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Nonlocal theories or gradient-type theories: a matter of convenience?(*)

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THE PURPOSE of this contribution is twofold: i) to point put obvious fields of application of nonlocal theories which need to be developed, and ii) to place in evidence, with the help of examples borrowed from different fields of physics (normal conduction in metals, superconductivity, radiative fluid dynamics, theory of dielectrics, surface phenomena such as surface tension and capillarity), the relationships that may exist between continuum approaches relying either on the consideration of higher order gradients of relevant variables (gradient-type theories) or on the use of genuine nonlocal theories. In most cases, however, it is conjectured that the use of oversimplified kernels in nonlocal theories leads many practitioners to prefer an approach using the concept of a "gradient theory" which yields "nice" (although of an increased order) differential equations instead of integrodifferential equations with seemingly and physically equivalent solutions at the output.

Cel pracy jest dwojaki: i) wskazać oczywiste dziedziny zastosowania teorii nielokalnych, które powinny być rozwijane, ii) wykazać za pomocą przykładów zapożyczonych z różnych dziedzin fizyki (zwykłe przewodnictwo w metalach, nadprzewodnictwo, dynamika cieczy z radiacyjną wymianą ciepła, teoria dielektryków, napięcie powierzchniowe i zjawisko kapilarne) związki, jakie mogą istnieć pomiędzy podejściami kontynualnymi, bądź wprowadzającymi do rozważań gradienty wyższego rzędu odpowiednich zmiennych (teoria gradientowa), bądź opartymi na teoriach istotnie nielokalnych. Jednakże można przypuszczać, że w większości przypadków, w konsekwencji użycia zbyt uproszczonych jąder w teoriach nielokalnych, praktycy preferują podejście wykorzystujące koncepcję „teorii gradientowej", które daje „ładne" (choć zwiększonego rzędu) równania różniczkowe, w miejsce równań różniczkowo-całkowych dających równoważne rozwiązania.

Работа имеет двойственную цель: (1) указать очевидные области приложения нелокальных теорий, которые следует развивать и (2) доказать с помощью примеров заимствованных из различных областей физики (обычная проводимость и сверхпроводимость, динамика излучающих жидкостей, теория диэлектриков, поверхностные эффекты такие как поверхностное натяжение и капиллярность) взаимосвязь, которая возможно существует между континуальными подходами основанными или на рассмотрении высших градиентов соответствующих переменных (градиентные теории) или же на использовании чисто нелокального подхода. В большинстве случаев однако, как следует предполагать, использование чрезмерно упрощенных ядер в нелокальных теориях приводит к тому, что многократно практически отдают предпочтение „градиентным теориям", которые приводят к „приятным" (хотя и повышенного порядка) дифференциальным уравнениям вместо интегро-дифференциальных уравнений, которые очевидно приводят к физически эквивалентным решениям.

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1. Some heuristic principles of formulation of constitutive equations(1)

1.1. Principle of antecedence or causality

If A and B are two time-dependent properties of a material and if A determines B causally (i.e. in a deterministic or stochastic manner), then B at time t is a functional of A on the time interval $(-\infty, t]$. For example for a linear functional

$$B(t) = \int_{-\infty}^{t} dt' K(t, t') A(t'). \tag{1.1}$$

Causality restricts the time interval of integration to $(-\infty, t]$. The uniformity of time (i.e. the invariance under time shifts) has K depend on the couple (t, t') only through the variable $\xi \equiv t-t' \geqslant 0$. The principle is also valid at the limit $A = B$. It applies practically to the whole of classical physics except in some controversial versions of electrodynamics. It accounts for *hereditary effects*. The kernel K measures the influence of past states of the independent variable A on the present value of the dependent variable B. The axiom of *fading memory* [2] is one possible formulation of the fact that only recent past states of A influence much the present value of B.

1.2. Principle of contiguity or local action

If A and B are two spatially dependent properties of a material and if A determines B, then B at $\mathbf{r}$ is a functional of A on the space region that surrounds $\mathbf{r}$, including $\mathbf{r}$ itself. Let D be an open of R^3 containing both $\mathbf{r}$ and $\mathbf{r}'$; then a particular formulation of this principle (linear functional) reads

$$B(\mathbf{r}) = \int_{D} d^3r' K(\mathbf{r}, \mathbf{r}') A(\mathbf{r}'), \tag{1.2}$$

where the value of the kernel K measures the infuence of the local contribution of A. The macroscopic homogeneity of the material accounts for the fact that K depends on the couple $(\mathbf{r}, \mathbf{r}')$ only through $\mathbf{r}-\mathbf{r}'$. Rotational invariance further imposes that this dependence reduces to that on $|\mathbf{r}-\mathbf{r}'|$ only. The application of this principle is, for instance, classical continuum mechanics where D is reduced to a neighbourhood of $\mathbf{r}$ (*local action*) and, more generally, the classical theory of fields. The axiom of *smooth neighbourhood* [3] leads then to the notion of *gradient-type-theories* for which Eq. (1.2) can be replaced by the relationship

$$B(r) = \mathscr{B}(A(\mathbf{r}), \nabla A(\mathbf{r}), \nabla\nabla A(\mathbf{r}), \ldots), \tag{1.3}$$

the gradient order being n if n-th order gradients of A, at the most, are taken into account in the usual function $\mathscr{B}$. The axiom fo *attenuating neighbourhood* [3] yields the notion of *nonlocal theories* [4] in which relationships of the general type (1.2) are kept but the kerne K assumes such a form as to privilege the influence of points $\mathbf{r}'$ not far from $\mathbf{r}$. As in the

(1) Because of the lack of proof for the foundation of such principles, M. BUNGE [1] qualifies them as "zero-logical". They have an ontological nature. Their heuristic value stems from the fact that they place in evidence the influence of what preceeds and what occurs in a neighbourhood, so that they suggest the use of integro-differential equations.

case of the fading memory hypothesis, the axiom of attenuating neighbourhood requires a clear mathematical statement since it obviously concerns functional continuity and differentiability.

1.3. Principle of space-time contiguity (or strict local causality [5])

Only regions of the field that can be interrelated by field perturbations can interact (i.e. regions of space-time which can be jointed by time-like paths). The precise mathematical formulation of this principle depends on the framework chosen. Its obvious field of application is relativistic physics [5]. In classical field theories it can be expressed in integral form as a pure space-time generalization of Principles 1.1 and 1.2, e.g.

$$B(\mathbf{r}, t) = \int_{-\infty}^{t} dt' \int_{D} d^3r K(\mathbf{r}, \mathbf{r}'; t, t') A(\mathbf{r}', t'). \tag{1.4}$$

Since the dependent variable B participates in differential field equations (balance laws) and only simple K's yields manageable equations (1.2), the question arises as to whether constitutive equations of the type (1.2) or (1.3) should be used, the former yielding integro-differential equations while the latter yields "nice" differential equations. Furthermore, simple K's may result in solutions of the field equations that do not differ much from, or even are identical to, the solutions obtained on the basis of a description (1.3). The present contribution, of a rather descriptive nature, aims (i) at pointing out some physical theories (mainly electrodynamics) where equations of the type (1.2) intervene, so that they offer a potential field of study to the tenants of nonlocal theories and (ii) at exhibiting some examples where the mathematical point just raised shows up. We offer no solution of this dilemma, the choice between gradient-like theories and nonlocal theories appearing in these examples as a matter of mathematical convenience and personal taste.

2. A model equation

Retrospectively, we should not force upon the genius of giants of science to find in their most hidden works the germ of all concepts arising now in science. However, the following fact is quite remarkable. Writing to H. A. Lorentz in 1909(2) about the application of the quantum concept to the photoelectric effect, A. Einstein outlined his thoughts of the moment in detail: In analogy with electrons surrounded by electrostatic fields, light quanta could be singular points (not necessarily mathematical singular points) which are surrounded by extended vector fields, diminishing with distance and somehow capable of superposition. The essence of the theory, however, would not be the assumption of singular points, but rather the assumption of *linear homogeneous field equations* whose solution would permit the propagation of small, localized, and directed bundles of energy at velocity c (photons). Einstein thought that such a goal should be obtainable by slightly modifying Maxwell's theory as, for instance, by considering to start with, in the case of

(2) Letter from Einstein to Lorentz dated May 23, 1909; See [6], pp. 48–50.

statics, the following fourth-order differential equation which clearly is a modification of Laplace's equation:

(2.1) $$\nabla^2\phi - \lambda^2\nabla^2\nabla^2\phi = 0.$$

The solution

(2.2) $$\phi = \varepsilon\,\frac{1-\exp(-r/\lambda)}{4\pi r}$$

of Eq. (2.1) is the only solution that goes over to the Coulomb potential $\phi = \varepsilon/4\pi r$ at large distances $r \gg \lambda$ and has no mathematical singularity at $\mathbf{r} = 0$. Einstein then speculated that the dynamical case would be obtained from Eq. (2.1) by replacing ∇^2 by the d'Alembertian operator $\Box$. Although Einstein's proposal (2.1) now appears quite futile as regards the photoelectric effect, we know that Eq. (2.1) can be considered as a model equation which, by taking account of higher order derivatives, (i) enables one to avoid field singularities at peculiar points; (ii) places in evidence the role played by a *characteristic length scale* λ; and (iii), in the dynamical case, yields a *dispersive character* for the medium (non-homogeneous polynomial of differentiation). Indeed, an example of mechanical theory where an equation of the type (2.1) is encountered is the indeterminate couple-stress theory of elasticity (strain-gradient theory or second-order gradient theory according to the formulation 1.3) in which the equation that governs transverse (or shear) elastic waves has the form [7]

(2.3) $$\nabla^2\phi - \lambda^2\nabla^2\nabla^2\phi = c_T^{-2}\,\frac{\partial^2\phi}{\partial t^2},$$

where the field ϕ is the vorticity vector $\boldsymbol{\omega} \equiv \frac{1}{2}\nabla\times\mathbf{u}$, $c_T \equiv \sqrt{\mu/\varrho}$ is the usual transverse-wave velocity and $\lambda \equiv \sqrt{\eta/\mu}$ is a characteristic length where η is a new material modulus that accounts for couple stresses. The dispersion effect arising from the presence of λ is of interest if λ is of the order of the wave-length of the wave.

However, the presence of the parameter λ in Eq. (2.1) and the type of solution (2.2) are also germane to nonlocal theories. Thus it is possible, for instance in electrostatics, that solutions of the type (2.2) arise not from a modification of Maxwell's equations as suggested by Einstein, but from a different assumption concerning the constitutive equation that is carried in the unmodified Maxwell field equations. We shall return to this special case in Sect. 4.

3. Early examples of nonlocal theories in physics

3.1. Normal electrical conduction in metals

First we note that in perfectly conducting metals where normal conductivity is due to conduction electrons the following characteristic length, λ_L, can be constructed:

(3.1) $$\lambda_L \equiv \left(\frac{mc^2}{4\pi ne^2}\right)^{1/2},$$

where m, e and n are the mass, charge and number density of electrons, respectively. Typically, $\lambda_L \simeq 10^{-6}$ cm. Another characteristic small length is the electronic mean free path, l. In isotropic normal conductors Ohm's constitutive equation can be written as

$$\mathbf{J}(\mathbf{r}) = \sigma(l)\mathbf{E}(\mathbf{r}). \tag{3.2}$$

This is valid only if $\mathbf{E}(\mathbf{r})$ varies slowly over a distance of the order of l. Otherwise, i.e. at low temperatures and high frequencies, Eq. (3.2) becomes inadequate. The current at the point $\mathbf{r}$ must then be the integrated effect of the field over distances of the order of l [8] (nonlocal theory with attenuating neighbourhood!). If σ_{dc} is the d.c conductivity, then REUTER and SONDHEIMER [9] have shown that Eq. (3.2) had to be replaced by ($\mathbf{R} \equiv \mathbf{r}-\mathbf{r}'$)

$$\mathbf{J}(\mathbf{r}) = \frac{3\sigma_{dc}}{4\pi l}\int_{R^3} \mathbf{R}^{-4}[\mathbf{R}(\mathbf{R}.\mathbf{E})\exp(-|\mathbf{R}|/l)]d^3r'. \tag{3.3}$$

More generally ([10], pp. 10–12), the current density $\mathbf{J}$ at $\mathbf{r}$ and time t is determined by the time-varying electric field $\mathbf{e}(\mathbf{r})$, not only by its value at the same point $(\mathbf{r}, \mathbf{t})$, but also at all other points and times. We have thus a representation of the type (1.4):

$$J_i(\mathbf{r}, t) = \int_{-\infty}^{t} dt' \int_{R^3} d^3r' K_{ij}(|\mathbf{r}-\mathbf{r}'|, t-t')e_j(\mathbf{r}', t'). \tag{3.4}$$

Using a monochromatic plane-wave representation of the type

$$A(\mathbf{r}, t) = (2\pi)^{-4}\int_{R^4} A(\mathbf{k}, \omega)\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]d^3k d\omega, \tag{3.5}$$

yields then an algebraic relationship between the Fourier components $\mathbf{J}(\mathbf{k}, \omega)$ and $\mathbf{e}(\mathbf{k}, \omega)$:

$$J_i(\mathbf{k}, \omega) = \sigma_{ij}(\mathbf{k}, \omega, \mathbf{H})e_j(\mathbf{k}, \omega), \tag{3.6}$$

where $\mathbf{H}$ is the static magnetic field and $\sigma_{ij}(\mathbf{k}, \omega, \mathbf{H})$ are the Fourier components of the conductivity tensor:

$$\sigma_{ij}(\mathbf{k}, \omega, \mathbf{H}) = \int_{R^3} d^3r \int_0^{\infty} dt K_{ij}(\mathbf{r}, t)\exp[i(\omega t-\mathbf{k}\cdot\mathbf{r})]. \tag{3.7}$$

The $\mathbf{k}$-dependence of σ_{ij} is called the *spatial dispersion* of the conductivity and is due to spatial inhomogeneities of the wave field, whereas the ω-dependence is called the *tempora dispersion* and is due to effects of retardation. An expression for $\sigma_{ij}(\mathbf{k}, \omega, \mathbf{H})$ can be found in electronic kinetic-theory arguments (Cf. [10], pp. 18–20): the spatial dispersion effect will be important if wavelengths are of the order of the electronic mean free path l.

3.2. Superconductivity

In London's phenomenological theory of superconductivity (see, e.g. [11]), the magnetic field $\mathbf{H}$ is governed in statics by an equation of the type

$$\mathbf{H} - \lambda_L^2\nabla^2\mathbf{H} = \mathbf{0}, \tag{3.8}$$

while the current $\mathbf{J}$ and the magnetic vector potential $\mathbf{A}$ are related by the equation

$$\mathbf{J}(\mathbf{r}) = -\frac{c}{4\pi\lambda_L}\mathbf{A}(\mathbf{r}), \tag{3.9}$$

where λ_L has been defined in Eq. (3.1). Note that Eq. (3.8) yields the model equation (2.1) if we apply to it the operator ∇^2. Equation (3.8) tells us that upon penetrating into a superconducting specimen, an externally applied field $\mathbf{H}$ decays exponentially (Meissner effect), i.e. $\mathbf{H}(\mathbf{r}) \approx \mathbf{0}$ for $|\mathbf{r}| \gg \lambda_L$. λ_L depends on temperature and has a behaviour of the type $\lambda_L \alpha (1-t)^{-1/2}$, $(t \equiv \theta/\theta_C)$, in the neighbourhood of the phase-transition critical temperature θ_C. The superconducting state is an ordered phase and a typical length known as the *range of coherence* ,ξ, can be introduced which corresponds to lengths over which the order parameter changes gradually (it is the typical size of Cooper's pairs in the microscopic BCS theory). In low mean-free-path alloys and in certain pure metals $\xi \ll 10^{-4}$ cm. In general, ξ is a function of the electronic mean free path l: $\xi = \xi(l)$. Equation (3.9) can be re-written as $\mathbf{J} = -(ne^2/m)\mathbf{A}(\mathbf{r})$ in virtue of Eq. (3.1). The influence of ξ can be introduced by modifying the latter equation so that it reads

$$\mathbf{J}(\mathbf{r}) = -\frac{ne^2}{m}\frac{\xi(l)}{\xi_0}\mathbf{A}(\mathbf{r}), \tag{3.10}$$

where ξ_0 is a constant of the superconductor. On this basis PIPPARD [12] proposed, in view of the analogy of Eqs. (3.10) and (3.2), a basic relation for the electromagnetic response of a superconductor of the form [compare Eq. (3.3)]

$$\mathbf{J}(\mathbf{r}) = -\frac{3ne^2}{4\pi\xi_0 m}\int\limits_{R^3} R^{-4}[\mathbf{R}\cdot(\mathbf{R}\cdot\mathbf{A})\exp(-|\mathbf{R}|/\xi)]d^3r'; \tag{3.11}$$

we have thus a *nonlocal theory* of superconductivity with a hypothesis of attenuating neighbourhood, the characteristic length involved being the range of coherence ξ. The validity of Eq. (3.11) is strongly supported (i) by the fact that it yields a reversal of the phase of the magnetic field penetrating into a superconductor, this effect was observed in 1962, and (ii) by the accepted microscopic theory of Bardeen, Cooper and Schrieffer (the so-called BCS theory), which yields a relationship entirely equivalent to Eq. (3.11) if one makes the appropriate identifications.

3.3. Radiative fluid dynamics

By its very nature radiative fluid dynamics also is a domain where nonlocal expressions appeared quite early. Studying the influence of heat propagation on convective instability and convective transport in stars, SPIEGEL [13] proposed in 1957 to add to the heat equation a term of the following form:

$$Q(\mathbf{r}) = \frac{1}{\tau}\int\limits_{R^3} K(|\mathbf{r}-\mathbf{r}'|)\theta(\mathbf{r}', t)d^3r', \tag{3.12}$$

where θ is the small perturbation in temperature, τ is the characteristic decay time of optically thin perturbations, and the kernel K has the expression

$$K(|\mathbf{r}|) = \left[\frac{\exp(-|\mathbf{r}|/\lambda)}{4\pi\lambda r^2}\right] - \delta(|\mathbf{r}|), \tag{3.13}$$

where λ is the mean free path of photons and δ is Dirac's generalized function. As λ goes to zero (all scales optically thick), Eq. (3.12) yields

$$Q = \varkappa \nabla^2 \theta, \quad \varkappa \equiv \lambda^2/3\tau, \tag{3.14}$$

while in the opposite limit ($\lambda \to \infty$), it yields

$$Q = -\theta/\tau. \tag{3.15}$$

The latter is none other than Newton's law of cooling while Eq. (3.14) merely describes heat diffusion for an optically thick gas and provides an expression for thermal diffusivity when it results from radiation. The intermediate case is more difficult to deal with but it represents a *nonlocal theory* with an hypothesis of attenuating neighbourhood, the characteristic length involved being the photon mean free path λ.

4. Gradient theory and nonlocal theory of dielectrics

Maxwell's form of Gauss' law reads (in Lorentz-Heaviside units):

$$\nabla \cdot \mathbf{D} = q, \tag{4.1}$$

where $\mathbf{D} = \mathbf{E}+\mathbf{P}$ is the electric displacement, $\mathbf{E}$ is the electric field, and P is the volume electrical polarization. q is the volume density of free charges. For electrostatics we have in supplement the equation $\nabla \times \mathbf{E} = \mathbf{0}$ from which there follows the existence of an electrostatic potential ϕ, such that $\mathbf{E} = -\nabla\phi$. In the usual theory of rigid isotropic dielectrics one has the linear relationship $\mathbf{P} = \chi\mathbf{E}$, hence $\mathbf{D} = \varepsilon\mathbf{E}$ with $\varepsilon \equiv 1+\chi > 0$. If q now is a point charge located at the origin, then Eq. (4.1) yields

$$\nabla^2\phi = -\left(\frac{q}{\varepsilon}\right)\delta(|\mathbf{r}|), \tag{4.2}$$

of which the solution $\phi(\mathbf{r}) = q/4\pi\varepsilon r$ is unbounded at $\mathbf{r} = 0$. Non-singular solutions at $\mathbf{r} = \mathbf{0}$, which practically do not differ from the Coulomb solution for sufficiently large r, can be obtained by considering different and more sophisticated models of rigid dielectrics, namely, a nonlocal theory of dielectrics with attenuating neighbourhood [14], a micromorphic theory of dielectrics (dielectrics with quadrupoles) [15]–[17], and a first-order gradient theory of dielectrics (dielectrics with polarization gradients) [18], [19], all these without modifying the original Gauss equation (4.1), but by making specific constitutive assumptions as regards $\mathbf{P}$.

4.1. Nonlocal theory of rigid isotropic dielectrics [14]

We then have a constitutive equation of the type

$$\mathbf{P}(\mathbf{r}) = \chi\mathbf{E}(\mathbf{r}) + \int_{R^3} K(|\mathbf{r}-\mathbf{r}'|)\mathbf{E}(\mathbf{r}')d^3r'. \tag{4.3}$$

Substituting from Eq. (4.3) in Eq. (4.1) for a point-like source at $\mathbf{r} = \mathbf{0}$, we obtain an integro-differential equation ($\varepsilon \equiv 1+\chi$):

$$\varepsilon\nabla^2\phi + \int_{R^3} \frac{\partial}{\partial r_i} K(|\mathbf{r}-\mathbf{r}'|)\frac{\partial\phi}{\partial r'_i}\, d^3r' = -q\delta(|\mathbf{r}|). \tag{4.4}$$

A plausible form for K that attributes a strong influence to points $\mathbf{r}'$ in the neighbourhood of $\mathbf{r}$ (attenuating neighbourhood) is

$$K(|\mathbf{r}-\mathbf{r}'|) = -\beta^2\nabla^2\delta(|\mathbf{r}'-\mathbf{r}|), \tag{4.5}$$

where ∇^2 is taken with respect to $\mathbf{r}$. Setting $\lambda = \sqrt{\beta/\varepsilon}$, the solution of Eq. (4.4) reads [compare (2.2)]

$$\phi(r) = \frac{q}{4\pi\varepsilon r}[1-\exp(-|\mathbf{r}|/\lambda)]. \tag{4.6}$$

Then

$$\lim_{|\mathbf{r}|\to 0}\phi(r) = q/4\pi\varepsilon\lambda. \tag{4.7}$$

Another example of a possible kernel K is the following one:

$$K(|\mathbf{r}-\mathbf{r}'|) = \frac{k}{4\pi|\mathbf{r}'-\mathbf{r}|}\exp\left(-\frac{|\mathbf{r}'-\mathbf{r}|}{\lambda}\right), \tag{4.8}$$

where λ is a characteristic length. Set $\lambda_L \equiv \sqrt{\varepsilon/(k+\lambda^{-2})}$. Then the corresponding solution of Eq. (4.4) is

$$\phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon\lambda^2}(1-\lambda^2\nabla^2)\{r^{-1}[1-\exp(-|\mathbf{r}|/\lambda_L)]\}. \tag{4.9}$$

Here we have two characteristic lengths, λ and λ_L, to describe the solution in the neighbourhood of the origin.

4.2. Rigid dielectrics with quadrupoles [16]

In that case we have separate constitutive equations for he electric dipole moment p_i and the electric quadrupole tensor Q_{ij}, in such a way that

$$p_i = \chi E_i, \quad Q_{ij} = K_3 E_{k,k}\delta_{ij} + K_4 E_{(i,j)}, \tag{4.10}$$

while the polarization per unit volume is given by

$$P_i = p_i - Q_{ij,j}. \tag{4.11}$$

It follows from Eqs. (4.10) and (4.11) that $\mathbf{D}$ is related to $\mathbf{E}$ by the equation

$$\mathbf{D} = \varepsilon\mathbf{E} - \varepsilon_1\nabla(\nabla\cdot\mathbf{E}) - \varepsilon_2[\nabla^2\mathbf{E} + \nabla(\nabla\cdot\mathbf{E})], \tag{4.12}$$

and Eq. (4.1) for a point-like source at $\mathbf{r} = \mathbf{0}$ reads

$$\nabla^2\phi - \lambda^2\nabla^2\nabla^2\phi = -\left(\frac{q}{\varepsilon}\right)\delta(r), \tag{4.13}$$

where

$$\varepsilon \equiv 1+\chi > 0, \quad \varepsilon_1 \equiv K_3, \quad \varepsilon_2 \equiv K_4/2 > 0, \quad \varepsilon_1 + 2\varepsilon_2 > 0, \tag{4.14}$$

and

$$\lambda^2 \equiv (\varepsilon_1 + 2\varepsilon_2)/\varepsilon > 0. \tag{4.15}$$

The inequalities follow from the positive definiteness of the potential energy. The solution of Eq. (4.13) is none other than Eq. (4.6). Another problem can be solved to place in evidence the skin effect inherent in the present theory. It is that of a spherical cavity with uniformly distributed surface charge, embedded in an infinite linear elastic dielectric. It is then found [17] that a part of the electrostatic potential dies out exponentially at far distances from the surface of the cavity, an effect quite similar to that due to the Debye potential in plasma physics.(3)

4.3. Isotropic dielectrics with polarization gradients [18]

In that case Eq. (4.1) yields

$$\nabla^2\phi - \nabla\cdot\mathbf{P} = -q\delta(\mathbf{r}). \tag{4.16}$$

But $\mathbf{P}$ must also satisfy a new balance law (known as the intromolecular force balance law [20]) which, on taking account of polarization gradients, reads for isotropy and electrostatics:

$$b\nabla^2\mathbf{P} - a\mathbf{P} - \nabla\phi = \mathbf{0}, \tag{4.17}$$

where b and a are material constants. Upon combining Eqs. (4.16) and (4.17), we arrive at an equation of the type (compare Eq. (3.8))

$$\psi - \lambda_L^2\nabla^2\psi = \frac{q}{(1+a)}\,\delta(\mathbf{r}), \tag{4.18}$$

where $\psi \equiv \nabla\cdot\mathbf{P}$ and the characteristic length λ_L is defined by $\lambda_L^2 = b/(1+a)$. The solution of Eq. (4.18) again is of the type (4.6). We thus see that three conceptually different models of rigid isotropic dielectrics yield the same solution (4.6). The relationship either with the model equation (2.1) or the model solution (2.2) is clear.

5. The example of surface tension

The examples examined in the foregoing sections dealt mainly with the problem of smoothing out singularities. However, the fact that characteristic lengths intervene makes it clear that both gradient-like theories and nonlocal theories are of interest in problems concerned with boundary layer phenomena, thin films, effects of dislocations in elastic solids, microcracks, the propagation of disturbances of short wavelengths, the penetration of surface waves and, obviously, surface phenomena of which surface tension and capillarity are the mechanical tenets. In solids surface tension results from electronic bondig, while Van der Waals' forces are responsible for the phenomenon in liquids. In the case of liquids a typical approach [21]–[24] relies upon the use of a density-gradient theory, hence a *first-order gradient theory* in the formalism of Eq. (1.3) if A is none other than the density ϱ. This sort of approach can be traced back to KORTEWEG [25]. Specializing the equations of Ref. [22] to the case of non-ferromagnetic liquids and using a potential energy per unit mass of the Cahn-Hilliard type [23], i.e.

$$\psi = f(\varrho) + \frac{\beta}{2\varrho}(\nabla\varrho)^2, \quad \beta = \text{const}, \tag{5.1}$$

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(3) Compare Eq. (5.106), p. 180 [32].

we obtain the boundary conditions at a surface Σ which has a discontinuous tangent plane (oriented edge Γ) in the form

$$\begin{aligned} [\![t_{ij}]\!]n_j &= -2\Omega\sigma n_i + D_i\sigma \quad \text{on} \quad \Sigma-\Gamma, \\ \mathbf{R}+\sigma\mathbf{n} &= \mathbf{0} \end{aligned} \tag{5.2}$$

while along Γ, ($\mathbf{v}$: binormal to Γ)

$$\mathbf{E}+[\![\sigma\mathbf{v}]\!] = \mathbf{0}. \tag{5.3}$$

Here $\mathbf{n}$ is the oriented unit normal to Σ, Ω is its mean curvature, and D_i indicates the tangential derivative on Σ. t_{ij} is the Cauchy stress tensor, $\mathbf{E}$ is a linear density of force, $\mathbf{R}$ is a so-called double normal force, and σ is the surface tension. t_{ij} and σ have constitutive equations derived from Eq. (5.1) in accordance with ($\partial/\partial n \equiv n\cdot\nabla$)

$$\begin{aligned} t_{ij} &= -(\bar{p}\delta_{ij}+\beta\varrho_{,i}\varrho_{,j}), \\ \bar{p} &= \varrho^2\frac{df}{d\varrho}+\varrho(f-\psi)-\varrho\beta\nabla^2\varrho, \qquad \sigma = \varrho\beta\frac{\partial\varrho}{\partial n}. \end{aligned} \tag{5.4}$$

The first term in the right-hand side of Eq. $(5.2)_1$ is the classical term of Laplace's theory of capillarity. The remaining term does not appear in Laplace's theory although it also pertains to a second-order geometrical description of the surface Σ. At rest, the "capillarity action" $\mathbf{R}$ imposes the value of σ. If $\mathbf{R}$ is constant on Σ, so is σ, hence $D_i\sigma = 0$. The value of the stress also follows, but not in an extremely thin boundary layer since σ being fixed by the data $\mathbf{R}$, the value of $\partial\varrho/\partial n$ follows and density will increase form the surface through this layer beyond which it will remain practically constant, so that t_{ij} will then reduce to a pressure term. But within this layer where density is strongly non-uniform, the stress is no longer spherical. The expression of this stress is corroborated by the kinetic theory of gases. The characteristic length involved in the problem above is given by $\lambda^2 \equiv \varrho_0^2\beta/\bar{p}_0$, where ϱ_0 and $\bar{p}_0$ are typical densities and pressures. The above development can be generalized to the case of ferrofluids [23]. In contrast to the present approach, ERINGEN [26] provides a phenomenological representation of surface effects in liquids on the basis of a *nonlocal theory*. His expression for σ is given by

$$\sigma(\mathbf{r}) = \varrho^{-1}\tau(\mathbf{r},\mathbf{r}')\frac{\partial}{\partial n'}(\nabla'\cdot\mathbf{u})|_{\Sigma}, \quad \mathbf{r}\in\Sigma, \tag{5.5}$$

where ∇' and $\partial/\partial n'$ are computed at $\mathbf{r}'$, $\tau(\mathbf{r},\mathbf{r}')$ is a nonlocal parameter whose dimension is that of $\varrho^3\beta$, and $\mathbf{u}$ is the displacement field. Similar arguments lead the same author to a phenomenological explanation of surface tension in elastic solids on the basis of a nonlocal theory [29]–[30], whereas MINDLIN [31] prefers a strain-gradient theory and BLINOWSKI [27] and GURTIN and MURDOCH [28] favour an approach using the notion of elastic surfaces.

6. Conclusion

The following conclusions can be drawn at once from the series of examples dealt with in the foregoing sections. First we witness the exemplary value of the model equation (2.1) and of the model solution (2.2), the former with respect to an approach dealing with gradient-type theories, and the latter with respect to both gradient-type and nonlocal theories. Closely related to the second point is the recurrence of typically simple kernels in the various nonlocal theories set forth [compare Eqs. (3.3), (3.11) and (4.8)]. It is the drastic simplicity (one could also say the rather obvious form) of these kernels that makes, in the cases described, the use of either a gradient-type theory or a nonlocal theory appear as a matter of personal taste. Thus, in our opinion of a non-specialist, the interest of nonlocal theories should stem from two facts: that it be possible to determine the expression of the kernel from a microscopic theory; this is the case for Eq. (3.7) on the basis of an electronic kinetic theory, and for Eq. (3.11) on the basis of the BCS theory of superconductivity; (ii) that the imagination of scientists be such that kernels be constructed so as to yield solutions readily different from those that can be obtained from a gradient-type theory or a micromorphic theory and therefore the relative value of each phenomenological representation could be determined. Should lack of imagination be the case, one can always try to build such kernels as to agree, for instance, with a dispersion curve obtained from a microscopic theory, and then use these kernels in other applications. This attitude is exemplified by the remarkable trianglewise kernel introduced by ERINGEN [29] in nonlocal elasticity, namely for a nonlocal elasticity modulus of dimension L^{-1}:

$$K_1'(x) = \frac{1}{\lambda}\left(1-\frac{|x|}{\lambda}\right) \quad \text{for} \quad \frac{|x|}{\lambda} < 1, \qquad = 0 \quad \text{for} \quad \frac{|x|}{\lambda} > 1, \tag{6.1}$$

which allows a perfect fitting of the dispersion curve over an entire Brillouin zone (of width 2λ) for elastic waves, and thereafter is conveniently used in solving problems such as that of Griffith's crack (see recent works by A. C. Eringen).

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