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Mathematical modelling of delamination effects in layered composites

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THE AIM of the paper is to propose and discuss a mathematical model of the interlaminae debonding process in layered composites. The proposed method of modelling leads to the time-dependent quasi-variational inequality for the displacement rates. The results obtained can be applied to composites made of elastic as well as elastic/viscoplastic materials subject to small strains.

Celem artykułu jest propozycja i dyskusja pewnego modelu matematycznego procesu rozwarstwiania (delaminacji) kompozytów warstwowych (laminatów). Proponowany model prowadzi do zależnej od czasu nierówności quasi-wariacyjnej dla pól prędkości przemieszczeń. Otrzymane rezultaty mogą być zastosowane zarówno do sprężystych jak i sprężysto/lepkoplastycznych laminatów poddanych małym odkształceniom.

Целью статьи является предложение и обсуждение некоторой математической модели процесса расслоения (деламинации) слоистых композитов (ламинатов). Продложенная модель приводит к независящему от времени квазивариационному неравенству для полей скоростей перемещений. Полученные результаты могут быть применены так к упрутим, как и купруго/вязкопластическим ламинатам, подвергнутым малым деформациям.

1. Introduction

IN THE PAPER we shall consider the multilayered composites (laminates) formed by a sequence of plane or curved sheets separated by the very thin layers of a bonding material. The sheets are assumed to have finite thickness and are modelled as anisotropic elastic or elastic/viscoplastic materials. The interlaminae layers of the bonding material are modelled as surfaces, their thickness being neglected, and are assumed to sustain only restricted values of the interlaminae tractions. Hence, during the deformation, discontinuities in the tangential and normal displacements across some parts of the interlaminae surfaces may arise. The main aim of the paper is to propose a certain general mathematical model of such interlaminae debonding process. It will be shown that the proposed method of modelling leads to a certain time-dependent quasi-variational inequality.

For the sake of simplicity we confine our considerations to the pure mechanical debonding processes; more general approach will be given separately. The basic notions and denotations which will be used in the subsequent parts of the paper are detailed below.

By Ω we define the known regular region in the Euclidean 3-space R^3 that is occupied by the undeformed composite body, $\Omega \subset \mathsf{R}^3$. The body is assumed to be made of S disjointed layers. Hence $\bar{\Omega} = \cup \bar{\Lambda}_K$, $K = 1, \ldots, S$, where Λ_K stand for the undeformed layers (laminae); the thickness of the layer Λ_K is constant and will be denoted by δ_K, cf. Fig. 1. Setting

$$\Lambda \equiv \bigcup_{K=1}^{S} \Lambda_K, \quad \Pi_K \equiv (\partial\Lambda_K \cap \partial\Lambda_{K+1}) \setminus \partial\Omega, \quad K = 1, \ldots, S-1,$$
$$\Pi \equiv \bigcup_{K=1}^{S-1} \Pi_K,$$

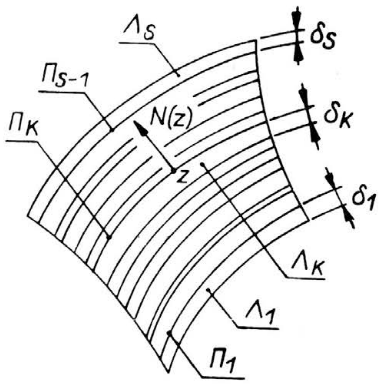

FIG. 1. Scheme of the undeformed layered body.

we see that $\Pi = \partial\Lambda \setminus \partial\Omega$ is the sum of all interlaminae surfaces separating the adjacent layers. We assume that every Π_K is a smooth surface and we assign to every point $z \in \Pi_K$ the unit normal vector $\mathbf{N}(z)$, which is outward to Λ_K and hence inward to Λ_{K+1}, $K = 1, \ldots, S-1$, cf. Fig. 1. At the same time to every point $\mathbf{x} \in \partial\Omega$ at which $\partial\Omega$ is smooth, the unit normal $\mathbf{n}(\mathbf{x})$ outward to Ω is assigned.

We shall deal with functions defined on Ω which may suffer discontinuities across Π. Let $\psi: \Lambda \to \mathrm{R}$ be a function such that every $\psi|_{\Lambda_K}$ (which is a function obtained from $\psi(\cdot)$ by the restriction of its domain to Λ_K) has well defined traces (boundary values) on $\partial\Lambda_K$, $K = 1, \ldots, S$. If $z \in \Pi$, then also $z \in \Pi_K \subset \Pi$ for some $K = 1, \ldots, S-1$. Thus the traces of $\psi|_{\Lambda_{K+1}}$, $\psi|_{\Lambda_K}$ at $\mathbf{z} \in \Pi_K$, where K runs over $1, \ldots, S-1$, can be uniquely denoted by $\psi^+(\mathbf{z})$, $\psi^-(\mathbf{z})$, respectively. Hence we shall define

$$[\![\psi]\!](\mathbf{z}) \equiv \psi^+(\mathbf{z}) - \psi^-(\mathbf{z}), \quad \mathbf{z} \in \Pi,$$

as the jumps of $\psi(\cdot)$ across Π. We shall also deal with functions defined on Π and for an arbitrary vector function $\mathbf{w}: \Pi \to \mathrm{R}^3$ we shall introduce the denotations

$$w_N(\mathbf{z}) \equiv \mathbf{w}(\mathbf{z}) \cdot \mathbf{N}(\mathbf{z}),$$
$$\mathbf{w}_T(\mathbf{z}) \equiv \mathbf{w}(\mathbf{z}) - \mathbf{N}(\mathbf{z})\,[\mathbf{w}(\mathbf{z}) \cdot \mathbf{N}(\mathbf{z})],$$

for the normal and tangent component of $\mathbf{w}(\mathbf{z})$, respectively. Obviously, $\mathbf{w}_T(\mathbf{z}) \cdot \mathbf{N}(\mathbf{z}) = 0$ for every $\mathbf{z} \in \Pi$.

The problems under consideration will be examined in the time interval $[\tau_0, \tau_f]$, with the initial time instant τ_0 related to the undeformed body. The time derivatives (for the constant $\mathbf{x} \in \Lambda$ or $\mathbf{z} \in \Pi$) will always be interpreted as the right-hand side derivatives and denoted by dots. The differentiation with respect to $\mathbf{x} \in \Lambda$ will be represented by the operator ∇. We assume that at every time instant $\tau \in [\tau_0, \tau_f]$ the displacement field $\mathbf{u}(\cdot, \tau): \Lambda \to \to R^3$ and the velocity field $\dot{\mathbf{u}}(\cdot, \tau): \Lambda \to R^3$ are well defined, where $\mathbf{u}(\mathbf{x}, \tau_0) = 0$, $\mathbf{x} \in \Lambda$. The investigations will be carried out under the assumptions of the small displacement gradients and the small velocity gradients. Thus the strain-displacement relation will be assumed in the form

$$\mathbf{E}(\mathbf{x}, \tau) = \frac{1}{2}[\nabla\mathbf{u}(\mathbf{x}, \tau) + \nabla\mathbf{u}^T(\mathbf{x}, \tau)], \quad \mathbf{x} \in \Lambda, \quad \tau \in [\tau_0, \tau_f], \tag{1.1}$$

where the superscript T stands for a transpose operation. The remaining denotations will be introduced in the subsequent sections of the paper.

We develop the subject from the known relations of the continuum mechanics, starting from the equations of motion (in Sect. 2) and introducing the constitutive relations in Sect. 3. The crucial point of the approach lies in Sect. 4 where the interlaminae conditions are detailed and a certain mathematical model of the possible debonding process is proposed. The results obtained will be summatrized in Sect. 5 in the form of the quasi-variational time-dependent inequality. On this basis we discuss some special models of the delamination effects in the composites under consideration. We belive that the introduced models provide the theoretical background for the experiments as well as for the analysis; some general conclusions on this subject are listed in Sect. 6.

2. Equations of motion

We assume that at any time instant $\tau \in [\tau_0, \tau_f]$ the body is subject exclusively to the body forces $\mathbf{b}(\mathbf{x}, \tau)$, $\mathbf{x} \in \Lambda$, and to the surface tractions $\mathbf{p}(\mathbf{z}, \tau)$, $\mathbf{z} \in \Gamma \subset \partial\Omega$, defined on the part Γ of the boundary $\partial\Omega$. For the sake of simplicity we assume that on the remaining part $\Gamma_0 = \partial\Omega \setminus \bar{\Gamma}$ of the boundary $\partial\Omega$, the traces (boundary values) of the displacement field are equal to zero:

$$\mathbf{u}(\mathbf{z}, \tau) = 0, \quad \mathbf{z} \in \Gamma_0, \quad \tau \in [\tau_0, \tau_f]. \tag{2.1}$$

By $\mathbf{T}(\mathbf{x}, \tau)$, $\mathbf{x} \in \Lambda$, $\tau \in [\tau_0, \tau_f]$, we define the Cauchy stress tensor and by $\mathbf{t}(z, \tau)$, $z \in \Pi$, $\tau \in [\tau_0, \tau_f]$, the interlaminae stress vector. We shall assume that the tractions $\mathbf{t}(\mathbf{z}, \tau)$, $\mathbf{z} \in \Pi$, are acting on the pertinent layers of the composite across the surface elements oriented by the outward normals $\mathbf{N}(\mathbf{z})$, $\mathbf{z} \in \Pi$. This means that if $\mathbf{z} \in \Pi_K \subset \partial\Lambda_{K+1} \cap \partial\Lambda_K$ for some $K = 1, \ldots, S-1$, then the traction $\mathbf{t}(\mathbf{z}, \tau)$ is acting on the layer Λ_K (cf. Fig. 1) and hence the traction $-\mathbf{t}(\mathbf{z}, \tau)$ is acting on the layer Λ_{K+1} (we bear in mind that no external forces are applied to the interlaminae surfaces). Finally, by $\varrho(\mathbf{x})$, $\mathbf{x} \in \Lambda$, we define the known mass density of the undeformed body. We shall postulate the equations of motion in the variational form

$$\int_\Lambda \mathrm{tr}[\mathbf{T}(\mathbf{x}, \tau)\nabla\mathbf{v}(\mathbf{x})]dV + \int_\Pi \mathbf{t}(\mathbf{z}, \tau) \cdot [\![\mathbf{v}]\!](\mathbf{z})dA = \int_\Lambda \varrho(\mathbf{x})[\mathbf{b}(\mathbf{x}, \tau) - \ddot{\mathbf{u}}(\mathbf{x}, \tau)] \cdot \mathbf{v}(\mathbf{x})dV + \int_\Gamma \mathbf{p}(\mathbf{x}, \tau) \cdot \mathbf{v}(\mathbf{x})dA, \quad \tau \in [\tau_0, \tau_f], \tag{2.2}$$

which has to hold for every sufficiently regular test function $\mathbf{v}: \Lambda \to \mathsf{R}^3$ the trace of which on Γ_0 attains values equal to zero. Under the known smoothness conditions, Eq. (2.2) is equivalent to the local equations

$$
(2.3) \qquad \begin{aligned}
&\operatorname{div}\mathbf{T}(\mathbf{x}, \tau) + \varrho(\mathbf{x})\mathbf{b}(\mathbf{x}, \tau) = \varrho(\mathbf{x})\ddot{u}(\mathbf{x}, \tau), \quad \mathbf{x} \in \Lambda, \\
&\mathbf{T}(\mathbf{x}, \tau)\mathbf{n}(\mathbf{x}) = \mathbf{p}(\mathbf{x}, \tau), \quad \mathbf{x} \in \Gamma, \\
&\mathbf{T}^+(\mathbf{z}, \tau)\mathbf{N}(\mathbf{z}) = \mathbf{T}^-(\mathbf{z}, \tau)\mathbf{N}(\mathbf{z}) = \mathbf{t}(\mathbf{z}, \tau), \quad \mathbf{z} \in \Pi,
\end{aligned}
$$

for every $\tau \in [\tau_0, \tau_f]$. The required regularity conditions concerning the fields introduced above will be specified in Sect. 5.

3. Constitutive relations

Every layer (lamina) of the composite body under consideration is supposed to consist of a family of fibres embedded in a ceratin matrix material. We are to assume below that every such layer has been modelled as a certain homogeneous anisotropic material; for the particulars the reader is referred to [1]. We shall confine our study to the two following special cases of the layers.

In the first case we shall assume that the layers are modelled as homogeneous anisotropic linear-elastic materials. Then the constitutive relations have the form of the well known linear mappings

$$
(3.1) \qquad \mathbf{T}(\mathbf{x}, \tau) = \mathbf{C}^K[\mathbf{E}(\mathbf{x}, \tau)], \quad \mathbf{x} \in \Lambda_K, \quad K = 1, \dots, S,
$$

with $\mathbf{C}^K$ as the known tensors of elastic modulae for the material of the K-th layer of the body, $K = 1, \dots, S$.

In the second case every layer is assumed to be modelled as the homogeneous elastic/viscoplastic material. To this end, for every $K = 1, \dots, S$, we have to introduce the closed convex set H_K in the stress space $\mathsf{R}^{3\times3}$(¹), such that for every $\mathbf{T}(\mathbf{x}, \tau) \in H_K$, $x \in \Lambda_K$, the material behaves as elastic with properties defined by the tensor $\mathbf{C}^K$ of elastic modulae. Moreover, for every layer the viscosity coefficient μ_K is known. Setting

$$
\mathbf{A}^K \equiv (\mathbf{C}^K)^{-1},
$$

$$
J_K(\mathbf{T}) \equiv \frac{1}{4\mu_K} \operatorname{tr}[(\mathbf{T} - \pi_K\mathbf{T})(\mathbf{T} - \pi_K\mathbf{T})], \quad \mathbf{T} \in \mathsf{R}^{3\times3},
$$

with $\pi_K\mathbf{T}$ as the orthogonal projection of $\mathbf{T}$ on H_K in the space $\mathsf{R}^{3\times3}$, we assume the following form of the constitutive relation for the layers (cf. [2], p. 234):

$$
(3.2) \qquad \dot{\mathbf{E}}(\mathbf{x}, \tau) = \mathbf{A}^K[\dot{\mathbf{T}}(\mathbf{x}, \tau)] + \frac{\partial J_K(\mathbf{T}(\mathbf{x}, \tau))}{\partial \mathbf{T}(\mathbf{x}, \tau)}, \quad \mathbf{x} \in \Lambda_K, \quad K = 1, \dots, S.
$$

It has to be emphasized that the modelling of sheets the layered composites are made of, as certain homogeneous materials, is a separate problem which will not be discussed here (for details cf. [1]). In the sequel we are to assume that the constitutive relations for every

(¹) For any nonempty set Ξ, the symbol $\Xi^{3\times3}$ stands for a set of all 3×3 symmetric matrices with elements belonging to Ξ.

layer are given a priori either in the form (3.1) or in the form (3.2). For the same reason the mass density $\varrho(\mathbf{x})$, $\mathbf{x} \in \Lambda$, in every layer is assumed to be constant: $\varrho(\mathbf{x}) = \varrho_K$ for $\mathbf{x} \in \Lambda_K$, $K = 1, \ldots, S$.

4. Interlaminae relations

4.1. Preliminaries

By the delamination effects in the layered composites under consideration, we shall mean the possible local discontinuities of the displacement field across the interlaminae surfaces Π_K, $K = 1, \ldots, S-1$, leading to debonding of the layers. Thus we shall analyse the delamination effects from the local point of view, i.e. for an arbitrary but fixed $\mathbf{z} \in \Pi$. Moreover, the points of the interlaminae surfaces can be treated as material points only before local debonding of the pertinent layers. After debonding we have to take into account the possible unilateral contact between the layers. This unilateral local contact implies the existence of the interlaminae contact tractions $\mathbf{r}(\mathbf{z}, \tau)$ across the interfaces of the layers. Setting

$$\mathbf{r}(\mathbf{z}, \tau) = \mathbf{r}_T(\mathbf{z}, \tau) + \mathbf{N}(\mathbf{z}) r_N(\mathbf{z}, \tau)$$

we assume that $r_N(\mathbf{z}, \tau)$ is responsible for the impenetrability of the layers and $\mathbf{r}_T(\mathbf{z}, \tau)$ is due to the friction between the layers. On the other hand, before the possible delamination there exist the interlaminae bonding tractions

$$\mathbf{s}(\mathbf{z}, \tau) = \mathbf{s}_T(\mathbf{z}, \tau) + \mathbf{N}(\mathbf{z}) s_N(\mathbf{z}, \tau)$$

due to the bonding material between the layers. We have assumed before that the bonding material is represented (before the possible delamination) by the surfaces Π_K, $K = 1, \ldots, S-1$. We shall also take into account the state of a partial delamination in which we deal both with the interlaminate contact tractions and the interlaminae bonding tractions. Hence, in general, the condition

$$\mathbf{t}(\mathbf{z}, \tau) = \mathbf{r}_T(\mathbf{z}, \tau) + \mathbf{s}_T(\mathbf{z}, \tau) + \mathbf{N}(\mathbf{z})\,[r_N(\mathbf{z}, \tau) + s_N(\mathbf{z}, \tau)] \tag{4.1}$$

has to hold for every $z \in \Pi$ and $\tau \in [\tau_0, \tau_f]$. In the subsequent parts of this section we are to interrelate the right-hand sides of Eq. (4.1) with the displacement jump field

$$[\![\mathbf{u}]\!]\,(\mathbf{z}, \tau) = [\![\mathbf{u}]\!]_T(\mathbf{z}, \tau) + \mathbf{N}(\mathbf{z})\,[\![\mathbf{u}]\!]_N(\mathbf{z}, \tau), \qquad \mathbf{z} \in \Pi, \tag{4.2}$$

and its time derivative. This interrelation will be based on the concept of the internal constraints which have been detailed in [4] and will be summarized below. We shall deal with the constraints given in the general form of the inclusion $w \in \Delta$, where Δ is the known nonempty closed convex set in the linear topological space W. Let W^* be the dual of W, i.e. the space of all linear continuous functionals defined on W(²). Define $\bar{\mathsf{R}} \equiv \mathsf{R} \cup \{\infty\} \cup \{-\infty\}$ and introduce

$$\langle \cdot, \cdot \rangle : W \times W^* \to \bar{\mathsf{R}}$$

(²) For the concepts of the convex analysis which will be needed in the sequel the reader is referred to the Chapters 1 and 2 of [3]; here we confine ourselves only to the general explanations of these concepts.

as the bilinear form which determines the dual pairing between W and W^*; hence $\langle w, {}^*w\rangle$ is the value of the functional $w^* \in W^*$ on the element $w \in W$. In the sequel we shall tacitly assume that every $\langle w, w^*\rangle$ represents a local or global value of the rate of work in the problem under consideration. This requirement imposes certain restrictions on the choice of the spaces W and W^*. By the internal reaction which can maintain the constraints $w \in \Delta$, we shall mean any element $w^* \in W^*$ such that

$$w^* \in \partial \operatorname{ind}_\Delta(w), \tag{4.3}$$

where $\operatorname{ind}_\Delta : W \to \overline{\mathrm{R}}$ is the indicator function of Δ defined by

$$\operatorname{ind}_\Delta(w) \equiv \begin{cases} 0 & \text{if} \quad w \in \Delta, \\ \infty & \text{if} \quad w \in W \setminus \Delta, \end{cases}$$

and where we have used the known denotation $\partial\varphi(w)$ for the subdifferential of a convex lower-semicontinuous function $\varphi : W \to \mathrm{R}$. The relation $w^* \in \partial\varphi(w)$ is equivalent to the variational inequality

$$\langle \overline{w} - w, w^*\rangle \leqslant \varphi(\overline{w}) - \varphi(w) \ \forall \overline{w} \in W,$$

provided that $\varphi(w) < \infty$. Hence the condition (4.3) is equivalent to the variational inequality

$$\langle \overline{w}, w^*\rangle \leqslant \langle w, w^*\rangle \ \forall \overline{w} \in \Delta$$

which has to hold together with the constraint inclusion $w \in \Delta$. This means that the rate of work of the internal reactions attains its maximum under the constraints $w \in \Delta$(3). Hence any internal reaction w^* which can maintain the constraints $w \in \Delta$ has to satisfy the condition

$$w^* \in \partial \operatorname{ind}_\Delta(w). \tag{4.4}$$

For a more detailed discussion of the concept of constraints, the reader in referred to [4] and to the references therein.

4.2. Impenetrability condition

The impenetrability of the adjacent layers is given by the condition

$$[\![\mathbf{u}]\!]_N(\mathbf{z}, \tau) \geqslant 0$$

which has to hold for every $\mathbf{z} \in \Pi$, $\tau \in [\tau_0, \tau_f]$. Under the denotation $\overline{\mathrm{R}}_+ \equiv \mathrm{R} \cup \{0\} \cup \{\infty\}$, we shall rewrite this condition in the form

$$[\![\mathbf{u}]\!]_N(\mathbf{z}, \tau) \in \overline{\mathrm{R}}_+, \quad \mathbf{z} \in \Pi, \quad \tau \in [\tau_0, \tau].$$

Setting

$$\mathrm{K}(w) \equiv \begin{cases} v \in \mathrm{R};\ v = \lambda(\overline{w} - w),\ \overline{w} \in \overline{\mathrm{R}}_+,\ \lambda \in \mathrm{R}_+ & \text{if} \quad w \in \overline{\mathrm{R}}_+, \\ \phi & \text{if} \quad w \in \overline{\mathrm{R}} \setminus \overline{\mathrm{R}}_+, \end{cases} \tag{4.5}$$

(3) If w^* is the internal reaction, then $-w^*$ is called the external reaction to the constraints. Thus the rate of work due to the external reactions attains its minimum under constraints $w \in \Delta$. The concept of external reactions will not be used throughout the paper.

we also obtain

$$(4.6)\qquad [\![\dot{\mathbf{u}}]\!]_N(\mathbf{z}, \tau) \in \mathsf{K}([\![\mathbf{u}]\!]_N(\mathbf{z}, \tau)), \quad \mathbf{z} \in \Pi, \quad \tau \in [\tau_0, \tau_f].$$

Equations (4.5) defines the set $\mathsf{K}(w)$ as a cone tangent to $\overline{\mathsf{R}}_+$ at $w \in \overline{\mathsf{R}}$. Now it can be seen that the inequality (4.6) represents constraints imposed on $[\![\dot{\mathbf{u}}]\!]_N(\mathbf{z}, \tau)$ for any fixed $[\mathbf{u}]_N(\mathbf{z}, \tau) \in \overline{\mathsf{R}}_+$. Using Eq. (4.3), for the internal reactions $r_N(\mathbf{z}, \tau)$ maintaining the constraints (4.6) we obtain the condition

$$(4.7)\qquad r_N(\mathbf{z}, \tau) \in \partial\,\mathrm{ind}_{\mathsf{K}([\![\mathbf{u}]\!]_N(\mathbf{z}, \tau))}([\![\dot{\mathbf{u}}]\!]_N(\mathbf{z}, \tau)),$$

which has to hold for $\mathbf{z} \in \Pi$, $\tau \in [\tau_0, \tau_f]$. It follows that

$$r_N(\mathbf{z}, \tau)[w - [\![\dot{\mathbf{u}}]\!]_N(\mathbf{z}, \tau)] \leqslant 0 \quad \forall w \in \mathsf{K}([\![\mathbf{u}]\!]_N(\mathbf{z}, \tau)),$$

provided that $[\![\dot{\mathbf{u}}]\!]_N(\mathbf{z}, \tau) \in \mathsf{K}([\![\mathbf{u}]\!]_N(\mathbf{z}, \tau))$. It can also be proved, [4], that the condition (4.7) is equivalent to the conditions

$$r_N(\mathbf{z}, \tau) \in \partial\,\mathrm{ind}_{\overline{\mathsf{R}}_+}([\![\mathbf{u}]\!]_N(\mathbf{z}, \tau)), \qquad [\![\dot{\mathbf{u}}]\!]_N(\mathbf{z}, \tau)\, r_N(\mathbf{z}, \tau) = 0,$$

which have a clear physical sense.

4.3. Friction condition

Let $\mu > 0$ stand for the coefficient of friction between any two adjacent layers of the composite; for the sake of simplicity μ is assumed to be independent of $\mathbf{z} \in \Pi$. Setting

$$(4.8)\qquad \mathsf{F}(\mathbf{z}; r_N) \equiv \{\mathbf{w} \in \mathsf{R}^3;\ ||\mathbf{w}|| \leqslant \mu |r_N|,\ \mathbf{w} \cdot \mathbf{N}(\mathbf{z}) = 0\},$$

we assume that the friction forces $\mathbf{r}_T(\mathbf{z}, \tau)$ are restricted by the known condition

$$(4.9)\qquad \mathbf{r}_T(\mathbf{z}, \tau) \in \mathsf{F}(\mathbf{z}; r_N(\mathbf{z}, \tau)), \quad \mathbf{z} \in \Pi, \quad \tau \in [\tau_0, \tau_f].$$

Interpreting Eq. (4.9) as the constraint inclusion (for any fixed $\mathbf{z} \in \Pi$, $r_N(\mathbf{z}, \tau) \in \overline{\mathsf{R}}_+$) and bearing in mind that $[\![\dot{\mathbf{u}}]\!]_T(\mathbf{z}, \tau) \cdot \mathbf{r}_T(\mathbf{z}, \tau)$ is the rate of work of the friction forces, from Eq. (4.5) we get

$$(4.10)\qquad [\![\dot{\mathbf{u}}]\!]_T(\mathbf{z}, \tau) \in \partial\,\mathrm{ind}_{\mathsf{F}(\mathbf{z}; r_N(\mathbf{z}, \tau))}(\mathbf{r}_T(\mathbf{z}, \tau)), \quad \mathbf{z} \in \Pi, \quad \tau \in [\tau_0, \tau_f].$$

Thus the internal reactions maintaining the constraints (4.9) are represented by the rates of the displacement jumps $[\![\dot{\mathbf{u}}]\!]_T(\mathbf{z}, \tau)$ across the laminae interfaces. It can be shown that Eq. (4.10) leads to the conditions describing Coulomb's friction law.

4.4. Debonding relations

We shall assume that the interlaminae bonding material (the initial configuration of which is represented by the surfaces Π), before the possible local debonding of layers, has linear elastic properties. At the same time we shall postulate that the bonding material under consideration can sustain only restricted values of the interlaminae bonding tractions $s(\mathbf{z}, \tau)$. Moreover, after debonding of layers, only unilateral contact between the adjacent sheets is assumed to occur.

In order to describe these facts, we shall introduce the function

$$(4.11)\qquad \pi(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \tau)) \equiv \frac{1}{2}\gamma_N |[\![\mathbf{u}]\!]_N(z, \tau)|^2 + \frac{1}{2}\gamma_T ||[\mathbf{u}]_T(z, \tau)||^2 - \alpha$$

with $||a|| \equiv \sqrt{a \cdot a}$ for every $a \in \mathsf{R}^3$ and where γ_N, γ_T, α are the positive constants. The functions $\pi(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \tau))$, $\mathbf{z} \in \Pi$, will be interpreted as the strain energy functions (related to the surface Π) of the bonding material before the possible debonding of layers; hence γ_N, γ_T stand for the longitudinal and shear modulus, respectively. After debonding of adjacent layers, the strain energy of the bonding material is assumed to be equal to zero; thus the positive constant α may be interpreted as the maximum value of the internal energy of the bonding material, measured from the state in which $[\![\mathbf{u}]\!](\mathbf{z}, \tau) = 0$. Thus, after introducing the functionals

$$\theta(\mathbf{z}, \tau) = \begin{cases} 1 & \text{if} \quad \pi(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \sigma)) < 0 \quad \text{for every} \quad \sigma \in [\tau_0, \tau), \\ 0 & \text{if} \quad \pi(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \sigma)) = 0 \quad \text{for some} \quad \sigma \in [\tau_0, \tau), \end{cases} \tag{4.12}$$

we shall postulate the debonding condition in the form

$$\pi(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \tau)) = 0 \quad \text{and} \quad \theta(\mathbf{z}, \tau) = 1, \tag{4.13}$$

which takes into account the fact that the debonding process is irreversible in time. Using the relations (4.11) and (4.12), we see that if the debonding condition (4.13) does not hold, then the interlaminae boundary tractions $\mathbf{s}(\mathbf{z}, \tau)$ are determined by

$$\mathbf{s}(\mathbf{z}, \tau) = \theta(\mathbf{z}, \tau) \frac{\partial \pi(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \tau))}{\partial [\![\mathbf{u}]\!](\mathbf{z}, \tau)}, \quad \mathbf{z} \in \Pi. \tag{4.14}$$

If the debonding condition (4.13) holds, then we shall assume that

$$\mathbf{s}(\mathbf{z}, \tau) = \lambda \frac{\partial \pi(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \tau))}{\partial [\![\mathbf{u}]\!](\mathbf{z}, \tau)}, \quad \mathbf{z} \in \Pi, \tag{4.15}$$

for some $\lambda \in [0, 1]$. This means that during (local) debonding, the interlaminae bonding tractions $\mathbf{s}(\mathbf{z}, \tau)$ are not uniquely determined but can change their values from those before the debonding (for $\lambda = 1$) to those after debonding (for $\lambda = 0$).

Equations (4.14) and (4.15) can be rewritten in the simple form (cf. Appendix at the end of the paper and Figs. 2 and 3):

$$\mathbf{s}(\mathbf{z}, \tau) \in S(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \tau), \theta(\mathbf{z}, \tau)), \tag{4.16}$$

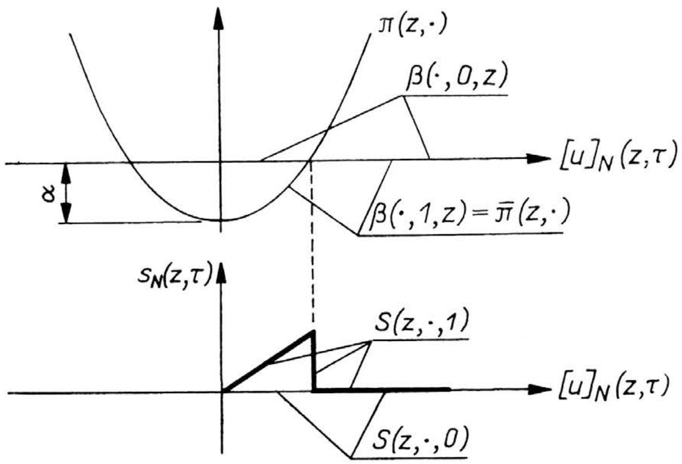

FIG. 2. Delamination in the normal direction.

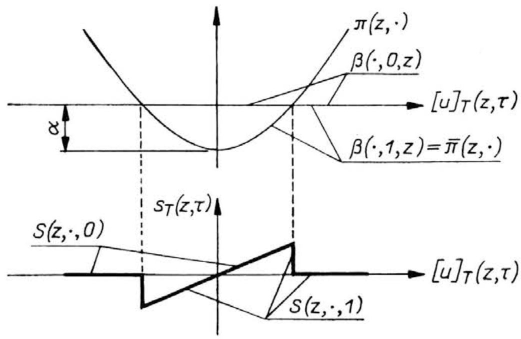

FIG. 3. Delamination in the tangential direction.

where $S(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \tau), \theta(\mathbf{z}, \tau))$ are the closed convex sets in R^3 defined by

$$
(4.17)\qquad S(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \tau), \theta(\mathbf{z}, \tau))
= \begin{cases} \left\{\theta(\mathbf{z}, \tau) \dfrac{\partial \pi(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \tau))}{\partial [\![\mathbf{u}]\!](\mathbf{z}, \tau)}\right\} & \text{if} \quad \pi(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \tau)) < 0 \quad \text{or} \quad \theta(\mathbf{z}, \tau) = 0, \\ \left\{\lambda \dfrac{\partial \pi(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \tau))}{\partial [\![\mathbf{u}]\!](\mathbf{z}, \tau)} : \lambda \in [0, 1]\right\} & \text{if} \quad \pi(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \tau)) = 0 \quad \text{and} \quad \theta(\mathbf{z}, \tau) = 1, \end{cases}
$$

and where $\theta(\mathbf{z}, \tau)$ is defined by Eq. (4.12). Hence it has to be emphasized that the sets $S(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \tau), \theta(\mathbf{z}, \tau))$ depend on the history of the displacement jump $[\![\mathbf{u}]\!](\mathbf{z}, \sigma)$, $\sigma \in [\tau_0, \tau)$.

Following the line of approach outlined in Sects. 4.2 and 4.3, we shall interpret the inclusion (4.16) as the constraints imposed on the interlaminae bonding tractions $\mathbf{s}(\mathbf{z}, \tau)$. Bearing in mind that $\mathbf{s}(\mathbf{z}, \tau) \cdot [\![\dot{\mathbf{u}}]\!](\mathbf{z}, \tau)$ is the rate of work of these tractions, we obtain the following form of the constituent law of the local debonding process:

$$
(4.18)\qquad [\![\dot{\mathbf{u}}]\!](\mathbf{z}, \tau) \in \partial \operatorname{ind}_{S(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \tau), \theta(\mathbf{z}, \tau))}(\mathbf{s}(\mathbf{z}, \tau)), \qquad \mathbf{z} \in \Pi,
$$

with $\theta(\mathbf{z}, \tau)$ determined by the condition (4.12). The physical sense of the obtained constituent law can be easily seen if we rewrite the condition (4.18) in the form of the variational inequality (cf. Sect. 4.1). Obviously, the models of delamination effects different from the one these introduced above can also be proposed.

4.5. Interlaminae traction relations

As it is known, [3], if $\varphi: W \to \bar{\mathsf{R}}$ is a proper, convex and lover semicontinuous function, then

$$
w^* \in \partial\varphi(w) \quad \text{if and only if} \quad w \in \partial\varphi^*(w^*),
$$

where $\varphi^*: W^* \to \mathsf{R}$ is the polar function defined by

$$
\varphi^*(w^*) = \sup_{w \in W} \{\langle w, w^* \rangle - \varphi(w)\}.
$$

Applying the forementioned proposition to Eqs. (4.10), (4.18) and taking into account Eqs. (4.7), we arrive at the relations

$$
(4.19)\qquad \begin{aligned} r_N(\mathbf{z}, \tau) &\in \partial \operatorname{ind}_{\mathsf{K}([\![\mathbf{u}]\!]_N(\mathbf{z}, \tau))}([\![\dot{\mathbf{u}}]\!]_N(\mathbf{z}, \tau)), \\ \mathbf{r}_T(\mathbf{z}, \tau) &\in \partial \operatorname{ind}^*_{\mathsf{F}(\mathbf{z}; r_N(\mathbf{z}, \tau))}([\![\dot{\mathbf{u}}]\!]_T(\mathbf{z}, \tau)), \\ \mathbf{s}(\mathbf{z}, \tau) &\in \partial \operatorname{ind}^*_{S(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \tau), \theta(\mathbf{z}, \tau))}([\![\dot{\mathbf{u}}]\!](\mathbf{z}, \tau)), \end{aligned}
$$

for $\mathbf{z} \in \Pi$, $\tau \in [\tau_0, \tau_f]$. Equations (4.19) will be referred to as the interlaminae traction relations. The families of sets $\mathrm{K}(\cdot)$, $\mathrm{F}(\cdot)$ are determined by Eqs. (4.5) and (4.8) respectively, and $S(\cdot)$ are defined by means of Eqs. (4.16). The interlaminae traction relations (4.19) have to be considered together with Eq. (4.1) for the interlaminae stress $\mathbf{t}(\mathbf{z}, \tau)$, $\mathbf{z} \in \Pi$, $\tau \in [\tau_0, \tau_f]$. Let us observe that

$$t_N(\mathbf{z}, \tau) = \begin{cases} r_N(\mathbf{z}, \tau) & \text{if} \quad t_N(\mathbf{z}, \tau) \leqslant 0, \\ s_N(\mathbf{z}, \tau) & \text{if} \quad t_N(\mathbf{z}, \tau) > 0, \end{cases} \tag{4.20}$$

for every $\mathbf{z} \in \Pi$, $\tau \in [\tau_0, \tau_f]$.

5. Law of motion with delamination effects

Summing up the results of Sects. 2–4, we conclude that the basic system of the governing relations of the debonding process for the composites under consideration is given by:

i. The equation of motion (2.2).

ii. The constitutive relations which can be assumed either in the form (3.1) or in the form (3.2), with $\mathbf{E}(\mathbf{x}, \tau)$ given by Eq. (1.1).

iii. The interlaminae traction relations (4.19) with the formula (4.1) for the interlaminae stress $\mathbf{t}(\mathbf{z}, \tau)$.

The forementioned relations have to be analysed together with the pertinent initial conditions as well as with the regularity conditions for the fields involved. From the forementioned governing relations, we can obtain relations where the basic unknowns are: $[\tau_0, \tau_f] \ni \tau \to \mathbf{u}(\cdot, \tau)$, $[\tau_0, \tau_f] \ni \tau \to \mathbf{T}(\cdot, \tau)$ and $[\tau_0, \tau_f] \ni \tau \to r_N(\cdot, \tau)$, where the displacement fields $\mathbf{u}(\cdot, \tau)$ and the stress tensor fields $\mathbf{T}(\cdot, \tau)$ are defined on Λ, while the reaction fields $r_N(\cdot, \tau)$ are defined on Π for every $\tau \in [\tau_0, \tau_f]$. For the elastic materials we can assume as the basic unknowns $[\tau_0, \tau_f] \ni \tau \to \mathbf{u}(\cdot, \tau)$ and $[\tau_0, \tau_f] \ni \tau \to r_N(\cdot, \tau)$. At the same time the displacement fields $\mathbf{u}(\cdot, \tau)$, $\tau \in [\tau_0, \tau_f]$, have to satisfy the boundary condition (2.1).

In this section we are to show that the equations of motion (2.2) with the interlaminae tractions determined by Eqs. (4.1) and (4.19) lead to a certain time-dependent quasi-variational inequality for the displacement field, which can be referred to as the law of motion with delamination effects (i.e. with possible discontinuities in displacements). We shall use the known denotations for the Sobolev spaces; for particulars the reader is referred to [2], pp. 37–46. The results we are going to obtain will hold under the following regularity assumptions:

$$\mathbf{u}|_{\Lambda_K}(\cdot, \tau) \in \big(H^1(\Lambda_K)\big)^3, \quad \dot{\mathbf{u}}|_{\Lambda_K}(\cdot, \tau) \in \big(H^1(\Lambda_K)\big)^3, \quad \ddot{\mathbf{u}}|_{\Lambda_K}(\cdot, \tau) \in \big(L^2(\Lambda_K)\big)^3,$$
$$\mathbf{T}|_{\Lambda_K}(\cdot, \tau) \in \big(L^2(\Lambda_K)\big)^{3\times 3}, \quad \operatorname{div}\mathbf{T}|_{\Lambda_K}(\cdot, \tau) \in \big(L^2(\Lambda_K)\big)^3,$$
$$\mathbf{b}(\cdot, \tau) \in \big(L^2(\Omega)\big)^3, \quad \mathbf{p}(\cdot, \tau) \in \big(L^2(\Gamma)\big)^3,$$

for $K = 1, \dots, S$ and $\tau \in [\tau_0, \tau_f]$. As the space of the test functions in Eq. (2.2) we introduce the space V given by

$$V \equiv \{\mathbf{v} \in \big(L^2(\Omega)\big)^3;\ \mathbf{v}|_{\Lambda_K} \in \big(H^1(\Lambda_K)\big)^3,\ K = 1, \dots, S,\ \mathbf{v}(\mathbf{x}) = 0 \quad \text{for} \quad \mathbf{x} \in \Gamma_0\}.$$

We shall also assume that $\mathrm{mes}(\Gamma_0 \cap \Lambda_K) > 0$ for $K = 1, \ldots, S$. Denoting by $(\cdot, \cdot)$ and $(\cdot, \cdot)_K$ the scalar products in $(L^2(\Lambda))^{3\times 3}$ and $(L^2(\Lambda_K))^3$, respectively, we obtain

$$(\nabla \mathbf{v}, \mathbf{T}(\tau)) \equiv \int_{\Lambda} \mathrm{tr}[\mathbf{T}(\mathbf{x}, \tau)\nabla \mathbf{v}(\mathbf{x})]dV,$$

$$(\mathbf{v}, \ddot{\mathbf{u}}(\tau))_K \equiv \int_{\Lambda_K} \ddot{\mathbf{u}}(\mathbf{x}, \tau) \cdot \mathbf{v}(\mathbf{x}) dV, \quad K = 1, \ldots, S,$$

with the extra denotations $\mathbf{T}(\tau) \equiv \mathbf{T}(\cdot, \tau)$, $\mathbf{u}(\tau) \equiv \mathbf{u}(\cdot, \tau)$. We also define the functionals $f(\tau)$, $t^*(\tau) \in V^*$ by means of

$$\langle \mathbf{v}, f(\tau)\rangle \equiv \int_{\Omega} \mathbf{b}(\mathbf{x}, \tau) \cdot \mathbf{v}(\mathbf{x}) dV + \int_{\Gamma} \mathbf{p}(\mathbf{x}, \tau) \cdot \mathbf{v}(\mathbf{x}) dA, \quad \mathbf{v} \in V,$$

$$\langle \mathbf{v}, t^*(\tau)\rangle \equiv \int_{\Pi} \mathbf{t}(\mathbf{z}, \tau) \cdot [\![\mathbf{v}]\!](\mathbf{z}) dA, \quad \mathbf{v} \in V,$$

for every $\tau \in [\tau_0, \tau_f]$(4). Under the foregoing denotations the variational condition (2.2) will take the form

$$(5.1) \quad (\nabla \mathbf{v}, \mathbf{T}(\tau)) + \langle \mathbf{v}, t^*(\tau)\rangle = \langle \mathbf{v}, f(\tau)\rangle - \sum_{K=1}^{S} \varrho_K (\mathbf{v}, \ddot{\mathbf{u}}(\tau))_K, \quad \forall \mathbf{v} \in V, \quad \tau \in [\tau_0, \tau_f].$$

The decomposition (4.1) yields

$$(5.2) \quad \langle \mathbf{v}, t^*(\tau)\rangle = \langle \mathbf{v}, r_N^*(\tau) + s^*(\tau) + r_T^*(\tau)\rangle, \quad \mathbf{v} \in V,$$

where we have denoted

$$\langle \mathbf{v}, r_N^*(\tau)\rangle \equiv \int_{\Pi} r_N(\mathbf{z}, \tau) [\![\mathbf{v}]\!]_N(\mathbf{z}) dA, \quad \langle \mathbf{v}, s^*(\tau)\rangle \equiv \int_{\Pi} \mathbf{s}(\mathbf{z}, \tau) \cdot [\![\mathbf{v}]\!](\mathbf{z}) dA,$$

$$\langle \mathbf{v}, r_T^*(\tau)\rangle \equiv \int_{\Pi} \mathbf{r}_T(\mathbf{z}, \tau) \cdot [\![\mathbf{v}]\!]_T(\mathbf{z}) dA, \quad \mathbf{v} \in V.$$

Setting

$$V_0 \equiv \{\mathbf{v} \in V; [\![\mathbf{v}]\!]_T(\mathbf{z}) = 0 \quad \text{for} \quad \mathbf{z} \in \Pi\}$$

we obtain from Eqs. (4.20) and (5.1), $(2.3)_1$ the following relations

$$r_N(\mathbf{z}, \tau) = \begin{cases} t_N(\mathbf{z}, \tau) & \text{if} \quad t_N(\mathbf{z}, \tau) < 0, \\ 0 & \text{if} \quad t_N(\mathbf{z}, \tau) \geq 0, \end{cases}$$

$$(5.3) \quad \int_{\Pi} t_N(\mathbf{z}, \tau) [\![\mathbf{v}]\!]_N(\mathbf{z}) dA = -\sum_{K=1}^{S} (\mathbf{v}, \operatorname{div} \mathbf{T}(\tau))_K - (\nabla \mathbf{v}, \mathbf{T}(\tau)) \quad \forall \mathbf{v} \in V_0,$$

which determine $r_N(\cdot, \tau)$ in terms of $T(\tau)$ for every $\tau \in [\tau_0, \tau_f]$.

In order to simplify the notation, define

$$(5.4) \quad \begin{aligned} \varphi([\![\dot{\mathbf{u}}]\!](\mathbf{z}, \tau); [\![\mathbf{u}]\!](\mathbf{z}, \tau), \theta(\mathbf{z}, \tau), \mathbf{z}) &\equiv \mathrm{ind}^*_{\mathbf{S}(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \tau), \theta(\mathbf{z}, \tau))}([\![\dot{\mathbf{u}}]\!](\mathbf{z}, \tau)), \\ \psi([\![\dot{\mathbf{u}}]\!](\mathbf{z}, \tau); r_N(\mathbf{z}, \tau), \mathbf{z}) &\equiv \mathrm{ind}^*_{\mathbf{F}(\mathbf{z}, r_N(\mathbf{z}, \tau))}([\![\dot{\mathbf{u}}]\!]_T(\mathbf{z}, \tau)). \end{aligned}$$

(4) It can be seen that $t^*(\tau)$ is an element of V^* because the jump operator $[\![\cdot]\!]$ represents the linear continuous mapping from V onto $(H^{1/2}(\Pi))^3$.

Then the interlaminae traction relations (4.19) become

$$(5.5)\qquad \begin{aligned} r_N(\mathbf{z},\tau) &\in \partial\,\mathrm{ind}_{\mathrm{K}([\![\mathbf{u}]\!]_N(\mathbf{z},\tau))}\big([\![\dot{\mathbf{u}}]\!]_N(\mathbf{z},\tau)\big),\\ r_T(\mathbf{z},\tau) &\in \partial\psi\big([\![\dot{\mathbf{u}}]\!]_T(\mathbf{z},\tau);\, r_N(\mathbf{z},\tau),\mathbf{z}\big),\\ \mathbf{s}(\mathbf{z},\tau) &\in \partial\varphi\big([\![\dot{\mathbf{u}}]\!](\mathbf{z},\tau);\, [\![\mathbf{u}]\!](\mathbf{z},\tau),\theta(\mathbf{z},\tau),\mathbf{z}\big), \end{aligned}$$

where the subdifferentials have to be taken with respect to the first argument of the functions $\psi(\cdot)$ and $\varphi(\cdot)$. Bearing in mind Eq. (2.1), define

$$(5.6)\qquad \mathscr{K}\big([\![\mathbf{u}]\!]_N(\tau)\big) \equiv \{\mathbf{v}\in V\colon [\![\mathbf{v}]\!]_N(\mathbf{z}) \in \mathrm{K}\big([\![\mathbf{u}]\!]_N(\mathbf{z},\tau)\big) \quad \text{for} \quad \mathbf{z}\in\Pi\}.$$

Hence the global form of the kinematical constraints (4.6) will be given by (here and in the sequel we define $f(\tau)\equiv f(\cdot,\tau)$)

$$\dot{\mathbf{u}}(\tau)\in\mathscr{K}\big([\![\mathbf{u}]\!]_N(\tau)\big), \qquad \tau\in[\tau_0,\tau_f].$$

Finally, we shall introduce the functions $\Psi(\cdot, r_N(\tau))$, $\Phi(\cdot\,;[\![\mathbf{u}]\!](\tau),\theta(\tau))$ defined on V, setting

$$(5.7)\qquad \begin{aligned} \Psi\big(\mathbf{v};r_N(\tau)\big) &\equiv \int_\Pi \psi\big([\![\mathbf{v}]\!]_T(\mathbf{z});\, r_N(\mathbf{z},\tau),\mathbf{z}\big)dA,\\ \Phi\big(\mathbf{v};[\![\mathbf{u}]\!](\tau),\theta(\tau)\big) &\equiv \int_\Pi \varphi\big([\![\mathbf{v}]\!](\mathbf{z});\,[\![\mathbf{u}]\!](\mathbf{z},\tau),\theta(\mathbf{z},\tau),\mathbf{z}\big)dA, \qquad \mathbf{v}\in V, \end{aligned}$$

provided that the pertinent integrands are elements of $L^1(\Pi)$, and assuming that

$$\Psi\big(\mathbf{v};r_N(\tau)\big)=\infty, \qquad \Phi\big(\mathbf{v};[\![\mathbf{u}]\!](\tau),\theta(\tau)\big)=\infty$$

if otherwise. It can be shown that (under certain conditions) the functions $\psi(\cdot\,;r_N(\mathbf{z},\tau),\mathbf{z})$ and $\varphi(\cdot\,;[\![\mathbf{u}]\!](\mathbf{z},\tau),\theta(\mathbf{z},\tau),z)$ are proper, convex and lower-semicontinuous. Hence also $\Psi(\cdot\,;r_N(\tau))$ and $\Phi(\cdot\,;[\![\mathbf{u}]\!](\tau),\theta(\tau))$ are proper, convex and lower-semicontinuous functions defined on V, [5], and their subdifferentials can be well defined. Moreover, if Eqs. (5.5) hold for almost every $z\in\Pi$, then the following inclusions are satisfied:

$$(5.8)\qquad \begin{aligned} r_N^*(\tau) &\in\partial\,\mathrm{ind}_{\mathscr{K}([\![\mathbf{u}]\!]_N(\tau))}\big(\dot{\mathbf{u}}(\tau)\big),\\ r_T^*(\tau) &\in \partial\Psi\big(\dot{\mathbf{u}}(\tau);r_N(\tau)\big),\\ s^*(\tau) &\in \partial\Phi\big(\dot{\mathbf{u}}(\tau);[\![\mathbf{u}]\!](\tau),\theta(\tau)\big). \end{aligned}$$

Now combining Eqs. (5.1), (5.2) and (5.8) and denoting

$$\xi\big(\mathbf{v};\mathbf{u}(\tau),\theta(\tau),r_N(\tau)\big)\equiv \mathrm{ind}_{\mathscr{K}([\![\mathbf{u}]\!]_N(\tau))}(\mathbf{v})+\Psi\big(\mathbf{v};r_N(\tau)\big)+\Phi\big(\mathbf{v};[\![\mathbf{u}]\!](\tau),\theta(\tau)\big),$$

we arrive at the inequality

$$(5.9)\qquad \begin{aligned} &\big(\nabla\mathbf{v}-\nabla\dot{\mathbf{u}}(\tau),\mathbf{T}(\tau)\big)+\xi\big(\mathbf{v};\mathbf{u}(\tau),\theta(\tau),r_N(\tau)\big)-\xi\big(\dot{\mathbf{u}}(\tau);\mathbf{u}(\tau),\theta(\tau),r_N(\tau)\big)\\ &\qquad\geqslant \langle\mathbf{v}-\dot{\mathbf{u}}(\tau),f(\tau)\rangle+\sum_{K=1}^{S}\varrho_R\big(\mathbf{v}-\dot{\mathbf{u}}(\tau),\ddot{\mathbf{u}}(\tau)\big)_K, \end{aligned}$$

which has to hold for every $\mathbf{v}\in V$ and $\tau\in[\tau_0,\tau_f]$. This inequality, with $r_N(\tau)$ determined by the relations (5.3), will be referred to as the variational law of motion for the layered composites subject to the possible debonding process (law of motion describing the delamination effects); it has to be considered together with the constitutive relations (3.1) or (3.2), the strain-displacement relation (1.1), the boundary conditions (2.1) and the pertinent initial conditions. For the elastic/viscoplastic constituent law (3.2), the basic unknowns are: the motion $[\tau_0, \tau_f] \in \tau \to \mathbf{u}(\cdot, \tau)$ and the evolution of stresses $[\tau_0, \tau_f] \in \tau \to \mathbf{T}(\cdot, \tau)$. For the elastic constituent law (3.1) the stress fields $\mathbf{T}(\tau)$ can be eliminated from the relations (5.9) and (5.3) by means of Eqs. (3.1) and (1.1) and hence we arrive at the form of the law of motion with $[\tau_0, \tau_f] \in \tau \to \mathbf{u}(\cdot, \tau)$ as the basic unknown.

If the friction between the laminae can be neglected, then $\mathbf{r}_T(\mathbf{z}, \tau) = 0$ for every $\mathbf{z} \in \Pi$, $\tau \in [\tau_0, \tau]$, and under the denotation

$$\eta(\mathbf{v}; \mathbf{u}(\tau), \theta(\tau)) \equiv \mathrm{ind}_{\mathscr{K}([\![\mathbf{u}]\!]_N(\tau))}(\mathbf{v}) + \Phi(\mathbf{v}; [\![\mathbf{u}]\!](\tau), \theta(\tau)),$$

the variational law of motion reduces to the form

$$(5.10)\qquad (\nabla\mathbf{v} - \nabla\dot{\mathbf{u}}(\tau), \mathbf{T}(\tau)) + \eta(\mathbf{v}; \mathbf{u}(\tau), \theta(\tau)) - \eta(\dot{\mathbf{u}}(\tau); \mathbf{u}(\tau), \theta(\tau)) \geqslant \langle \mathbf{v} - \dot{\mathbf{u}}(\tau), f(\tau)\rangle - \sum_{K=1}^{S} \varrho_K (\mathbf{v} - \dot{\mathbf{u}}(\tau), \ddot{\mathbf{u}}(\tau))_K$$

which has to hold for every $\mathbf{v} \in V$ and $\tau \in [\tau_0, \tau_f]$. The obtained inequality (5.10) has to be considered with the constitutive relations (3.1) or (3.2), the strain-displacement relation (1.1), the boundary conditions (2.1) and with the pertinent initial conditions. Since the inequality (5.10) is independent of reactions $r_N(\tau)$, then Eqs. (5.3) do not enter into the description of problems.

6. Comments

From the physical and engineering point of view, the evolutional model of delamination proposed in Sect. 4 and described by the formula (4.18) with the denotations (4.17) and (4.11) seems to be the simplest matter (if we put aside the "static" model introduced in [7]) since it involves only one delamination parameter α. However, from the mathematical point of view, this model leads to the time-dependent inequalities (5.9) or (5.10) of the quasi-variational type, which are non-local in time. The existence and the properties of possible solutions to the pertinent boundary problems is an open question. Thus the obtained general relations describing the delamination processes have to be treated rather as the basis for various formulations of approximative models of special problems, then as the final description of the debonding processes in laminates. For the quasi-stationary problems we can apply a procedure similar to that proposed in [7] and pass to the incremental form of the governing relations. On the other hand, the known multilayered composites are made, as a rule, of a repeating sequence of a certain basic unit of layers, i.e. they have a periodic material structure. The modelling of such composites leads to so-called effective or homogenized models. The relations obtained in the paper may constitute the basis for the passage to the effective models of the debonding processes in laminates. To this aid the nonstandard homogenization approach, [6], can be applied; this method will be examined in the forthcoming paper [8].

Appendix

It can be easily observed that the proposed model of debonding effects is based on the condition (4.16) where the notations (4.17), (4.11) and (4.12) have been used. Hence the multifunction $S(\mathbf{z}, \cdot, \theta(\mathbf{z}, \tau)) \subset \mathrm{R}^3$ plays the crucial role in modelling. Taking into account the approach given in [7], we shall now explain the mathematical meaning of this multifunction. To this end, firstly we introduce the functions $\bar{\pi}(\mathbf{z}, \cdot): R^3 \to R$ by means of

$$\bar{\pi}(\mathbf{z}, \mathbf{w}) \equiv \begin{cases} \pi(\mathbf{z}, \mathbf{w}) & \text{if} \quad \pi(\mathbf{z}, \mathbf{w}) \leqslant 0, \\ 0 & \text{if} \quad \pi(\mathbf{z}, \mathbf{w}) > 0, \quad \mathbf{w} \in R^3, \end{cases}$$

where $\pi(\mathbf{z}, \mathbf{w})$ is given by Eq. (4.11), and secondly we define

$$\beta(\mathbf{z}; \theta, \mathbf{w}) \equiv \bar{\pi}(\mathbf{z}, \mathbf{w})\theta, \qquad \theta = \theta(\mathbf{z}, \tau), \qquad \mathbf{w} \in R^3,$$

where $\theta(\mathbf{z}, \tau)$ satisfies the functionals (4.12). Here $\beta(\cdot; \theta, \mathbf{z})$ are non-smooth and non-convex functions (for $\theta = 1$) but it can be shown that they are regular in the sense of CLARK, [7]. Debonding may take place only for the values of $[\![\mathbf{u}]\!](\mathbf{z}, \tau)$ where $\beta([\![\mathbf{u}]\!](\mathbf{z}, \tau), \theta, \mathbf{z})$ is non-smooth. In any other case $\beta(\cdot, \theta, \mathbf{z})$ is smooth and the values of $\mathbf{s}(\mathbf{z}, \tau)$ can be obtained as the derivatives of $\beta(\cdot)$ with respect to $[\![\mathbf{u}]\!](\mathbf{z}, \tau)$. However, using the notion of the generalized Clarke's gradient, [5], we can obtain the general interrelation between $\mathbf{s}(\mathbf{z}, \tau)$ and $[\mathbf{u}](\mathbf{z}, \tau)$ in the form

$$s(\mathbf{z}, \tau) \in \bar{\partial}\beta([\![\mathbf{u}]\!](\mathbf{z}, \tau); \theta(\mathbf{z}, \tau), \mathbf{z}) \tag{A.1}$$

given in [7] where the generalized Clarke's gradient is taken with respect to the first argument of $\beta(\cdot)$. At the same time it can be observed that the formula (A.1) coincides with the assumption (4.16). This means that

$$S(\mathbf{z}, [\![\mathbf{u}]\!](\mathbf{z}, \tau), \theta(\mathbf{z}, \tau)) = \bar{\partial}\beta([\![\mathbf{u}]\!](\mathbf{z}, \tau); \theta(\mathbf{z}, \tau), \mathbf{z}) \tag{A.2}$$

holds. Equation (A.2) yields the explanation of the mathematical structure of the multifunctions $S(\mathbf{z}, \cdot, \theta(\mathbf{z}, \tau))$, which were introduced in the paper. The scheme of interrelations between the interlaminae forces and the displacement jumps is shown in Fig. 2 (for $[\![\mathbf{u}]\!]_T(\mathbf{z}, \tau) = 0$) and in Fig. 3 (for $[\![\mathbf{u}]\!]_N(\mathbf{z}, \tau) = 0$) where the diagrams of the pertinent functions $\pi(z, \cdot)$ and $\beta(\cdot, \cdot, z)$ are also presented.

References

1. R. JONES, *Mechanics of composite materials*, McGraw Hill, New York 1975.
2. G. DUVAUT and J. L. LIONS, *Les inéquations en mécanique et en physique*, Dunod, Paris 1972.
3. I. EKELAND and R. TEMAM, *Convex analysis and variational problems*, North Holland–American Elsvier, Amsterdam–Oxford–New York 1976.
4. CZ. WOŹNIAK, *Materials with generalized constraints*, Arch. Mech., **36**, 539–551, 1984.
5. P. D. PANAGIOTOPOULOS, *Inequality problems in mechanics and applications*, Birkhäuser, Boston 1985.
6. CZ. WOŹNIAK, *A nonstandard method of modelling of thermoelastic periodic composites*, Int. J. of Engng. Sci., **25**, 483–498, 1987.
7. Z. NANIEWICZ and CZ. WOŹNIAK, *On the quasi-stationary models of debonding processes in layered composites*, Ing. Arch., **58**, 403–412, 1988.
8. CZ. WOŹNIAK, *Nonstandard modelling of certain qusi-stationary debonding processes in laminates*, Trends in Appl. of Mathematics to Mechanics, Springer-Verlag, Berlin-Heidelberg, 1988, 186–192.

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