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Unilateral contact with dry friction: Time and space discrete variables formulation

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ONE CONSIDERS a continuous medium coming into contact with a rigid obstacle or another deformable body. Quasi-static evolution problems (i.e. with negligible inertia) are considered as well as proper dynamical problems. Formulations of unilateral contact are proposed for these two cases. Dry friction is taken into account through Coulomb's law. A system of equations for the time and space discretized problem is proposed together with an algorithm for solving this system. The derivation of equations when performing space variable discretization is specially developed in this paper.

Przedmiotem pracy jest zagadnienie kontaktu ośrodka ciągłego ze sztywną przeszkodą lub z innym ciałem odkształcalnym. Rozważono quasi-statyczne, jak i dynamiczne problemy ewolucji. Zaproponowano sformułowanie problemu kontaktu jednostronnego dla tych dwóch przypadków przy uwzględnieniu suchego tarcia ujętego przez prawo Coulomba. Przedstawiono podstawowy układ równań problemu po dokonaniu jego dyskretyzacji w odniesieniu do zmiennych czasu i przestrzeni. Podano algorytm rozwiązania tego układu. Szczególną uwagę poświęcono dyskretyzacji ze względu na zmienną przestrzenną.

Предметом работы является задача контакта сплошной среды с жесткой преградой или с другим деформируемым телом. Рассмотрены квазистатические и динамические задачи эволюции. Предложена формулировка задачи одностороннего контакта для этих двух случаев, при учете сухого трения описанного законом Кулона. Представлена основная система уравнений задачи после проведения ее дискретизации по отношению к переменным времени и пространства. Приведен алгоритм решения этой системы. Особенное внимание посвящено дискретизации из-за пространственной переменной.

1. Introduction

WHEN DEALING from a numerical point of view with a continuous medium, a finite-dimensional subspace of admissible velocity field $\{U\}$ is selected (space variable discretization), together with its dual space $\{F\}$ whose elements are representatives of the applied forces. These representatives are constructed through the principle of virtual power. When unilateral contact and dry friction are involved, it is usual to select a finite number of particles, the candidates to contact, for which the contact and friction laws are invoked. Actually the boundary in contact is submitted to a density of reaction forces from the obstacle, satisfying the contact and friction laws. In the above discretization procedure, this distributed reaction is represented by some $R \in \{F\}$, while the considered velocity field is $U \in \in \{U\}$. The contact and friction laws assumed to hold for every particle of the boundary induce some relations between R and U, which might be called the discretized contact and friction laws. These laws are different from those obtained when concentrating the reaction forces on candidates to contact. The object of this paper is to propose a way of exhibiting the discretized laws between R and U. A comparison will be made between

the usual procedure based on concentrated forces applied to candidates, and the above discretized laws in the case of $T3$ finite elements (linear interpolation on a triangle mesh).

Section 2 presents a choice of unilateral contact conditions for the quasi-static case and for the dynamic case. The friction law adopted is Coulomb's law.

Section 3 deals with time discretization and results are presented concerning space variable discretization of contact and friction laws, see also [6]. The method of constructing such laws is developed in Sect. 4, with examples and numerical comparison.

2. Contact and friction equations

2.1. Unilateral contact equations

Let the position of a particle P of the deformable body relative to the boundary of the obstacle be measured by the real number $q_{\mathcal{N}}$: this is the distance of P to this boundary, counted as negative if P lies inside the obstacle. Impenetrability is thus expressed by $q_{\mathcal{N}} \geqslant 0$. The obstacle is supposed to be nonadhesive, i.e. the normal component of the reaction is non-negative. When P lies clear from the obstacle, then $\mathcal{R}_{\mathcal{N}} = 0$. In short,

$$(2.1)_1 \qquad q_{\mathcal{N}} \geqslant 0, \quad \mathcal{R}_{\mathcal{N}} \geqslant 0, \quad q_{\mathcal{N}}\mathcal{R}_{\mathcal{N}} = 0.$$

This has the well-known form of a complementarity relation, commonly accepted for unilateral constraints. An equivalent form is

$$(2.1)_2 \qquad \mathcal{R}_{\mathcal{N}} \geqslant 0, \quad \forall \mathcal{S}_{\mathcal{N}} \geqslant 0, \quad (\mathcal{S}_{\mathcal{N}} - \mathcal{R}_{\mathcal{N}})q_{\mathcal{N}} \geqslant 0.$$

In the case of dynamical problems one has to provide more information about *shock conditions*. The velocity $\mathcal{U}$ of P relative to the obstacle may be discontinuous; we shall assume it to be a function of the time t with locally bounded variation (l.b.v.). This secures the existence of the right and left limits of $\mathcal{U}$, respectively denoted by $\mathcal{U}^+(t)$ and $\mathcal{U}^-(t)$. The reaction can no more be expected to be a finite-valued function of time. As a mathematical formalization of the traditional concept of *percussion*, we describe this reaction as a *measure* on the considered time interval. In the case of smooth motion, this measure possesses a density, relative to Lebesgue's measure, which is nothing but the force denoted above by $\mathcal{R}_{\mathcal{N}}$. On the contrary, if a shock occurs at some instant t, the reaction measure presents an *atom* at t, i.e. it involves a Dirac mass at this instant.

Generally, a non-negative measure μ is sure to exist (non-uniquely defined) relative to which the reaction measure possesses a density function [1]. By $\mathcal{R}_{\mathcal{T}}$ and $\mathcal{R}_{\mathcal{N}}$ in the sequel, we shall denote the values of such density functions at the considered instant.

With a view to such applications as metal forming, one proposes the *inelastic* shock law:

At every instant such that $q_{\mathcal{N}} = 0$:

$$(2.1)_3 \qquad \mathcal{U}^+_{\mathcal{N}} = 0$$

together with the relations $(2.1)_1$, where $\mathcal{U}^+_{\mathcal{N}}$ is the normal component of $\mathcal{U}^+$. One may relax the previous law and consider the inelastic shock law only in the case of strict contact,

i.e. when $\mathscr{R}_{\mathscr{N}} > 0$; otherwise $\mathscr{U}^{+}_{\mathscr{N}} \geqslant 0$. Together with $\mathscr{R}_{\mathscr{N}} \geqslant 0$ and $\mathscr{R}_{\mathscr{N}} q_{\mathscr{N}} = 0$, these equations may be summarized as (the superscript + is now omitted).

$$(2.1)_4 \qquad \psi_{R^-}(q_{\mathscr{N}}) + \mathscr{U}_{\mathscr{N}} \geqslant 0, \quad \mathscr{R}_{\mathscr{N}} \geqslant 0, \quad (\psi_{R^-}(q_{\mathscr{N}} + \mathscr{U}_{\mathscr{N}} \mathscr{R}_{\mathscr{N}}) = 0,$$

where ψ_{R^-} is the indicator function of R^- (namely $\psi_{R^-}(x) = 0$ if $x \leqslant 0$, and ∞ otherwise); the convention $\infty \times 0 = 0$ is made. The choice of the relations $(2.1)_1$, $(2.1)_3$ and $(2.1)_4$ to express unilateral contact depends on the contemplated problem. Formulation $(2.1)_1$ is suitable for a quasi-static situation, and the relations $(2.1)_3$ or $(2.1)_4$ for dynamical problems. One shall not make systematic comparisons. Observe that if $t \to \mathscr{U}(t)$ is l.b.v. and $t \to \mathscr{R}_{\mathscr{N}}(t)$ is continuous, the formulation $(2.1)_1$ implies the relations $(2.1)_4$. Also, the solutions of the relation $(2.1)_3$ satisfies the relations $(2.1)_4$.

2.2. Coulomb's friction law

One uses Coulomb's friction law under the form given by J. J. MOREAU [2],

$$\mathscr{R}_{\mathscr{T}} \in \partial \psi^{*}_{C}(-\mathscr{U}_{\mathscr{T}}),$$

where $\mathscr{R}_{\mathscr{T}}$ is the tangential component of the reaction and $\mathscr{U}_{\mathscr{T}}$ is the sliding velocity. C is the convex set

$$C = \mathscr{R}_{\mathscr{N}} \mathscr{D} \text{ where } \mathscr{D} \text{ is the unit section of the friction cone.}$$

This formulation is also equivalent to a variational inequality expressing the "principle of maximum dissipation"

$$(2.2)_1 \qquad \mathscr{R}_{\mathscr{T}} \in \mathscr{R}_{\mathscr{N}} \mathscr{D}, \quad \forall \mathscr{S} \in \mathscr{R}_{\mathscr{N}} \mathscr{D}, \quad (\mathscr{S} - \mathscr{R}_{\mathscr{T}}) \mathscr{U}_{\mathscr{T}} \geqslant 0.$$

In the case of a one-dimensional motion with a friction coefficient μ, the relations $(2.2)_1$ take the form

$$\mathscr{D} = [-\mu, \mu], \quad \mathscr{R}_{\mathscr{N}} \mathscr{D} = [-\mu \mathscr{R}_{\mathscr{N}}, \mu \mathscr{R}_{\mathscr{N}}],$$

$$\mathscr{R}_{\mathscr{T}} \in [-\mu \mathscr{R}_{\mathscr{N}}, \mu \mathscr{R}_{\mathscr{N}}] \quad \forall \mathscr{S} \in [-\mu \mathscr{R}_{\mathscr{N}}, \mu \mathscr{R}_{\mathscr{N}}], \quad (\mathscr{S} - \mathscr{R}_{\mathscr{T}}) \mathscr{U}_{\mathscr{T}} \geqslant 0.$$

3. Discrete variables and algorithms

3.1. Discrete space variables

For the purpose of numerical computation, the configuration of a continuous medium Ω is approximately described through a n-dimensional variable $X = (X_1, \ldots, X_m, \ldots)$; for instance this may consist of the displacements of the mesh nodes in a finite element method. The position of every particle P of the medium is approximated by $q(\xi) = X_m e_m(\xi)$ where e_m are interpolation functions, ξ is some parameter labelling the particle. The tangential and normal components of the density of reaction on the contact boundary $\partial_1 \Omega$ may be approximated as

$$\mathscr{R}_{\mathscr{T}}(\xi) = r_{th} \mathscr{F}_h(\xi),$$

$$\mathscr{R}_{\mathscr{N}}(\xi) = r_{nh} \mathscr{F}_h(\xi),$$

where $\mathscr{F}_h$, $h = 1, \ldots, p$ are *non-negative* functions. It will be shown in Sect. 4, using a virtual power formula, that one may construct a $n \times p$ matrix H such that

$$(3.1)_1 \qquad \begin{aligned} R &= Hr = H_t r_t + H_n r_n, \\ r_t &= (r_{t1}, \ldots, r_{th}, \ldots), \quad r_n = (r_{n1}, \ldots, r_{nh}, \ldots), \end{aligned}$$

R is the representative of the reaction forces for the parametrization X. It will be shown also that, under some conditions, an approximation of the relations $(2.2)_1$ may be chosen as

$$(3.1)_2 \qquad \begin{aligned} &r_n \geqslant 0, \\ &r_t \in r_n \mathscr{D} \quad \forall s \in r_n \mathscr{D}, \quad (s - r_t) \cdot H_t^*(\dot{X} - \dot{X}_{0t}) \geqslant 0, \end{aligned}$$

where

$$\begin{aligned} r_n \geqslant 0 \qquad &\text{means} \quad \forall h \quad r_{nh} \geqslant 0, \\ r_t \in r_n \mathscr{D} \qquad &\text{means} \quad \forall h \quad r_{th} \in r_{nh} \mathscr{D}, \end{aligned}$$

$\dot{X}_{0t}$ is explicitly known and appears when dealing with obstacles moving with given velocities.

3.2. Discrete time variables

One adopts the usual implicit discretization during a time step

$$X(i+1) = X(i) + h\dot{X}(i+1),$$

where $X(i)$, $X(i+1)$, $\dot{X}(i+1)$ are approximants of $X(t_i)$, $X(t_{i+1})$, $\dot{X}(t_{i+1})$, $h = t_{i+1} - t_i$. One may also accept for the normal component of the relative position of a particle with respect to the obstacle, considered in its position at time t_{i+1}, the following approximation:

$$q_{\mathscr{N}}(i+1, \xi) = q_{\mathscr{N}}(i, \xi) + h\mathscr{U}_{\mathscr{N}}(i+1, \xi).$$

A discretization of the formulation $(2.1)_1$ is readily found to be

$$\begin{aligned} \forall \xi \quad &\mathscr{U}_{\mathscr{N}}(i+1, \xi) + q_{\mathscr{N}}(i, \xi)/h \geqslant 0, \quad \mathscr{R}_{\mathscr{N}}(i+1, \xi) \geqslant 0, \\ &\big(\mathscr{U}_{\mathscr{N}}(i+1, \xi) + q_{\mathscr{N}}(i, \xi)/h\big) \cdot \mathscr{R}_{\mathscr{N}}(i+1, \xi) = 0. \end{aligned}$$

This may also be considered as a discrete form of the relations $(2.1)_4$.

One may as well write, omitting the index $i+1$,

$$(3.2)_1 \qquad \forall \xi \quad \mathscr{R}_{\mathscr{N}}(\xi) \geqslant 0 \quad \forall \mathscr{S} \geqslant 0, \quad \big(\mathscr{S} - \mathscr{R}_{\mathscr{N}}(\xi)\big) \cdot \big(\mathscr{U}_{\mathscr{N}}(\xi) + q^i_{\mathscr{N}}(\xi)/h\big) \geqslant 0.$$

It will be shown in Sect. 4 that an approximation of the inequalities $(3.2)_1$ is

$$(3.2)_2 \qquad r_n \geqslant 0 \quad \forall s \geqslant 0 \quad (s - r_n) \cdot H_n^*(\dot{X} - \dot{X}_{0n}) \geqslant 0,$$

where $\dot{X}_{0n}$ is explicitly known from data at step i.

One may notice that the inequalities $(3.1)_2$ and $(3.2)_2$ are quite similar to the expressions found for a system of a finite number of particles [3, 4].

3.3. Numerical algorithm

In linear quasi-static problems as well as in dynamic problems, most numerical implicit or explicit schemes, used in order to solve the equilibrium equations, yield an expression such as

$$(3.3)_1 \qquad \dot{X}(i+1) = \dot{X}(i) + WhA(i) + WhR(i+1),$$

where $A(i)$ represents external loads explicitly known at time t_{i+1} and internal forces at time t_i. In a quasi-static problem the matrix h^2W is the inverse of the rigidity matrix; in a dynamic problem W is the inverse of a mass matrix. In the nonlinear case W^{-1} might be considered as a tangent matrix and Eq. $(3.3)_1$ is written for equilibrium subiterations, [4]. The set of equations for the discretized problem may now be summarized as follows, omitting the index $i+1$:

The equilibrium equation

$$(3.3)_1 \qquad \dot{X} = \dot{X}(i) + WhA(i) + WhR.$$

The change of variable formula

$$(3.3)_2 \qquad R_t = H_t r_t, \quad R_n = H_n r_n.$$

The unilateral constraint inequality

$$(3.3)_3 \qquad r_n \geqslant 0 \quad \forall s_n \geqslant 0, \quad (s_n - r_n) \cdot H_n^*(\dot{X} - \dot{X}_{0n}) \geqslant 0.$$

Coulomb's friction law

$$(3.3)_4 \qquad r_t \in r_n \mathscr{D} \quad \forall s_t \in r_n \mathscr{D}, \quad (s_t - r_t) \cdot H_t^*(X - X_{0t}) \geqslant 0,$$

where the unknowns are $\dot{X}$, R_t, R_n, r_t, r_n at step $i+1$. The variables $\dot{X}$, R_t, R_n may be eliminated using the inequalities $(3.3)_1$ and $(3.3)_2$. The inequalities $(3.3)_3$ and $(3.3)_4$ become

$$(3.3)_5 \qquad r_n \geqslant 0 \quad \forall s_n \geqslant 0, \quad (s_n - r_n) \cdot (W_{nn} r_n + W_{nt} r_t - Y_{0n}) \geqslant 0,$$

$$(3.3)_6 \qquad r_t \in r_n \mathscr{D} \quad \forall s_t \in r_n \mathscr{D}, \quad (s_t - r_t) \cdot (W_{tn} r_n + W_{tt} r_t - Y_{0t}) \geqslant 0,$$

where Y_{0n}, Y_{0t} are data and

$$W_{nn} = H_n^*(Wh)^{-1}H_n, \quad W_{nt} = H_n^*(Wh)^{-1}H_t,$$
$$W_{tn} = H_t^*(Wh)^{-1}H_n, \quad W_{tt} = H_t^*(Wh)^{-1}H_t$$

are influence matrices. The system $(3.3)_5$ and $(3.3)_6$ is a system of coupled variational inequalities and may be solved by an iterative procedure such as the Gauss–Seidel method. It may be proved that such a system has a unique solution when the friction coefficient is less than a critical value. The next section is to emphasize the paragraph (3.1).

4. Discrete space variables

4.1. Approximation of the relative velocity and relative position

For the purpose of numerical computation the configuration of a continuous medium is approximately described through an n-dimensional variable $X = (X_1, \ldots, X_m, \ldots)$. For instance, this may consist of the displacements of a N nodes mesh in a finite element

method; there $n = \nu N$, with $\nu = 2$ in the two-dimensional case, $\nu = 3$ in the three-dimensional case. The position of every particle of the medium is approximated by

$$q(t, \xi) = \sum_{m=1}^{n} X_m(t)\, e_m(\xi),$$

where ξ is a parameter labelling the particle and $\xi \to e_m(\xi) \in \mathsf{R}^\nu$ are interpolation functions. The components of the relative velocity in a local frame may be written as

$$\mathcal{U}(t, \xi) = \sum_{m=1}^{n} \dot{X}_m(t)\, \mathcal{E}_m(\xi) + \mathcal{U}_0(t, \xi),$$

where the functions $\mathcal{E}_m$ are deduced from e_m by a change of variables. The term $\mathcal{U}_0(t, \xi)$ appears when dealing with obstacles moving with given velocities. It is assumed that $\mathcal{U}_0(t, \xi)$ may be approximated as

$$\mathcal{U}_0(t, \xi) = \sum_{m=1}^{n} \mathcal{E}_m(\xi) \dot{X}_{0m}(t).$$

An expression of the normal and tangential components of the relative velocity is thus

$$(4.1)_1 \qquad \begin{aligned} \mathcal{U}_{\mathcal{T}}(t, \xi) &= \sum_{m=1}^{n} \mathcal{E}_{\mathcal{T}m}(\xi)\big(\dot{X}_m(t) - \dot{X}_{0m}(t)\big), \\ \mathcal{U}_{\mathcal{N}}(t, \xi) &= \sum_{m=1}^{n} \mathcal{E}_{\mathcal{N}m}(\xi)\big(\dot{X}_m(t) - \dot{X}_{0m}(t)\big). \end{aligned}$$

As mentioned in the previous section about time discretization, the expressions of $\mathcal{U}(t_{i+1}, \xi)$ will be used in the approximation process. One thus omits the variable t_{i+1}

$$\mathcal{U}_{\mathcal{T}}(\xi) = \sum_{m=1}^{n} \mathcal{E}_{\mathcal{T}m}(\xi)\,(\dot{X}_m - \dot{X}_{0m}),$$

$$\mathcal{U}_{\mathcal{N}}(\xi) = \sum_{m=1}^{n} \mathcal{E}_{\mathcal{N}m}(\xi)\,(\dot{X}_m - \dot{X}_{0m}).$$

For the normal component of the relative position of the particle, one uses the approximation

$$q_{\mathcal{N}}(t_{i+1}, \xi) = q_{\mathcal{N}}(t_i, \xi) + h\mathcal{U}_{\mathcal{N}}(t_{i+1}, \xi).$$

An approximant of $q_{\mathcal{N}}(t_i, \xi)$ is supposed to be

$$q_{\mathcal{N}}(t_i, \xi) = \sum_{m=1}^{n} \mathcal{E}_{\mathcal{N}m}(\xi)\, Y_{0m}.$$

Omitting the variable t_{i+1}, one obtains

$$(4.1)_2 \qquad \begin{aligned} q_{\mathcal{N}}(\xi)/h &= \sum_{m=1}^{n} \mathcal{E}_{\mathcal{N}m}(\xi)\,(\dot{X}_m - Y'_{0m}), \\ Y'_{0m} &= \dot{X}_{0m} - Y_{0m}/h. \end{aligned}$$

4.2. Approximation of the density of reaction

One intends to represent tangential and normal densities of reactions exerted on particles on a part $\partial_1 \Omega$ of the boundary as

$$(4.2)_1 \qquad \mathscr{R}_{\mathscr{T}}(\xi) = \sum_{h=1}^{P} r_{th} \mathscr{F}_h(\xi), \qquad \mathscr{R}_{\mathscr{N}}(\xi) = \sum_{h=1}^{P} r_{nh} \mathscr{F}_h(\xi),$$

where $\mathscr{F}_h$ are functions defined on $\partial_1 \Omega$ and with value in R; $r_{nh} \in \mathsf{R}$ and $r_{th} \in \mathsf{R}^{\nu-1}$. One deduces from the virtual power formula an expression relating the components r_{th} of $\mathscr{R}_{\mathscr{T}}$ with respect to the generating functions $\mathscr{F}_h$ and the representation R_t of this reaction with respect to the parametrization X:

$$\int_{\partial_1 \Omega} \mathscr{R}_{\mathscr{T}}(\xi)\, \delta \mathscr{U}_{\mathscr{T}}(\xi)\, d\xi = \int_{\partial_1 \Omega} \sum_{h=1}^{P} r_{th} \mathscr{F}_h(\xi) \sum_{m=1}^{n} \mathscr{E}_{\mathscr{T}m}(\xi)\, \delta \dot{X}_m\, d\xi = \delta \dot{X} \cdot H r_t = \delta \dot{X} \cdot R_t$$

so that

$$R_t = H_t r_t,$$

where H_t is the n-row $\times$ p-column block matrix with m-row, h-column element,

$$\int_{\partial_1 \Omega} \mathscr{F}_h(\xi) \mathscr{E}_{\mathscr{T}m}(\xi)\, d\xi \in \mathsf{R}^{\nu-1},$$

H_n is similarly defined,

$$\mathsf{R}_n = H_n r_n,$$

H_n is the n-row $\times$ p-column matrix with m-row, h-column element

$$\int_{\partial_1 \Omega} \mathscr{F}_h(\xi) \mathscr{E}_{\mathscr{N}m}(\xi)\, d\xi \in \mathsf{R}.$$

4.3. Approximation of contact and friction laws

The unilateral condition $(2.1)_2$ and Coulomb's law $(2.2)_1$ are expressed using densities of reactions (for the sake of simplicity they will be assumed to be L^2 functions as well as $q_{\mathscr{N}}$ and $\mathscr{U}_{\mathscr{T}}$, $\mathscr{U}_{\mathscr{N}}$).

$$(4.3)_1 \quad \text{a.e.}\xi \quad \mathscr{R}_{\mathscr{N}}(\xi) \geqslant 0 \quad \forall \mathscr{S}_{\mathscr{N}} \geqslant 0, \quad (\mathscr{S}_{\mathscr{N}} - \mathscr{R}_{\mathscr{N}}(\xi)) q_{\mathscr{N}}(\xi) \geqslant 0,$$

$$(4.3)_2 \quad \text{a.e.}\xi \quad \mathscr{R}_{\mathscr{T}}(\xi) \in \mathscr{R}_{\mathscr{N}}(\xi)\mathscr{D} \quad \forall \mathscr{S}_{\mathscr{T}} \in \mathscr{R}_{\mathscr{N}}(\xi)\mathscr{D}, \quad (\mathscr{S}_{\mathscr{T}} - (\xi)) \cdot \mathscr{U}_{\mathscr{T}}(\xi) \geqslant 0.$$

A similar formulation using density with respect to a positive measure has already been introduced by J. J. MOREAU [1] for time-varying reactions and relative velocities. Using Rockafellar's convex integrand theorem, these inequalities are, respectively, equivalent to

$$\mathscr{R}_{\mathscr{N}} \geqslant 0 \quad \forall \mathscr{S}_{\mathscr{N}} \geqslant 0 \qquad \int_{\partial_1 \Omega} (\mathscr{S}_{\mathscr{N}}(\xi) - \mathscr{R}_{\mathscr{N}}(\xi)) q_{\mathscr{N}}(\xi)\, d\xi \geqslant 0,$$

$$\mathscr{R}_{\mathscr{T}} \in \mathscr{R}_{\mathscr{N}}\mathscr{D} \quad \forall \mathscr{S}_{\mathscr{T}} \in \mathscr{R}_{\mathscr{N}}\mathscr{D} \qquad \int_{\partial_1 \Omega} (\mathscr{S}_{\mathscr{T}}(\xi) - \mathscr{R}_{\mathscr{T}}(\xi)) \mathscr{U}_{\mathscr{T}}(\xi)\, d\xi \geqslant 0,$$

where $\mathscr{R}_{\mathscr{N}}\mathscr{D} = \{\mathscr{R}_{\mathscr{T}} \in L^2 : \text{a.e.}\xi\ \mathscr{R}_{\mathscr{T}}(\xi) \in \mathscr{R}_{\mathscr{N}}(\xi)\mathscr{D}\}$ is a closed convex bounded set.

One supposes now that $\mathscr{F}_h$ are *positive functions*. Using the approximations $(4.1)_1$, $(4.1)_2$ and $(4.2)_1$, one may choose as approximations of the previous inequalities for the unilateral conditions

$$(4.3)_3 \qquad r_n \geqslant 0 \quad \forall s_n \geqslant 0,\ (s_n - r_n) \cdot H_n^*(\dot{X} - \dot{Y}_{0n}) \geqslant 0$$

for Coulomb's law

$$(4.3)_4 \qquad r_t \geqslant r_n \mathscr{D} \quad \forall s_t \in r_n \mathscr{D}, \quad (s_t - r_t) H_n^*(\dot{X} - \dot{X}_{0t}) \geqslant 0,$$

$$r_n \geqslant 0 \quad \text{means} \quad \forall h \quad r_{nh} \geqslant 0,$$

$$r_t \in r_n \mathscr{D} \quad \text{means} \quad \forall h \quad r_{th} \in r_{nh} \mathscr{D}.$$

The dot denotes the scalar product in $\mathbb{R}^p$. It may be proven that this approximation is suitable in the following sense.

PROPOSITION 1.

The $\mathscr{F}_h$ are supposed to be the characteristic functions of a partition S^k of $\partial_1 \Omega$. Let r_n^k, s_n^k, $\dot{X}^k$, verify the inequalities $(4.3)_3$ and $(4.3)_4$, and suppose that the sequences

$$\mathscr{R}_{\mathscr{N}}^k = r_{nh}^k \mathscr{F}_h, \qquad \mathscr{F}_{\mathscr{T}}^k = r_{th}^k \mathscr{F}_h,$$

$$q_{\mathscr{N}}^k = \left(H_n^{k*}(\dot{X}^k - \dot{X}_{0n}^k)\right)_m \mathscr{E}_{\mathscr{N}m}^k,$$

$$\mathscr{U}_{\mathscr{T}}^k = \left(H_t^{k*}(\dot{X}^k - \dot{X}_{0t})\right)_m \mathscr{E}_{\mathscr{T}m}^k.$$

converge to some $\mathscr{R}_{\mathscr{N}}^*$, $\mathscr{R}_{\mathscr{T}}^*$, $q_{\mathscr{N}}^*$, $\mathscr{U}_{\mathscr{T}}^*$ when the partition $\mathscr{S}^k$ is refined. Then $\mathscr{R}_{\mathscr{N}}^*$, $\mathscr{R}_{\mathscr{T}}^*$, $q_{\mathscr{N}}^*$, $\mathscr{U}_{\mathscr{T}}^*$ satisfy the inequalities $(4.3)_1$ and $(4.3)_2$.

The mathematical framework to discuss this formulation and the results is to be found in [5].

4.4. Examples

One considers the two-dimensional case where the generating functions e_m are associated with the nodes $h = 1, \ldots, N$ of a mesh

$$e_m(\xi) = k_j f_s(\xi), \qquad k_1 = (1, 0), \qquad k_2 = (0, 1),$$

$$j = 1, 2, \qquad s = 1, \ldots, N, \qquad m = 2s - 2 + j.$$

One supposes that the obstacle occupies the lower half plane $x_2 \leqslant 0$, with boundary, the x_1 axis, Fig. 1. The obstacle has a translative motion with constant velocity (E_1, E_2). One considers a part $\partial_1 \Omega$ of the boundary of a continuous medium, which is the union of successive $p-1$ segments of a straight line, parallel to the x_1 axis, with extremities the node h and the node $h+1$, for $h = 1, \ldots, p-1$. The parameter ξ labelling the particles of $\partial_1 \Omega$ may be chosen as the x_1 coordinate of the particle; the node h has for x_1 coordinate, ξ_h. The normal and tangential components of the relative velocity of a particle of $\partial_1 \Omega$ (with x_1 coordinate ξ) may be written as

$$\mathscr{U}_{\mathscr{N}} = \sum_{s=1}^{N} \dot{X}_{2s} f_s(\xi) - E_1,$$

$$\mathscr{U}_{\mathscr{T}} = \sum_{s=1}^{N} \dot{X}_{2s-1} f_s(\xi) - E_2.$$

One assumes that the constant functions may be generated by f_s:

$$\mathscr{U}_{\mathscr{N}} = \sum_{s=1}^{N} (\dot{X}_{2s} - \dot{X}_{0Ns}) f_s(\xi) = \sum_{l=1}^{p'} \mathscr{E}_{\mathscr{N}l}(\xi)\,(\dot{X}_l - \dot{X}_{0l}),$$

$$\mathscr{U}_{\mathscr{T}} = \sum_{s=1}^{N} (\dot{X}_{2s-1} - \dot{X}_{0Ts}) f_s(\xi) = \sum_{l=1}^{p'} \mathscr{E}_{\mathscr{T}l}(\xi)\,(\dot{X}_l - \dot{X}_{0l}),$$

with

$$p' = 2N,$$
$$\mathscr{E}_{\mathscr{N}l} = 0 \quad \text{if } l \text{ is odd}, \quad \mathscr{E}_{\mathscr{N}l} = f_s \quad \text{if} \quad l = 2s, \quad s = 1, \ldots, N,$$
$$\mathscr{E}_{\mathscr{T}l} = 0 \quad \text{if } l \text{ is even}, \quad \mathscr{E}_{\mathscr{T}l} = f_s \quad \text{if} \quad l = 2s-1,$$
$$\dot{X}_{0l} = \dot{X}_{0Ns} \quad \text{if} \quad l = 2s, \quad \dot{X}_{0l} = \dot{X}_{0Ts} \quad \text{if} \quad l = 2s-1.$$

One deduces then the value of the $p \times p'$ matrices H_n^*, H_t^*

$$H_{nhl}^* = \int_{\partial_1 \Omega} \mathscr{E}_{Nl}(\xi) \mathscr{F}_h(\xi)\, d\xi, \quad H_{thl}^* = \int_{\partial_1 \Omega} \mathscr{E}_{\mathscr{T}l}(\xi) \mathscr{F}_h(\xi)\, d\xi.$$

EXAMPLE 1

The reactions are described by the generating functions $\mathscr{F}_h$, $h = 1, \ldots, p$, associated with the nodes. Those are the step functions vanishing on $\partial_1 \Omega$ except on the interval $[\xi_h - \varepsilon/2, \xi_h + \varepsilon/2]$ for $\mathscr{F}_h$, $[\xi_1, \xi_1 + \varepsilon/2]$ for $\mathscr{F}_1$, $[\xi_p - \varepsilon/2, \xi_p]$ for $\mathscr{F}_p$, where they take the value $1/\varepsilon$ for $\mathscr{F}_h$, $2/\varepsilon$ for $\mathscr{F}_1$ and $\mathscr{F}_p$. When ε tends to zero, this is a way to describe concentrated forces on the nodes $h = 1, \ldots, p$. Let us assume that f_s are continuous functions and $f_s(\xi_h) = 1$. One finds the limits of H_n^*, H_t^* when $\varepsilon \to 0$,

$$H_n^* = L_n^*, \quad H_t^* = L_t^*$$

with

$$L_n^* = \begin{bmatrix} 0 & 1 & \ldots & & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ldots & & & & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & & \ldots & 0 & 1 & 0 & \ldots & 0 \end{bmatrix} \begin{matrix} \uparrow \\ \\ \\ \downarrow \end{matrix} p,$$

$$\overbrace{\qquad\qquad}^{2p} \qquad \overbrace{\qquad\qquad\qquad\qquad}^{p' = 2N}$$

$$L_t^* = \begin{bmatrix} 1 & 0 & \ldots & & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ldots & & & & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & & \ldots & 1 & 0 & 0 & \ldots & 0 \end{bmatrix} \begin{matrix} \uparrow \\ \\ \\ \downarrow \end{matrix} p.$$

$$\overbrace{\qquad\qquad}^{2p} \qquad \overbrace{\qquad\qquad\qquad\qquad}^{p' = 2N}$$

When the boundary of the obstacle is not a straight line, the terms of these two matrices are replaced by 2×2 blocks corresponding to the components of the unit normal and

tangent vectors in the local frame. For this kind of generating functions $\mathscr{F}_h$ (Dirac functions), it is irrelevant to discuss now the question of convergence within the scope of proposition 1.

EXAMPLE 2

The mesh is a $T3$ mesh; the relative velocities $\mathscr{U}_\mathscr{T}$, $\mathscr{U}_\mathscr{N}$ are approximated by continuous affine functions. The generating functions f_h have a graph as depicted in Fig. 2. The density of reactions are described by generating functions χ_h, $h = 1, \ldots, p$, associated with the nodes. They are the characteristic functions χ_h of the intervals $\left[\frac{1}{2}(\xi_{h-1}+\xi_h), \frac{1}{2}(\xi_h+\right.$ $\left.+\xi_{h+1})\right]$, $h \neq 1, p$, χ_1 of $\left[\xi_1, \frac{1}{2}(\xi_1+\xi_2)\right]$, χ_p of $\left[\frac{1}{2}(\xi_{p-1}+\xi_p), \xi_p\right]$. One finds for the expression of H_n^*, H_t^*

$$H_n^* = \mathscr{M}^* P^* L_n^*, \quad H_t^* = \mathscr{M}^* P^* L_t^*.$$

$\mathscr{M}^*$ is the $p \times p$ diagonal matrix with the element $\mathscr{M}_h$ equal to the measure of the support of χ_h, $\left(\frac{1}{2}(\xi_{h+1}-\xi_{h-1}), \ldots\right)$

$$P^* = \begin{bmatrix} 3/4 & 1/4 & 0 & \ldots & & & 0 \\ 1/8 & 3/4 & 1/8 & 0 & \ldots & & 0 \\ 0 & 1/8 & 3/4 & 1/8 & 0 & \ldots & 0 \\ & \ldots & & & & \ldots & \\ & \ldots & 0 & 1/8 & 3/4 & 1/8 & 0 \\ & & \ldots & 0 & 1/8 & 3/4 & 1/8 \\ & & & & & 1/4 & 3/4 \end{bmatrix} \begin{matrix} \uparrow \\ \\ \\ \\ \\ \\ \downarrow \end{matrix} p.$$

$$\longleftrightarrow p \longrightarrow$$

This approximation process is consistent within the scope of proposition 1. The unknowns r_{nh}, r_{th}, have the dimension of a density of reaction. The images by $\mathscr{M}$, $\mathscr{M}r_n$ and $\mathscr{M}r_t$ have the dimension of a force and might be as well used as unknowns. The terms r'_{nh}, r'_{th}, with

$$r'_n = P\mathscr{M}r_n, \quad r'_t = P\mathscr{M}r_t$$

may be considered as the concentrated forces to nodes h, equivalent (in the sense of the finite element method) to the densities of reactions $\sum_{h=1}^{p} r_{nh}\chi_h$, $\sum_{h=1}^{p} r_{th}\chi_h$. One may wonder if the solutions of examples 1 and 2 may be compared. The following propositions allow one to answer this question.

PROPOSITION 2.

Let B be a mapping from R^p into itself represented by a matrix with non-negative elements, i.e.

$$\forall i = 1, \ldots, p \quad \forall j = 1, \ldots, p \quad B_{ij} \geqslant 0,$$

and rows different from zero

$$\forall i \quad \exists j \quad B_{ij} > 0.$$

Then,

i) $$r_n \in (\mathbf{R}^+)^p \Rightarrow Br_n \in (\mathbf{R}^+)^p.$$

If $r \in (\mathbf{R}^+)^p$,

ii) $$(Br_n)\mathscr{D} \subset B(r_n\mathscr{D}).$$

Furthermore, if B is a diagonal matrix, the converse of i) is true, the equality in ii) is true.

Proof

The implication is obvious. The proof of the inclusion is as follows. Let $r_t \in B(r_n\mathscr{D})$,

$$r_{th} = \sum_{k=1}^{p} B_{hk} s_k, \qquad s_k \in r_{nk}\mathscr{D},$$

s_k may be written, $s_k = r_{kn}\gamma_k$ with $\gamma_k \in \mathscr{D}$.

$$r_{th} = \sum_{k=1}^{p} B_{hk} r_{nk} \gamma_k.$$

If $r_n = 0$, the assertion is true. If $r_n \neq 0$, the term

$$\sum_{k=1}^{p} B_{hk} r_{nk}$$

is different from zero. One may write

$$r_{th} = \Big(\sum_{k=1}^{p} B_{hk} r_{nk}\gamma_k \Big/ \sum_{k=1}^{p} B_{hk} r_{nk}\Big) \sum_{k=1}^{p} B_{hk} r_{nk}.$$

The quotient term appears as a linear convex combination of $\gamma_j \in \mathscr{D}$. It is an element of $\mathscr{D}$. So $r_t \in (Br_n)\mathscr{D}$.

The matrix P of example 2 satisfies the assumptions of proposition 2.

PROPOSITION 3

Let P satisfy the assumptions of proposition 2. Let $\bar{r}_n$, $\bar{r}_t$ be such that

$$\bar{r}_n > 0 \quad \forall \bar{s}_n \geqslant 0, \quad (\bar{s}_n - \bar{r}_n) \cdot L_n^*(\dot{X} - \dot{X}_{0n}) \geqslant 0.$$

If r_n is an inverse image of $\bar{r}_n$ by P, $\bar{r}_n = Pr_n$, such that $r_n \geqslant 0$, then

$$r_n \geqslant 0 \quad \forall s_n \geqslant 0 \quad (s_n - r_n) \cdot P^* L_n^*(\dot{X} - \dot{X}_{0n}) \geqslant 0,$$

In other words, if the inverse images r_n, r_t of the solution of a problem with concentrated reactions satisfy the constraints equations, they are solution of the problem with weighted reactions.

Proof

Let $s_n \geqslant 0$, then

$$(s_n - r_n) \cdot P^* L_n^*(\dot{X}_n - \dot{X}_{0n}) = (Ps_n - Pr_n) \cdot L_n^*(\dot{X} - \dot{X}_{0n}) \geqslant 0$$

since $Pr_n \geqslant 0$, $Ps_n \geqslant 0$.

Let $s_t \in r_n\mathscr{D}$

$$(s_t - r_t) P^* L_t^*(\dot{X} - \dot{X}_{0n}) = (Ps_t - Pr_t) L_t^*(\dot{X} - \dot{X}_{0t}) \geqslant 0$$

since $Pr_t = \bar{r}_t \in \bar{r}_n\mathscr{D}$, $Ps_t \in P(r_n\mathscr{D}) \subset (Pr_n) = r_n\mathscr{D}$.

4.5. Numerical examples

The following examples are derived from a benchmark used in GRECO Grandes Déformations et Endommagement.

An elastic isotropic parallelepiped with $E = 0.13 \cdot 10^{12}$ Pa, $\nu = 0.2$, is considered within the plane deformations hypothesis. The dimensions are $4 \cdot 10^{-3}$ m $\times 4 \cdot 10^{-3}$ m $\times 1$ m (unit length thickness). The boundary conditions are as depicted in Fig. 3a. The parallelepiped is set on a plane obstacle with the friction coefficient $\mu = 1$. A monotone increasing loading is applied to the free edges with final state as depicted in the figure. Computation is made using the quasi-static hypothesis. For this kind of loading it is thus possible to use only one step of loading. Two computations are made, with concentrated reactions and with weighted reactions (using the matrix P). The accuracy required for the computation of the reaction force is 10^{-4}.

On the tables Figs. 3b are depicted for each node candidate to contact from number 1 to 16:

The horizontal and vertical displacements $Q1$, $Q2$, with respect to the reference configuration.

The tangential and normal components $\mathscr{R}_{\mathscr{T}}$, $\mathscr{R}_{\mathscr{N}}$ of the reaction forces, (N/m, Newton/unit length of thickness). In the first table, a computation with concentrated reaction is performed; the concentrated reactions are shown and also the inverse images of $\mathscr{R}_{\mathscr{T}}$, $\mathscr{R}_{\mathscr{N}}$, by P (weighted reactions equivalent to concentrated reactions). In the second table a computation with weighted reaction is performed; the weighted reactions are shown and also the images of $\mathscr{R}_{\mathscr{T}}$, $\mathscr{R}_{\mathscr{N}}$, by P (concentrated forces equivalent to weighted reactions).

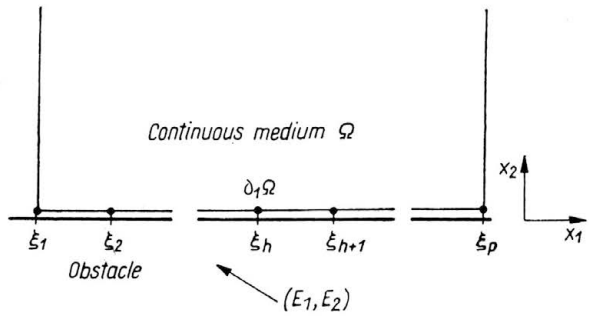


FIG. 1.

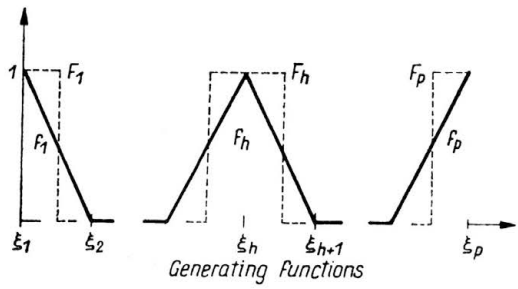
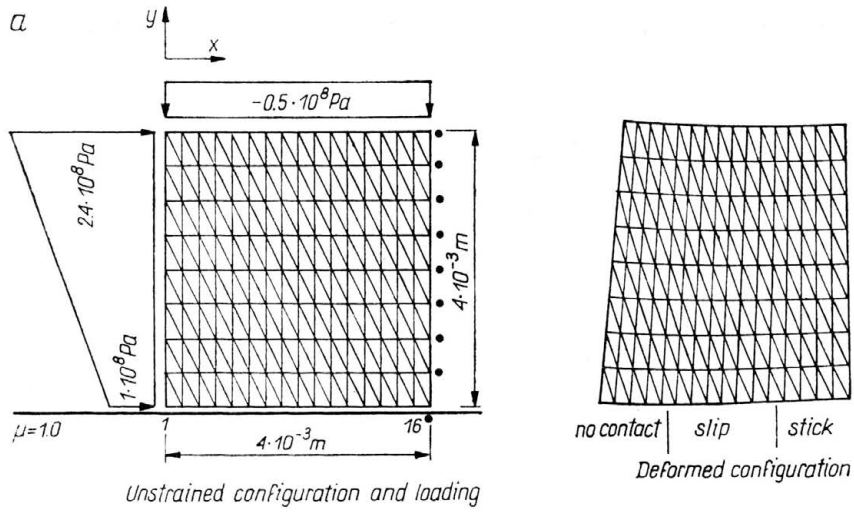


FIG. 2.

$\mathcal{R}_x$ and $\mathcal{R}_y$ are the horizontal and vertical components of the resulting reaction.

The deformed configuration with displacements amplified by a factor of 10^4 is displayed in Fig. 3a. The nodes 1 to 5 are not in contact; the nodes 5 to 12 are sliding and the nodes 5 to 16 are sticking. While the two computations give good agreement with respect to the



NPAS= 1 TPS= 0.10000D+01 IS= 62 ITT= 61 INN= 62 H= 0.10000D+01

	D1	D2	RT	RN	RT	RN
			Concentrated reactions		Whei. reac.	equ. conc. reac.
1	0.22015D-04	0.72137D-05	0.00000D+00	0.00000D+00	0.33656D+01	-0.33682D+01
2	0.20169D-04	0.50655D-05	0.00000D+00	0.00000D+00	-0.20193D+02	0.20209D+02
3	0.18367D-04	0.31504D-05	0.00000D+00	0.00000D+00	0.11443D+03	-0.11452D+03
4	0.16532D-04	0.15878D-05	0.00000D+00	0.00000D+00	-0.66638D+03	0.66690D+03
5	0.14562D-04	0.48193D-06	0.00000D+00	0.00000D+00	0.38839D+04	-0.38869D+04
6	0.12310D-04	-0.31046D-09	-0.27669D+05	0.27683D+05	-0.22637D+05	0.22654D+05
7	0.98259D-05	-0.71824D-09	-0.83808D+05	0.83817D+05	-0.89416D+05	0.89426D+05
8	0.74151D-05	-0.78711D-09	-0.11171D+06	0.11171D+06	-0.11134D+06	0.11133D+06
9	0.51681D-05	-0.67232D-09	-0.13597D+06	0.13597D+06	-0.13626D+06	0.13632D+06
10	0.31771D-05	-0.48273D-09	-0.15849D+06	0.15848D+06	-0.15889D+06	0.15851D+06
11	0.15512D-05	-0.28817D-09	-0.18013D+06	0.18012D+06	-0.17832D+06	0.18046D+06
12	0.43084D-06	-0.12944D-09	-0.20050D+06	0.20049D+06	-0.21226D+06	0.19972D+06
13	-0.71149D-10	-0.23030D-10	-0.14698D+06	0.22461D+06	-0.15210D+06	0.22517D+06
14	-0.51152D-10	0.25630D-11	-0.62609D+05	0.24258D+06	-0.50985D+05	0.24615D+06
15	-0.33412D-10	0.45696D-11	-0.40996D+05	0.24587D+06	-0.42863D+05	0.23854D+06
16	-0.17018D-10	-0.33881D-20	-0.20211D+05	0.24698D+06	-0.19805D+05	0.28955D+06

Rx Ry
-0.11691D+07 0.18583D+07

NPAS= 1 TPS= 0.10000D+01 IS= 85 ITT= 85 INN= 85 H= 0.10000D+01

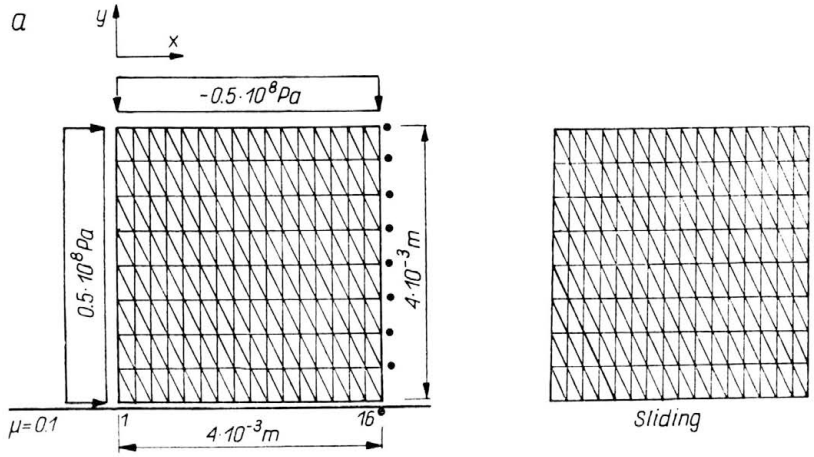
	D1	D2	RT	RN	RT	RN
			Wheighted reactions		Conc. reac.	equ. whei. reac.
1	0.22042D-04	0.71653D-05	0.00000D+00	0.00000D+00	0.00000D+00	0.00000D+00
2	0.20196D-04	0.50212D-05	0.00000D+00	0.00000D+00	0.00000D+00	0.00000D+00
3	0.18395D-04	0.31089D-05	0.00000D+00	0.00000D+00	0.00000D+00	0.00000D+00
4	0.16561D-04	0.15476D-05	0.00000D+00	0.00000D+00	0.00000D+00	0.00000D+00
5	0.14595D-04	0.44082D-06	0.00000D+00	0.00000D+00	-0.12022D+04	0.12002D+04
6	0.12364D-04	-0.74087D-07	-0.96178D+04	0.96015D+04	-0.19661D+05	0.19652D+05
7	0.98309D-05	0.11814D-07	-0.99583D+05	0.99609D+05	-0.89521D+05	0.89537D+05
8	0.74188D-05	-0.14609D-08	-0.10905D+06	0.10903D+06	-0.11137D+06	0.11136D+06
9	0.51647D-05	0.12290D-09	-0.13705D+06	0.13706D+06	-0.13624D+06	0.13624D+06
10	0.31675D-05	0.36018D-10	-0.15853D+06	0.15852D+06	-0.15856D+06	0.15856D+06
11	0.15343D-05	0.39268D-10	-0.18025D+06	0.18026D+06	-0.18012D+06	0.18012D+06
12	0.40443D-06	-0.97150D-11	-0.20093D+06	0.20093D+06	-0.19616D+06	0.20096D+06
13	-0.69381D-07	0.23745D-10	-0.18342D+06	0.22184D+06	-0.16593D+06	0.22279D+06
14	0.11895D-07	-0.26392D-11	-0.26007D+05	0.25035D+06	-0.48741D+05	0.24517D+06
15	-0.20381D-08	0.37773D-11	-0.50469D+05	0.23738D+06	-0.43351D+05	0.24557D+06
16	0.33927D-09	-0.62955D-12	-0.17986D+05	0.28996D+06	-0.19798D+05	0.24714D+06

Rx Ry
-0.11729D+07 0.18946D+07

FIG. 3.

values of $Q1$, $Q2$, and the resulting reactions $\mathscr{R}_x$, $\mathscr{R}_y$, and while some similarity may be noticed between the values of $\mathscr{R}_\mathscr{T}$, $\mathscr{R}_\mathscr{N}$, these values are not to be compared since $P^{-1}\bar{r}_n \notin \notin (\mathrm{R}^+)^p$ and also $P^{-1}\bar{r}_t \notin P^{-1}(\bar{r}_n \mathscr{D})$.

In Figs. 4a and 4b a similar example is treated with a different final loading state and $\mu = 0.1$. In this example, the nodes 1 to 16 are sliding. The two computations provide the same values of $Q1$, $Q2$ and of $\mathscr{R}_\mathscr{T}$, $\mathscr{R}_\mathscr{N}$, with respect to the change of variable by P



NFAS= 1 TFS= 0.10000D+01 IS= 42 ITT= 41 INN= 42 H= 0.10000D+01

	Q1	Q2	RT	RN	RT	RN
			Concentrated reactions		Whei. reac.	equ. conc. reac.
1	0.90963D-05	0.33282D-09	-0.40952D+04	0.40945D+05	-0.35443D+04	0.35431D+05
2	0.83546D-05	-0.11685D-09	-0.10944D+05	0.10945D+06	-0.11496D+05	0.11497D+06
3	0.76545D-05	0.17365D-09	-0.11548D+05	0.11547D+06	-0.11490D+05	0.11489D+06
4	0.69801D-05	0.34581D-09	-0.11916D+05	0.11915D+06	-0.11944D+05	0.11944D+06
5	0.63244D-05	0.41693D-09	-0.12166D+05	0.12166D+06	-0.12172D+05	0.12172D+06
6	0.56842D-05	0.41047D-09	-0.12352D+05	0.12352D+06	-0.12357D+05	0.12357D+06
7	0.50584D-05	0.34916D-09	-0.12500D+05	0.12500D+06	-0.12503D+05	0.12503D+06
8	0.44469D-05	0.25711D-09	-0.12625D+05	0.12625D+06	-0.12627D+05	0.12627D+06
9	0.38505D-05	0.16086D-09	-0.12736D+05	0.12736D+06	-0.12737D+05	0.12737D+06
10	0.32703D-05	0.78356D-10	-0.12839D+05	0.12839D+06	-0.12839D+05	0.12840D+06
11	0.27083D-05	0.21654D-10	-0.12938D+05	0.12939D+06	-0.12938D+05	0.12938D+06
12	0.21675D-05	-0.65229D-11	-0.13038D+05	0.13038D+06	-0.13040D+05	0.13040D+06
13	0.16516D-05	-0.14278D-10	-0.13141D+05	0.13141D+06	-0.13128D+05	0.13128D+06
14	0.11666D-05	-0.12097D-10	-0.13247D+05	0.13247D+06	-0.13315D+05	0.13315D+06
15	0.72068D-06	-0.58080D-11	-0.13351D+05	0.13351D+06	-0.12959D+05	0.12958D+06
16	0.32607D-06	0.67763D-20	-0.13428D+05	0.13428D+06	-0.15744D+05	0.15744D+06

Rs: Rv
-0.19286D+06 0.19286D+07

NFAS= 1 TFS= 0.10000D+01 IS=146 ITT=146 INN=146 H= 0.10000D+01

	Q1	Q2	RT	RN	RT	RN
			Wheighted reactions		Conc. reac.	equ. whei. reac.
1	0.90955D-05	0.14158D-08	-0.35645D+04	0.35636D+05	-0.41058D+04	0.41054D+05
2	0.83544D-05	-0.96408D-09	-0.11460D+05	0.11461D+06	-0.10925D+05	0.10926D+06
3	0.76542D-05	0.42332D-09	-0.11516D+05	0.11515D+06	-0.11560D+05	0.11560D+06
4	0.69800D-05	-0.19221D-09	-0.11930D+05	0.11931D+06	-0.11909D+05	0.11909D+06
5	0.63243D-05	0.54583D-10	-0.12176D+05	0.12176D+06	-0.12168D+05	0.12167D+06
6	0.56842D-05	-0.26062D-10	-0.12355D+05	0.12355D+06	-0.12351D+05	0.12351D+06
7	0.50584D-05	0.23965D-11	-0.12503D+05	0.12503D+06	-0.12500D+05	0.12500D+06
8	0.44470D-05	-0.53240D-11	-0.12627D+05	0.12627D+06	-0.12625D+05	0.12625D+06
9	0.38505D-05	-0.57421D-12	-0.12737D+05	0.12737D+06	-0.12736D+05	0.12736D+06
10	0.32703D-05	-0.23196D-11	-0.12840D+05	0.12840D+06	-0.12839D+05	0.12839D+06
11	0.27084D-05	-0.47256D-12	-0.12939D+05	0.12939D+06	-0.12939D+05	0.12939D+06
12	0.21675D-05	-0.94924D-12	-0.13040D+05	0.13040D+06	-0.13038D+05	0.13038D+06
13	0.16516D-05	-0.33129D-12	-0.13129D+05	0.13129D+06	-0.13141D+05	0.13141D+06
14	0.11666D-05	-0.27656D-12	-0.13315D+05	0.13315D+06	-0.13247D+05	0.13247D+06
15	0.72069D-06	-0.97491D-13	-0.12959D+05	0.12959D+06	-0.13351D+05	0.13351D+06
16	0.32608D-06	0.16248D-13	-0.15744D+05	0.15744D+06	-0.13428D+05	0.13428D+06

Rs: Rv
-0.19483D+06 0.19483D+07

FIG. 4.

or P^{-1}. One may observe on the basis of these two examples that a computation with weighted reactions needs more iterations than a computation with concentrated reactions. This is probably due to the fact that in the first case there is more coupling than in the second case between the unknown reactions.

Conclusion

Other examples have shown that in general there is good agreement between the two ways of computation. They even coincide in the special case of full sliding or full sticking of the boundary candidates to contact. The two ways of computation seem thus acceptable. This is a particular situation due to the fact the matrix P is positive. For other kinds of finite elements further investigation is needed.

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