physical justification of the assumption (2.5), see Refs. [7, 8].

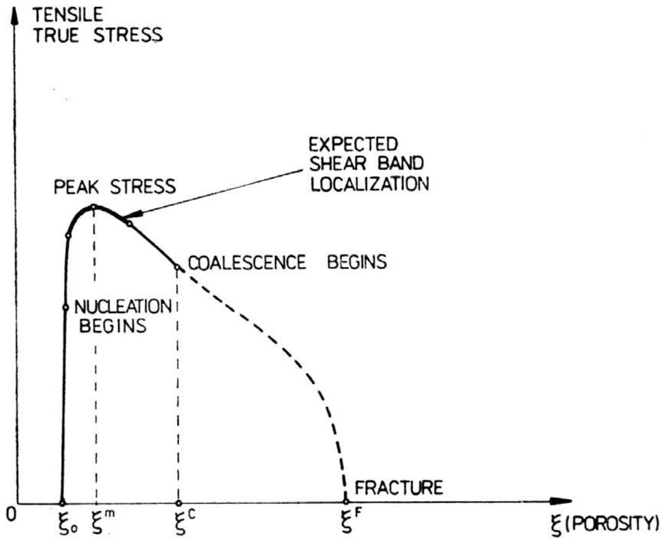

FIG. 1.

trajectory marked by a bold face line. The growth mechanism is assumed to occur during plastic deformation, so it may also influence the criterion of localization.

This heurestic consideration explains why the micro-damage process (i.e. nucleation and growth mechanisms) in its initial stage only can have an influence on the localization of plastic deformations within shear bands.

The evolution equation for the porosity parameter ξ was derived in [3] in the form (cf. with GURSON [4])

$$\dot{\xi} = \frac{k(\varepsilon^p, \xi)}{1-\xi}\,\mathrm{tr}(\tilde{\boldsymbol{\tau}}\mathbf{D}^p) + l(\tilde{J}_1, \xi)\dot{\tilde{J}}_1 + \Xi(\varepsilon^p, \xi)(1-\xi)\,\mathrm{tr}\,\mathbf{D}^p, \tag{2.8}$$

where k and l denote the nucleation material functions and Ξ denotes the growth material function. The first term in the right-hand side of Eq. (2.8) describes debonding of second-phase particles from the matrix, the second term is responsible for the cracking of the second-phase particles and the third term describes the microvoid growth process.

Postulating a flow law associated with the yield function (2.2), we obtain the rate of the plastic deformation tensor as follows:

$$D^p_{ij} = \Lambda\left(\frac{\tilde{\tau}'_{ij}}{2\sqrt{\tilde{J}_2}} + A\delta_{ij}\right), \tag{2.9}$$

where

$$A = n_1 + n_2\xi \tag{2.10}$$

and the scalar coefficient Λ has to be determined from the consistency condition, $\dot{\tilde{f}} = \dot{\varkappa}$. This leads to the relation

$$D^p_{ij} = \frac{1}{H}\left[\frac{\tilde{\tau}'_{ij}}{2\sqrt{\tilde{J}_2}} + A\delta_{ij}\right]\left[\frac{\tau'_{kl}}{2\sqrt{\tilde{J}_2}} + (A+C)\,\delta_{kl}\right](\overset{\triangledown}{\tau}_{kl} - \overset{\triangledown}{\alpha}_{kl}), \tag{2.11}$$

where $\overset{\nabla}{\boldsymbol{\tau}}$ denotes the Zaremba–Jaumann flux of the stress tensor $\boldsymbol{\tau}$,

$$C = \left[n_2 |\tilde{J}_1| + \frac{\varkappa_0}{\xi^F} (1 + k_0 \varepsilon^p) \right] l(\tilde{J}_1, \xi) \tag{2.12}$$

and the isotropic hardening modulus H is as follows:

$$H = \varkappa_0 k_0 \left(1 - \frac{\xi}{\xi^p}\right) \left[\frac{3}{2} \left(\frac{\tilde{\tau}'_{ij}}{2\sqrt{J_2}} + A\delta_{ij} \right) \left(\frac{\tilde{\tau}'_{ij}}{2\sqrt{\tilde{J}_2}} + A\delta_{ij} \right) \right]^{\frac{1}{2}} - \left[n_2 |\tilde{J}_1| + \frac{\varkappa_0}{\xi^p} (1 + k_0 \varepsilon^p) \right] \left[\frac{k}{1-\xi} \tilde{\tau}_{ij} + \Xi(1-\xi)\delta_{ij} \right] \left(\frac{\tilde{\tau}'_{ij}}{2\sqrt{\tilde{J}_2}} + A\delta_{ij} \right). \tag{2.13}$$

We postulate the evolution equation for the kinematic hardening internal state variable $\boldsymbol{\alpha}$ in the form (²)

$$\overset{\nabla}{\alpha}_{ij} = \frac{H^*}{(H + H^*)[\sqrt{\tilde{J}_2} + (A+C)\tilde{J}_1]} \left\{ \left[\frac{\tilde{\tau}'_{kl}}{2\sqrt{\tilde{J}_2}} + (A+C)\delta_{kl} \right] \overset{\nabla}{\tau}_{kl} \right\} \tilde{\tau}_{ij}, \tag{2.14$_1$}$$

where

$$H^* = \frac{r}{2} [1 + 6A(A+C)] \tag{2.15}$$

and r is the material constant.

The evolution equation $(2.14)_1$ in comparison to the postulate

$$\overset{\nabla}{\alpha}_{ij} = Q \left[\left(\frac{\tilde{\tau}'_{kl}}{2\sqrt{\tilde{J}_2}} + A\delta_{kl} \right) \overset{\nabla}{\tau}_{kl} \right] \tilde{\tau}_{ij} \tag{2.14$_2$}$$

assumed in the paper [3] has some advantages. It satisfies the condition that $\overset{\nabla}{\boldsymbol{\alpha}}$ vanishes when $\mathbf{D}^p = 0$ and for the states near to the neutral state, $\overset{\nabla}{\boldsymbol{\alpha}}$ is small.

Making use of Eq. $(2.14)_1$, the relation (2.11) can be written in the form

$$D^p_{ij} = L^p_{ijkl} \overset{\nabla}{\tau}_{kl}, \tag{2.16}$$

where

$$L^p_{ijkl} = \frac{1}{h} \left(\frac{\tilde{\tau}'_{ij}}{2\sqrt{\tilde{J}_2}} + A\delta_{ij} \right) \left[\frac{\tilde{\tau}'_{ij}}{2\sqrt{\tilde{J}_2}} + (A+C)\delta_{kl} \right] \tag{2.17}$$

and the isotropic-kinematic hardening modulus rate

$$h = H + H^* \tag{2.18}$$

depends explicitly on the kinematic hardening constant r.

The loading criterion is as follows:

$$\tilde{f} = \varkappa \quad \text{and} \quad \frac{1}{h} \left[\frac{\tilde{\tau}'_{kl}}{2\sqrt{\tilde{J}_2}} + (A+C)\delta_{kl} \right] \overset{\nabla}{\tau}_{kl} > 0. \tag{2.19}$$

(²) The kinematic hardening rule $(2.14)_1$ has been developed in the forthcoming paper of the authors.

From Eqs. (2.16)–(2.18) it follows that, in general, for $C \neq 0$ the normality rule does not apply. In view of the consitutive assumptions: $n_2 \geqslant 0$, $\varkappa_0 > 0$, $\xi^F > 0$, $k_0 > 0$, $\varepsilon^p \geqslant 0$ and Eq. (2.12), the necessary condition for $C \neq 0$ is $l(\tilde{J}_1, \xi) \neq 0$. Therefore we can conclude that the nucleation mechanism generated by cracking of second-phase particles is responsible for the deviation of the plastic deformation rate tensor $\mathbf{D}^p$ from the direction normal to the yield surface.

For the evolution equation for $\boldsymbol{\alpha}$ applied in [3] in the form $(2.14)_2$, the constitutive matrix L^p_{ijkl} has the form

$$L^p_{ijkl} = \frac{1}{h}\left(\frac{\tilde{\tau}'_{ij}}{2\sqrt{\tilde{J}_2}} + A\delta_{ij}\right)\left[\frac{\tilde{\tau}'_{kl}}{2\sqrt{\tilde{J}_2}} + \left(A + \frac{C}{1-Q\left[\sqrt{\tilde{J}_2} + (A+C)\tilde{J}_1\right]}\right)\delta_{kl}\right]. \tag{2.20}$$

Then the deviation of the plastic deformation rate tensor $\mathbf{D}^p$ from the direction normal to the yield surface depends also on the kinematic hardening parameter Q.

For small elastic strains(3), assuming the additivity of elastic and plastic rates of deformation and the generalized Hooke's law in the elastic range, we finally have

$$D_{ij} = D^e_{ij} + D^p_{ij} = L_{ijkl}\overset{\nabla}{\tau}_{kl}, \tag{2.21}$$

where

$$L_{ijkl} = \frac{1}{4G}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \left(\frac{1}{9K} - \frac{1}{6G}\right)\delta_{ij}\delta_{kl} + \frac{1}{h}\left(\frac{\tilde{\tau}'_{ij}}{2\sqrt{\tilde{J}_2}} + A\delta_{ij}\right)\left[\frac{\tilde{\tau}'_{kl}}{2\sqrt{\tilde{J}_2}} + (A+C)\delta_{kl}\right]. \tag{2.22}$$

The inverse of Eq. (2.22) has the form

$$\overset{\nabla}{\tau}_{ij} = M_{ijkl}D_{kl}, \tag{2.23}$$

where

$$M_{ijkl} = G(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}) + \left(K - \frac{2}{3}G\right)\delta_{ij}\delta_{kl} - \frac{1}{h+G+9KA(A+C)}\left(\frac{G}{\sqrt{\tilde{J}_2}}\tilde{\tau}'_{ij} + 3KA\delta_{ij}\right)\left[\frac{G}{\sqrt{\tilde{J}_2}}\tau'_{kl} + 3K(A+C)\delta_{kl}\right]. \tag{2.24}$$

Equations (2.23)–(2.24) together with the evolution equations: (2.8) for the porosity parameter ξ and $(2.14)_1$ for the residual stress tensor $\boldsymbol{\alpha}$, provide a set of the constitutive relations for elastic-plastic solids whose response involves the kinematic hardening as well as the intrinsic micro-damage process.

Neglecting the kinematic hardening, i.e. $H^* = 0$, and making the following formal identifications:

$$A = \frac{1}{3}\beta, \quad A + C = \frac{1}{3}\mu, \tag{2.25}$$

(3) The assumption concerning the small elastic strain is not restrictive for the consideration of the localization phenomenon in metallic materials.

where β is called the dilatancy factor and μ the internal friction coefficient, the constitutive equations (2.23)–(2.24) yield the constitutive relation obtained by Rudnicki and Rice (cf. Eq. (13) in [14]).

For $A = C = 0$ and $H^* = 0$ the constitutive relations (2.23)–(2.24) simplify to the form of the Prandtl–Reuss equations.

3. Alternative constitutive formulation

In the literature the classical Huber–Mises yield criterion is applied in two equivalent forms: $J_2 = \varkappa_0^2$ or $\sqrt{J_2} = \varkappa_0$ (e.g. HILL [5] and PRAGER [10]). The generalization of the Huber–Mises yield criterion for dilatant, pressure-sensitive or damaged materials can be made starting from each of these forms; however, final results then obtained are no more identical.

In Sect. 2 the yield criterion (2.1)–(2.2) obtained by the generalization of the Huber–Mises condition in the form $\sqrt{J_2} = \varkappa_0$ was applied. This approach was used in [10, 3]. The other approach which has been applied by GURSON [4] in the formulation of the yield criterion for porous materials will be utilized now and the results obtained will be compared with those derived in Sect. 2.

The yield criterion is postulated in the form

$$\hat{f}(\cdot) = \hat{\varkappa}, \tag{3.1}$$

where the yield function $\hat{f}(\cdot)$ and the isotropic hardening-softening parameter $\hat{\varkappa}$ are assumed as follows:

$$\hat{f}(\cdot) = \tilde{J}_2\left[1 + (n_1 + n_2\xi\,\frac{\tilde{J}_1^2}{\tilde{J}_2}\right], \tag{3.2}$$

$$\tilde{\varkappa} = \varkappa_0^2(1 + k_0\,\varepsilon^p)\left(1 - \frac{\xi}{\xi^F}\right). \tag{3.3}$$

Application of the same procedure as in Sect. 2 and the evolution equations (2.8) and $(2.14)_1$ for the internal state variables ξ and $\boldsymbol{\alpha}$, lead to the constitutive relations in the form

$$D_{ij}^p = \hat{L}_{ijkl}^p \overset{\nabla}{\tau}_{kl} \tag{3.4}$$

and

$$\hat{L}_{ijkl}^p = \frac{1}{\hat{h}}\left[\frac{\tilde{\tau}'_{ij}}{2\sqrt{\tilde{J}_2}} + \hat{A}\delta_{ij}\right]\left[\frac{\tilde{\tau}'_{kl}}{2\sqrt{\tilde{J}_2}} + (\hat{A} + \hat{C})\,\delta_{kl}\right], \tag{3.5}$$

where

$$\hat{h} = \hat{H} + \hat{H}^*, \tag{3.6}$$

$$\hat{H} = \varkappa_0^2 k_0\left(1 - \frac{\xi}{\xi^F}\right)\left[\frac{3}{2}\left(\frac{\tilde{\tau}'_{ij}}{2\sqrt{\tilde{J}_2}} + \hat{A}\delta_{ij}\right)\left(\frac{\tilde{\tau}'_{ij}}{2\sqrt{\tilde{J}_2}} + \hat{A}\delta_{ij}\right)\right]^{\frac{1}{2}}$$
$$- \left[n_2\tilde{J}_1^2 + \frac{\varkappa_0^2}{\xi^F}(1 + k_0\,\varepsilon^p)\right]\left[\frac{k}{1-\xi}\,\tilde{\tau}_{ij} + \Xi(1-\xi)\,\delta_{ij}\right]\left(\frac{\tilde{\tau}'_{ij}}{2\sqrt{\tilde{J}_2}} + \hat{A}\delta_{ij}\right) \tag{3.7}$$

$$\hat{A} = (n_1 + n_2 \xi) \frac{\tilde{J}_1}{\sqrt{\tilde{J}_2}}, \tag{3.8}$$

$$\hat{C} = \left[\tilde{J}_1^2 n_2 + \frac{\varkappa_0^2}{\xi^F} (1 + k_0 \varepsilon^p) \right] \frac{l(J_1, \xi)}{2\sqrt{\tilde{J}_2}}, \tag{3.9}$$

$$\overset{\nabla}{\alpha}_{kl} = \frac{H^*}{(H+H^*)[\sqrt{\tilde{J}_2} + (A+C)\tilde{J}_1]} \left\{ \left[\frac{\tilde{\tau}'_{ij}}{2\sqrt{\tilde{J}_2}} + (\hat{A} + \hat{C}) \delta_{ij} \right] \overset{\nabla}{\tau}_{ij} \right\} \tilde{\tau}_{kl}, \tag{3.10}$$

$$\hat{H}^* = \frac{r}{2} [1 + 6\hat{A} (\hat{A} + \hat{C})]. \tag{3.11}$$

Comparison of Eq. (3.5) with Eq. (2.17) indicates an analogical form of the constitutive matrices L^p_{ijkl} and $\hat{L}^p_{ijkl}$ for both forms of the yield criterion. The forms of the material functions are, however, slightly different. This difference can be detected by the identification procedure for particular materials.

4. Conditions for localization

Making use of the constitutive relations (2.23) and (2.24) for the elastic-plastic damaged solid with isotropic and kinematic hardening effects, we shall look for the conditions for localization of plastic deformations into a shear band. Our aim is to determine the direction of the shear band and the critical hardening modulus.

The theory of localization of plastic deformations into a shear band was developed mainly by RICE [11, 12], RICE and RUDNICKI [13], RUDNICKI and RICE [14] and NEEDLEMAN and RICE [6]. We use here the notation and follow the consideration presented by RUDNICKI and RICE [14] and in our previous paper [3].

Let **n** be the unit normal to the surface of the shear band across which certain components of the velocity gradient may admit jumps. Let us introduce rectangular Cartesian coordinates x_i in such a way that **n** is in the x_2-direction (cf. Fig. 2).

For the constitutive relation (2.23), the necessary condition for a localized shear band to be formed is

$$\det[M_{2jk2}] = 0. \tag{4.1}$$

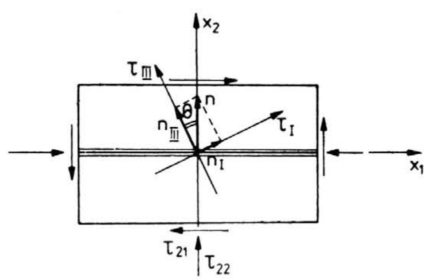

FIG. 2.

Solving Eq. (4.1) for the hardening modulus yields

$$\frac{h}{G+9KA(A+C)} = \frac{\left(G\tau'_{22}+3AK\sqrt{\tilde{J}_2}\right)\left[G\tilde{\tau}'_{22}+3K(A+C)\sqrt{\tilde{J}_2}\right]+\left(\frac{4}{3}G+K\right)G\left(\tilde{\tau}'^2_{21}+\tilde{\tau}'^2_{23}\right)}{\tilde{J}_2\left(\frac{4}{3}G+K\right)[G+9KA(A+C)]}-1. \tag{4.2}$$

Let $n_{\rm I}$, $n_{\rm II}$, $n_{\rm III}$ denote the corresponding components of the unit normal to the plane of shear band localization in the principal directions of tensor $\tilde{\boldsymbol{\tau}}$.

Denoting by $\tau_{\rm I}$, $\tau_{\rm II}$, $\tau_{\rm III}$ the principal stresses, we assume

$$\tau_{\rm I} \geqslant \tau_{\rm II} \geqslant \tau_{\rm III}. \tag{4.3}$$

The orientation of the plane within which shear band localization first takes place can be found from the requirement of h to be maximum with respect to n_K.

The solution has the form

$$\begin{aligned} n_{\rm I}^2 &= (\tilde{\tau}'_{\rm I}-\tilde{\tau}'_{\rm III})^{-1}\left[(2A+C)(1+\nu)\sqrt{\tilde{J}_2}-\tilde{\tau}'_{\rm II}(1-\nu)-\tilde{\tau}'_{\rm III}\right],\\ n_{\rm II} &= 0,\\ n_{\rm III}^2 &= (\tau'_{\rm III}-\tilde{\tau}'_{\rm I})^{-1}\left[(2A+C)(1+\nu)\sqrt{\tilde{J}_2}-\tilde{\tau}'_{\rm II}(1-\nu)-\tilde{\tau}'_{\rm I}\right]. \end{aligned} \tag{4.4}$$

Denoting by θ the angle between the vector $\mathbf{n}$ and the $\tilde{\tau}_{\rm III}$ direction, we obtain

$$\tan\theta = \frac{n_{\rm I}}{n_{\rm III}} = \left(\frac{\zeta+\mathscr{A}_{\rm min}-T_{\rm min}}{T_{\rm max}-\zeta-\mathscr{A}_{\rm max}}\right)^{1/2}, \tag{4.5}$$

where

$$\zeta = (2A+C)(1+\nu)-(T-\mathscr{A})(1-\nu) \tag{4.6}$$

and

$$\frac{\tau'_{\rm I}}{\sqrt{\tilde{J}_2}} = T_{\rm max}, \quad \frac{\tau'_{\rm II}}{\sqrt{\tilde{J}_2}} = T, \quad \frac{\tau'_{\rm III}}{\sqrt{\tilde{J}_2}} = T_{\rm min}, \tag{4.7}$$

$$\frac{\alpha'_{\rm I}}{\sqrt{\tilde{J}_2}} = \mathscr{A}_{\rm max}, \quad \frac{\alpha'_{\rm II}}{\sqrt{\tilde{J}_2}} = \mathscr{A}, \quad \frac{\alpha'_{\rm III}}{\sqrt{\tilde{J}_2}} = \mathscr{A}_{\rm min}. \tag{4.8}$$

The critical hardening modulus for a localized shear band to be formed is finally obtained in the form

$$\frac{h_{\rm cr}}{G} = \left(\frac{1+\nu}{1-\nu}\right)C^2-\frac{1+\nu}{2}(T-\mathscr{A}+2A+C)^2. \tag{4.9}$$

The critical hardening modulus h_{cr}/G that was obtained by the authors in the previous paper [3] for the evolution equation for $\boldsymbol{\alpha}$ postulated by Eq. $(2.14)_2$ has the following form

$$\frac{h_{cr}}{G} = \left(\frac{1+\nu}{1-\nu}\right)\left(\frac{C}{P}\right)^2 - \frac{1+\nu}{2}\left(T-\mathscr{A}+2A+\frac{C}{P}\right)^2, \tag{4.10}$$

where

$$P = 1-Q\left[\sqrt{\tilde{J}_2}+(A+C)\tilde{J}_1\right].$$

In that case, since P is a function of Q, A and C, the influence of two cooperative phenomena, namely the kinematic hardening and the micro-damage process, is different then for the case considered in the present paper when h_{cr}/G is described by Eq. (4.9). This explains why in the previous case we had such a pronounced synergetic effect.

On the other hand, the result (4.9) shows that the influence of kinematic hardening on the critical value of the hardening modulus, for the evolution equation for $\boldsymbol{\alpha}$ postulated in the form $(2.14)_1$, is mainly described by the relation (2.18).

Of course, for $Q = 0$ (i.e. when the kinematic hardening is neglected) Eq. (4.10) coincides with Eq. (4.9).

5. Discussion and comments

The solution of Eq. (4.9), for the critical hardening modulus as a function of the dimensionless mean principal value of the stress deviator T, may be represented by the family of parabolas, as it has been plotted in Fig. 3 by a solid line.

The origin of the parabola described by Eq. (4.9) moves along the trajectory t given by the parametric equations (see Fig. 3),

$$\left(\frac{h_{cr}}{G}\right)_{\max} = \left(\frac{1+\nu}{1-\nu}\right)C^2, \tag{5.1}$$

$$T = \mathscr{A}-(2A+C), \tag{5.2}$$

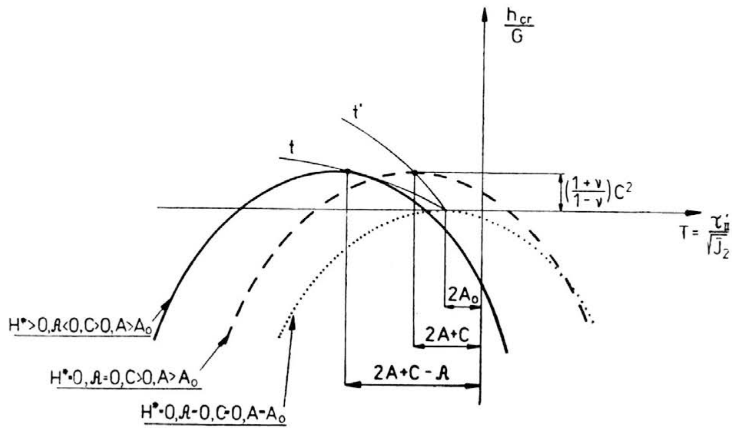

FIG. 3

where, for $t = 0$, it is assumed:

$$A = A_0 = n_1 + n_2 \xi_0, \quad C = 0, \quad \mathscr{A} = 0. \tag{5.3}$$

When kinematic hardening is neglected (i.e. $H^* = 0$ and $\mathscr{A} = 0$), Eq. (4.9) reduces to the form

$$\frac{h_{cr}}{G} = \frac{H_{cr}}{G} = \left(\frac{1+\nu}{1-\nu}\right) C^2 - \frac{1+\nu}{2} (T + 2A + C)^2. \tag{5.4}$$

This result is analogous to Eq. (20) obtained by Rudnicki and Rice [13] for a "pressure-sensitive dilatant material".

The parabola described by Eq. (5.4) is plotted in Fig. 3 by a broken line. Its origin moves along the trajectory t' given by the parametric equations

$$\left(\frac{h_{cr}}{G}\right)_{\max} = \left(\frac{1+\nu}{1-\nu}\right) C^2, \quad T = -2A - C, \tag{5.5}$$

where, for $t = 0$, the initial conditions (5.3) are satisfied.

When both the kinematic hardening and the micro-damage process are neglected (i.e. Eqs. (5.3) are satisfied for any instant of time t), then Eq. (4.9) simplifies to the form

$$\frac{h_{cr}}{G} = \frac{H_{cr}}{G} = -\frac{1+\nu}{2} (T + 2A_0)^2. \tag{5.6}$$

This equation describes the parabola plotted in Fig. 3 by the dotted line.

It follows from Eq. (4.9) that the critical hardening modulus takes the maximum value

$$\left(\frac{h_{cr}}{G}\right)_{\max} = \left(\frac{1+\nu}{1-\nu}\right) C^2 \tag{5.7}$$

for

$$T = \mathscr{A} - 2A - C. \tag{5.8}$$

Therefore it is clear that the micro-damage process makes the parabolas shift up and the value of h_{cr} increases almost for any state of stress, thus the material is more inclined to instability by localization of plastic deformations.

Kinematic hardening has no influence on the maximum value of h_{cr} but it affects the value of T for which h_{cr} reaches its maximum. The parabola shifts left for $\mathscr{A} > 0$ and right for $\mathscr{A} < 0$. Therefore the influence of kinematic hardening on localization depends strongly on the actual state of stress represented by the dimensionless mean principal value of the stress deviator T (see Fig. 3). On the other hand, as it follows from Eq. (2.18), because $h \geqslant H$, the localization may set in earlier (for smaller strains) than in the case when the material is hardening isotropically only.

For $C = 0$ (i.e. when there is no cracking of the second phase particles), Eq. (4.9) takes the form

$$\frac{h_{cr}}{G} = -\frac{1+\nu}{2} (T - \mathscr{A} + 2A)^2. \tag{5.9}$$

Thus, in this case the parabolas are shifted down below the T axis and $h_{cr} < 0$. This means that the material is less sensitive to localization. The onset of localization of plastic deformations into a shear band may occur when the material is unstable (falling part of stress-strain curve).

For the alternative form of the yield condition described by Eqs. (3.1)–(3.3) and the constitutive relations (3.4)–(3.9), the critical hardening modulus for localization, h_{cr}, is given by the relation (analogical to Eq. (4.9)):

$$\frac{h_{cr}}{G} = \left(\frac{1+\nu}{1-\nu}\right)\hat{C}^2 - \frac{1+\nu}{2}(T-\mathscr{A}+2\hat{A}+\hat{C})^2, \tag{5.10}$$

where $\hat{A}$, $\hat{C}$ are defined by Eqs. (2.8) and (3.9).

Hence all results and conclusions obtained for the situation described by Eq. (4.9) are valid in the case of the alternative formulation of the constitutive relations (3.4)–(3.9) when the critical hardening modulus is governed by Eq. (5.10).

6. Particular cases

In this Section we intend to consider several particular cases of the state of stress which are of great importance in practical applications.

6.1. Axially-symmetric "extension"

Let us assume

$$\tau_I > \tau_{II} = \tau_{III}. \tag{6.1}$$

The assumption (6.1) is satisfied for the states of stress as follows:

6.1.1. Uniaxial extension. We have

$$\tau_I > 0, \quad \tau_{II} = \tau_{III} = 0. \tag{6.2}$$

6.1.2. Plain stress axially-symmetric compresion. Let us consider a flat sheet compressive specimen and assume

$$\tau_I = 0, \quad \tau_{II} = \tau_{III} < 0. \tag{6.3}$$

According to the relations $(4.7)_2$ and (6.1), we obtain

$$\tau'_{II} = \tau'_{III} = T\sqrt{\tilde{J}_2} \leqslant 0, \quad \tau'_I = -2T\sqrt{\tilde{J}_2} \geqslant 0. \tag{6.4}$$

Substitution of the relations (6.4) into the relation $\tilde{J}_2 = \frac{1}{2}(\tilde{\tau}'^2_I + \tilde{\tau}'^2_{II} + \tilde{\tau}'^2_{III})$ yields:

$$T = -1/\sqrt{3} + \mathscr{A}. \tag{6.5}$$

The critical hardening modulus for localization can be determined from Eq. (4.9) as follows:

$$\frac{h_{cr}}{G} = \left(\frac{1+\nu}{1-\nu}\right)C^2 - \left(\frac{1+\nu}{2}\right)\left(-\frac{1}{\sqrt{3}} + 2A + C\right)^2. \tag{6.6}$$

Depending on the values A and C, localization may set in either while the material is continually hardening ($h_{cr} > 0$) or when it is softening ($h_{cr} < 0$).

Assuming that for $J_1 < 0$ second phase particles do not fracture and therefore $C = 0$, then for the plain stress axially-symmetric compression we obtain

$$\frac{h_{cr}}{G} = -\frac{1+\nu}{2}\left(-\frac{1}{\sqrt{3}}+2A\right)^2 \leqslant 0. \tag{6.7}$$

Localization may set in only when the material is perfectly plastic or softening.

6.2. Axially-symmetric "compression"

We assume

$$\tau_{I} = \tau_{II} > \tau_{III}. \tag{6.8}$$

This assumption is satisfied for the following important cases:

6.2.1. Uniaxial compression. We have

$$\tau_{I} = \tau_{II} = 0, \quad \tau_{III} < 0. \tag{6.9}$$

6.2.2. Plane stress axially-symmetric extension. This case is characterized by

$$\tau_{I} = \tau_{II} > 0, \quad \tau_{III} = 0. \tag{6.10}$$

The relation $(4.7)_2$ for the assumption (6.8) yields

$$\tau'_{I} = \tau'_{II} = T\sqrt{\tilde{J}_2} > 0, \quad \tau'_{III} = -2T\sqrt{\tilde{J}_2} < 0 \tag{6.11}$$

and therefore

$$T = 1/\sqrt{3}+\mathscr{A}. \tag{6.12}$$

The critical hardening modulus h_{cr} takes the value

$$\frac{h_{cr}}{G} = \left(\frac{1+\nu}{1-\nu}\right)C^2 - \frac{1+\nu}{2}\left(\frac{1}{\sqrt{3}}+2A+C\right)^2 \tag{6.13}$$

which, for the most practical situations, is negative.

Again, assuming that for $J_1 < 0$ we have $C = 0$, then Eq. (6.13) simplifies to the form

$$\frac{h_{cr}}{G} = -\left(\frac{1+\nu}{2}\right)\left(\frac{1}{\sqrt{3}}+2A\right)^2 < 0. \tag{6.14}$$

The onset of localization can occur only for a negative value of the hardening modulus (when the material is progressively softening).

6.3. Pure shear state

We assume the state of stress as follows:

$$\tau_{III} = -\tau_{I}, \quad \tau_{II} = 0, \tag{6.15}$$

then

$$J_1 = 0, \quad C = 0, \quad T = 0 \quad \text{and } \mathscr{A} = 0. \tag{6.16}$$

Equation (4.9) reduces to

$$\frac{h_{cr}}{G} = -2(1+\nu)A^2 \leqslant 0. \tag{6.17}$$

The onset of localization may take place only when the material is perfectly plastic or softening.

6.4. Plane plastic strain state

We require

$$D^p_{II} = 0. \tag{6.18}$$

From Eq. (2.9) it follows that the condition (6.18) is satisfied when

$$\frac{\tilde{\tau}'_{II}}{2\sqrt{\tilde{J}_2}} + A = 0. \tag{6.19}$$

Making use of Eqs. (2.4), $(4.7)_2$ and $(4.8)_2$, Eq. (6.19) leads to

$$T = \mathscr{A} - 2A. \tag{6.20}$$

Substituting Eq. (6.20) into Eq. (4.9), we find

$$\frac{h_{cr}}{G} = \frac{(1+\nu)^2}{2(1-\nu)} C^2 = \frac{1}{2}\left(\frac{h_{cr}}{G}\right)_{max} \geqslant 0. \tag{6.21}$$

A non-negative value of the critical hardening rate indicates that the states, like plane strain, are particularly sensitive for localization. This is the expected result since in this case the direction of maximum shear stress lies in the plane of the shear band.

7. Final conclusions

It has been confirmed (cf. conclusions in [3]) that two cooperative phenomena, namely kinematic hardening and the micro-damage process, account for the fact that the material is more inclined to instability.

The results obtained are different for axially-symmetric compression than for axially-symmetric tension.

It has been found that the form of the evolution equation for the residual stress $\boldsymbol{\alpha}$ has a very important influence on the criterion of localization within a shear band.

The results obtained are qualitatively in good accord with the experimental observations of initiation of localization (cf. CHAKRABARTI and SPRETNAK [2] and ANAND and SPITZIG [1]).

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