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Intrinsic physical limits to the theory of materials with memory

W. KOSIŃSKI and K. C. VALANIS (IOWA CITY)

ON THE BASIS of the response of a material to three "thought" experiments the class of dissipative materials is divided into three subclasses: hydromorphic, viscoelastic and viscoplastic materials. In the first part of the paper it is shown that the material possessing the relaxation property may be, at the most, viscoelastic; it cannot be viscoplastic. In the second part hydromorphic materials are investigated. Primitive and permanent natural states are defined. The mathematical and physical conditions under which a material has an infinite number of natural states are formulated in terms of the proper and almost relaxed states.

Na podstawie reakcji materiału na trzy myślowe eksperymenty klasę materiałów dysypatywnych podzielono na trzy podklasy: materiały hydromorficzne, lepkosprężyste i lepkoplastyczne. W pierwszej części pracy pokazano, że materiał z własnością relaksacji może być co najwyżej lepkosprężysty; nie może być lepkoplastyczny. W drugiej części zbadano materiały hydromorficzne. Zdefiniowano pierwotne i trwałe stany naturalne. Sformułowano warunki matematyczne i fizyczne, przy których materiał posiada nieskończenie dużo stanów naturalnych. Warunki te podano przy użyciu stanów właściwych i stanów prawie zrelaksowanych.

На основе реакции материала на три мыслимых эксперимента класс диссипативных материалов разделен на три подкласса: гидроморфические, вязкоупругие и вязкопластические материалы. В первой части работы показано, что материал со свойством релаксации может быть по крайней мере вязкоупругим; не может быть вязкопластическим. Во второй части исследованы гидроморфические материалы. Определены первичные и устойчивые натуральные состояния. Сформулированы математические и физические условия, при которых материал имеет бесконечно много натуральных состояний. Эти условия приведены при использовании удельных состояний и почти релаксационных состояний.

Preface

THE ESSENTIAL character of "plastic materials" is the significant effect of the past history on their subsequent mechanical response. Classical plasticity was developed in the latter part of the last century in a form that reflects this fact. Various schools of plasticity have emerged since then and their contributions have multiplied diffusely. For example, one such school has stipulated the existence of a convex yield surface, the separation of an increment of strain into elastic and plastic parts and the normality of the latter to the yield surface. It is interesting to note that these ideas, which were intended to be rough approximations to the observed behaviour, are now treated as axioms. Other schools elaborated on the original ideas to account for other engineering observations without, however, deviating substantially from the original concepts. (1)

(1) A significant departure from the above views was set forth by the second author who introduced the concept of history depedence with respect to a "time" scale which is a characteristic material property. This theory, called endochronic, was shown to predict more naturally and simply plastic material response. However, this approach will not be elaborated upon any further in this paper.

In essence, plastic media belong to the class of materials with memory. A smaller class, which is per force a member of the above class, is that of materials with fading memory. It is our aim to show that the model of a material with memory cannot predict "plastic response" in a sense which will be more precisely defined below.

In general, given a certain material one does not know a priori its precise constitutive characteristics. Before one begins to develop a phenomenological description of real materials, the typical and basic properties of the material in question should be known. Here we propose that material properties be examined by means of suitable experiments, as a result of which one can form a table of physical (mechanical) properties of the material at hand.

We do not intend to investigate one particular material and to compile a table of its properties. We are going, rather, to perform on a class of materials a few "thought" experiments which, in our opinion, are simple yet crucial to the characterization of materials with plastic effects.

The experiments are of three types:

A) A "reference" experiment in which the deformation[2] $\mathbf{C}$ has been kept constant and equal to $\mathbf{1}$ for all past time. Here $\mathbf{1}$ represents the unity tensor.

B) An arbitrary deformation history $\mathbf{C}(t)$ of which the terminal value, ${}_{-\infty}\mathbf{C}$, has been kept constant in the time interval $(-\infty, 0)$. Evidently, ${}_{-\infty}\mathbf{C} = \mathbf{C}(0) \equiv {}_0\mathbf{C}$. More precisely this history is an infinite long static continuation of any arbitrary history with a terminal value equal to ${}_0\mathbf{C}$, i.e. if $\mathbf{G}(\tau)$ represents any arbitrary history defined on the interval $(-\infty, -a)$, $a > 0$, such that $\mathbf{G}(-a) = {}_0\mathbf{C}$, then for $a \to \infty$ we defined $\mathbf{C}(\tau) = \mathbf{G}(\tau)$ when $\tau \in (-\infty, -a]$ and $\mathbf{C}(\tau) = \mathbf{G}(-a) \equiv {}_{-\infty}\mathbf{C}$ for $\tau > -a$.

C) An arbitrary deformation history of which the terminal value $\mathbf{1}$ has been kept constant in the time interval $(-\infty, 0)$. More precisely this history is an infinite long static continuation of any arbitrary history with a terminal value equal to $\mathbf{1}$.

REMARK. Experiments C) form a subclass of experiments B) in which ${}_{-\infty}\mathbf{C} = \mathbf{1}$.

For the purposes of further discussion we introduce the following definitions. A "prehistory" is a deformation process which originates at $\tau = -\infty$ and ends at $\tau = 0$. A "subsequent response" is a material response in the interval $[0, t]$, for any $t > 0$. A prehistory is said to have *an effect on the subsequent response* if a material response following experiments B) or C) is different from a response following experiment A).

To illustrate the physical motivation for these experiments, consider a material whose stress response σ (calculated per unit undeformed area) in a uniaxial stress field is given by the simple linear functional equation

$$\sigma(t) = \int_{-\infty}^{t} E(t-\tau)\frac{\partial\lambda}{\partial\tau}\, d\tau, \tag{1}$$

where λ is the extension ratio in the direction of stress calculated with respect to a specific reference configuration $\varkappa$. We further stipulate that the material has "fading" memory in the sense that $E(t) > 0$ is a monotonically decreasing function of time and that $E(\infty) = 0$.

We pose the question: To what extent does test B) *influence the response of this material to a subhistory in the interval* $[0, t]$? By this we mean to what extent is the material response

[2] The tensor $\mathbf{C}$ denotes the right Cauchy-Green strain tensor.

to a certain deformation subhistory in the interval $[0, t]$ different when this subhistory follows test B) instead of test A). More specifically, we inquire as to the effect of test B) on the material response to a constant strain rate history in the interval $[0, t]$. To this end let X be the material coordinate of a particle in the configuration $\varkappa$ and $Y(\tau)$ is the spatial coordinate at time τ; a constant strain rate history is given by Eq. (2),

$$Y(\tau) = Y(0)(1+k\tau), \tag{2}$$

where $Y(0)$ is the spatial coordinate of the particle at the continuation of test B), i.e. at $\tau = 0$. Obviously,

$$\lambda(\tau) = \frac{\partial Y(\tau)}{\partial X} = \frac{\partial Y(0)}{\partial X}(1+k\tau) = \lambda_0(1+k\tau), \tag{3}$$

$$\dot{\lambda} = \lambda_0 k. \tag{4}$$

Evidently, as a result of the stipulations on $E(\tau)$, the stress is zero at the end of test B), i.e. $\sigma(0) = 0$. In fact, if $\lambda_1(\tau)$ represents the uniaxial deformation history $\mathbf{C}(\tau)$ of experiment B), then on the interval $(-a, 0)$ we have

$$\frac{\partial \lambda_1(\tau)}{\partial \tau} = 0$$

and

$$\sigma(0) = \int_{-\infty}^{0} E(-\tau)\frac{\partial \lambda_1(\tau)}{\partial \tau}\, d\tau = \lim_{a\to\infty}\int_{a}^{0} E(-\tau)\frac{\partial \lambda_1(\tau)}{\partial \tau}\, d\tau = 0,$$

whenever $\lim\limits_{\tau\to-\infty}\dfrac{\partial \lambda_1(\tau)}{\partial \tau}$ is finite. (Note that we have used the fact $\lim\limits_{a\to\infty} E(a) = E(\infty) = 0$). Hence the stress response to a constant strain rate history following test B) is given by the equation

$$\sigma(t) = k\lambda_0 \int_0^t E(t-\tau)\,d\tau. \tag{5}$$

Equation (5) shows that σ is a function, in fact linear, of the end value λ_0 of the prehistory associated with test B). To the extent that λ_0 is a "residual deformation" it can be called "plastic" and in so far as σ is a function of λ_0 one can say that the stress response is "affected" by the previous plastic deformation. On the basis of these criteria the material may be called plastic, quite justifiably.

Yet if test C) precedes the constant strain rate history, then $\lambda_0 = 1$ and, as Eq. (5) readily indicates, the prehistory associated with test C) has no effect on the subsequent material response. This behaviour is not characteristic of metals which show strong changes in behaviour following prehistories associated with test C).

We conclude that Eq. (1) *is not suitable for the constitutive representation of the mechanical response of metals.*

Pursuing our discussion in the same vein, we are cognizant of materials known as "simple fluids" whose subsequent response in the sense of our previous discussion is not influenced by prehistories associated with test B). Insofar as these materials are concerned, Eq. (1) is also unsuitable. On the other hand, a constitutive equation of the type

$$\sigma(t) = \int_{-\infty}^{t} E(t-\tau) \frac{\partial \log \lambda(\tau)}{\partial \tau} d\tau \tag{6}$$

satisfies this property of simple fluids as a simple calculation using Eqs. (3) and (4) will readily indicate.

Taking into account the above observations we divide the class of dissipative materials into three classes on the basis of a positive or negative influence of tests B) and C) on their subsequent mechanical response. Materials on which tests B) as well as C) have "no influence" we call *hydromorphic*; materials on which test B) has an influence, but test C) does not, we call *viscoelastic*; materials on which both tests B) and C) have influence, we call *viscoplastic*. We summarize this division in the following table:

	Test B	Test C
Hydromorphic	−	−
Viscoelastic	+	−
Viscoplastic	+	+

REMARK: Plastic materials are simply a subclass of viscoplastic materials and are characterized by the invariance of their response to time scales which are isomorphic to the Newtonian time scale measured by simple clock.

The aim of the first part is to show that all materials described by the model of simple material with memory belong to the class of viscoelastic or hydromorphic materials. Furthermore, we want to derive the most general form of the constitutive equation for simple materials of hydromorphic type.

The results of the present part have a direct application in the proof that viscoplastic materials cannot be described by the model of the material with memory and with the relaxation property[3].

Part I

1. Introduction

IN THIS part we show that a general material with memory and with the relaxation property may, at the most, be a viscoelastic one in the sense of the classification given above. The proof of this proposition will be given in the case when the history space of the material with memory is the general normed function space, namely, the Köthe-Toeplitz space. This space is the most general function space and contains all the familiar Lebesgue spaces L_p.

It will be shown that the relaxation property implies that a subsequent response of the material following test C) is the same as the response following test A). Furthermore,

(3) In the forthcoming paper we give the proof of this fact for materials without the relaxation property.

the more general proposition is true: if we introduce another reference experiment in which the deformation **C** has been kept constant for all past time, but equal to $\mathbf{C} \neq \mathbf{1}$, then subsequent responses following this new reference experiment and test B) will be the same.

2. The history space

Different materials and different physical situations are distinguished by different constitutive assumptions. The constitutive assumptions consists not only of the so-called constitutive equations, i.e. the relations between the response of a material and the input applied, but also the domain of definition of the functionals (or the functions) which appear in the constitutive equations. The domain of definitions is the set of all inputs possible and admissible from the physical viewpoint.

In the theory of simple materials with memory the responses are the stress, the free energy, the heat flux and the entropy; the inputs, however, are the histories of the deformation and the temperature (and additionally the temperature gradient). The histories are the functions defined on the positive half-line $[0, \infty)$ with the values in a subset of some vector space. For example: the value of the history of the deformation gradient at each point s from $[0, \infty)$ must be an invertible tensor and therefore this vector space will be the tensor space. In each case, however, the primitive notion is the process, i.e. a function $p(t)$ over the real line $(-\infty, \infty)$. The argument t of the function p is called the time and the function p^t defined only for non-negative numbers s by the relation $p^t(s) = p(t-s)$, is the history of p up to t. The independent variable s of histories is called the elapsed time. The value $p^t(0)$ by the definition equal to $p(t)$ is the present value of p^t. For the simple material with memory the response at time t, i.e. $r(t)$, is given by the history of p up to time t:

$$r(t) = \mathfrak{r}(p^t). \tag{2.1}$$

Here $\mathfrak{r}$ represents the constitutive (or the so-called response) functional. It should be noticed that the variable $r(t)$ may represent a collection of responses as well as p^t represents a collection of inputs (histories).

Since temperature fields are usually functions with positive values, but the Cauchy-Green strain tensor fields are the positive definite and symmetric tensor functions, it is often the case that the values of p^t are restricted to a cone(4).

In recent years several topologies have been proposed as appropriate for sets of histories p^t(cf. [1–7]). Here we used Coleman and Mizel's general approach to the theory of fading memory [6]. First of all it should be noticed that in the present paper only a mechanical theory is investigated. It follows that we deal with one stress response functional Π through each history Φ of some deformation measure which prescribes a stress tensor (the Cauchy stress or the Piola-Kirchoff stress).

Without specifying which kind of deformation measure is assumed, we suppose only that its values lie in a cone V_9^c of the second order tensors. (In the case of the deformation gradient this cone will be the set of all invertible tensors, but in the case of the right Cauchy-Green strain — the set of all positive definite and symmetric tensors).

(4) A subset $\mathscr{C}$ of a vector space is a cone if $P \in \mathscr{C}$ and $b > 0$ imply that $bP \in \mathscr{C}$.

The most general normed function space in which all possible and known history spaces are contained is the so-called the Köthe-Toeplitz space(5).

In order to define the domain of definition of the constitutive functional we start with a non-trivial, non-negative, sigma-finite, regular Borel measure μ on $[0, \infty)$. The measure μ and a function norm ν defined below will be basic to introducing [8] a norm in the history space.

DEFINITION 1. *Let $\mathcal{T}$ be the set of all μ-measurable functions ϕ mapping* $[0, \infty)$ *into* $[0, \infty)$. *A function ν defined on $\mathcal{T}$ will be called the function norm, relative to μ, with the Fatou property, if for all ϕ (or ϕ_i)*

a) $0 \leqslant \nu(\phi) \leqslant \infty$ *and* $\nu(\phi) = 0$ *if and only if* $\phi(s) = 0$, μ — *a.e.*;(6)

b) $\nu(\phi_1+\phi_2) \leqslant \nu(\phi_1)+\nu(\phi_2)$ *and* $\nu(a\phi) = a\nu(\phi)$ *for all* $a \geqslant 0$;

c) *if* $\phi_1(s) \leqslant \phi_2(s)$ μ — *a.e., then* $\nu(\phi_1) \leqslant \nu(\phi_2)$;

d) *there is at least one* $\gamma \in \mathcal{T}$ *with* $0 < \nu(\gamma) < \infty$;

e) *if* $\phi, \phi_1, \phi_2, \ldots$ *are in* $\mathcal{T}$ *and* $\phi_n(s)\uparrow\phi(s)$ μ — *a.e., then* $\nu(\phi_n)\uparrow\nu(\phi)$.

One can extend the domain of definition of function norm to the whole set $\mathcal{T}^{\pm}$ of μ — measurable real functions (7) on $[0, \infty)$ by setting $\nu(\phi) = \nu(|\phi|)$ for any $\phi \in \mathcal{T}^{\pm}$. Now, if we identify μ — almost equal functions in the usual way(8), then the set $\mathcal{L}_\nu$ of all functions $\phi \in \mathcal{T}^{\pm}$ satisfying $\nu(\phi) < \infty$ will be a vector (linear) space with ν as its norm. Because ν has the Fatou property, the normed linear space will be norm complete (i.e. $\mathcal{L}_\nu$ is a Banach space). Any normed linear space of this kind is sometimes called a normed Köthe (-Toeplitz) space or a *Banach function space*. The space $\mathcal{L}_\nu$ are a generalization of the familiar Lebesgue space $\mathcal{L}_p(1 \leqslant p \leqslant \infty)$ where $\nu_p(\phi) = \left(\int_0^\infty |\phi|^p d\mu\right)^{\frac{1}{p}}$ for $1 \leqslant p < \infty$ and $\nu_\infty(\phi) = \text{ess sup}|\phi|$.

Let us consider the set of all μ — measurable functions mapping $[0, \infty)$ into V_9 — the set of all second order tensors. Let $\|\cdot\|$ be the function defined on this set in the following way:

$$\|\Phi\| = \nu(|\Phi|_9), \tag{2.2}$$

where $|\cdot|_9$ is the norm in the tensor space (i.e. for each tensor P its norm is defined as follows, $|P|_9 = \operatorname{tr}(PP)^T$, with P^T as the transpose of the tensor P). We denote by V the collection of all functions Φ satisfying $\nu|\Phi|_9 < \infty$, i.e.

$$\mathsf{V} = \{\Phi\colon \Phi\colon [0, \infty) \to V_9, \quad \mu \text{ — measurable}, \quad \nu(|\Phi)|_9) < \infty\}. \tag{2.3}$$

DEFINITION 2. *The function space $\mathfrak{B}$ obtained by calling the same functions in V which are μ — almost equal is called a history space. The space is a Banach one.*

Let V_9^c be a cone in the tensor space V_9. Let $\mathscr{C}$ be the set of functions in V with range in the cone V_9^c (i.e. $\Phi \in \mathscr{C}$ if $\Phi \in \mathsf{V}$, $\Phi(s) \in V_9^c$ for all $s \geqslant 0$). The set $\mathfrak{C}$ obtainable by calling the same functions in $\mathscr{C}$ which μ — a.e. are equal is a cone in $\mathfrak{B}$. The domain of definition of the constitutive-functional will be the set $\mathfrak{C}$.

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(5) A. C. ZAANEN in [8] called it the normed Köthe space. Cf. also the series of papers by W. A. J. LUXEMBURG and A. C. ZAANEN published in Proc. Acad. Sci. Amsterdam, 66, 1963.

(6) μ — a.e. means μ — almost everywhere. For example, $\phi(s) \geqslant 0$ μ — a.e. if the set of all s such that $\phi(s) < 0$ has the μ — measure zero.

(7) It may be extended to all μ — measure complex functions.

(8) Two functions ϕ_1 and ϕ_2 are called μ — almost equal if $\phi_1(s) = \phi_2(s)$ μ — a.e.

3. Constitutive assumptions

The history space $\mathfrak{B}$ introduced above with the norm $\|\cdot\|$ is too general in the discussion of the material with memory. On the other hand, physical principles require that the constitutive functional be defined for certain special functions; hence these functions must be in $\mathfrak{B}$. In order to express physical requirements we introduce two families of operators acting in $\mathfrak{B}$.

Given a function Φ on $[0, \infty)$ and a number $\sigma > 0$ one may define two functions $E^\sigma\Phi$ and $S_\sigma\Phi$ by

$$(3.1) \qquad (E^\sigma\Phi)(s) = \begin{cases} \Phi(0) & s \in [0, \sigma] \\ \Phi(s-\sigma) & s \in (\sigma, \infty), \end{cases} \qquad (S_\sigma\Phi) = \Phi(s+\sigma), \quad s \in [0, \infty).$$

The function $E^\sigma\Phi$ is called *the static continuation of* Φ by the amount σ and $S_\sigma\Phi$ is called the σ — *section of* Φ. If, for example, Φ is the history up to t of the deformation gradient $\mathbf{F}$(at a fixed material point in some particular process), i.e. $\Phi = \mathbf{F}^t$, then $E^\sigma\Phi$ is the history of $\mathbf{F}$ up to $t+\sigma$, which is constant from t to $t+\sigma$, with the present value of $\mathbf{F}^t$(i.e. $\mathbf{F}^{t+\sigma}(s) = \mathbf{F}^t(0)$ for $s \in [0, \sigma]$); whereas $S_\sigma\Phi$ is the history of $\mathbf{F}$ up to $t-\sigma$(i.e. $S_\sigma\Phi = \mathbf{F}^{t-\sigma}$).

Coleman and Mizel make the following assumption(9):

A_0) *If Φ is in $\mathbf{V}$, then $E_\sigma\Phi$ and $S_\sigma\Phi$ are in $\mathbf{V}$, for all $\sigma > 0$. Furthermore, if Φ and Ψ are in $\mathbf{V}$ and $\|\Phi-\Psi\| = 0$, then $\|E^\sigma\Phi-E^\sigma\Psi\| = 0$ for all $\sigma > 0$.*

As the consequences of the assumption A_0) one can receive the following results (cf. [3, 6]):

a. *The measure μ must have an atom at $s = 0$ (i.e. $\mu(\{0\}) > 0$) and be absolutely continuous on $(0, \infty)$ with respect to the Lebesgue measure.*

b. *Either $\mu((0, \infty)) = 0$ or the Lebesgue measure is absolutely continuous on $(0, \infty)$ with respect to μ.*

Thus the μ-measure of the singleton $\{0\}$ cannot be zero and arbitrary subset of $(0, \infty)$ has zero μ-measure if it and only if has zero Lebesgue measure (after the assumption that $\mu((0, \infty))$ is not zero). We remember that for given history Φ from $\mathbf{V}$ the value $\Phi(0)$ is called the present value, and the past values $\Phi(s)$ are those for which $0 < s < \infty$. Roughly speaking, the results a. and b. tell us that the norm ($\|\Phi\| = \nu(|\Phi|_9)$) places greater emphasis on the present value of Φ than on any individual past value but does not "ignore" any interval of past time.

This fact suggests to us that we should introduce the space of the past histories $\mathfrak{B}_r$ in the following way. If Φ is a history in $\mathbf{V}$ we denote by $_r\Phi$ the restriction of Φ to the

(9) This assumption expresses the following physical requirements (cf. [3]): Given an arbitrary history p^t and a positive number σ, we should be able to discuss a process $\bar{p}$ which has the history p^t up to time t and is held constant in the interval $[t, t+\sigma]$. Similarly, if we can discuss the history p^t of p up to a time t, then we should be able to discuss histories of p corresponding to earlier times, t-σ, $\sigma \geqslant 0$. These two conditions mean that corresponding static continuations and σ-section should be elements of our history space. The last condition in the assumption A_1) means, furthermore, that the static continuation of an element of $\mathfrak{B}$ should be well defined even if we identify the elements with the set of functions at zero distance from it.

open interval $(0, \infty)$ and call it the past history of Φ. Let V_r be the set $\{{}_r\Phi : \Phi \in \mathsf{V}\}$. The function $||\cdot||_r$ on V_r defined by

$$(3.2) \qquad ||{}_r\Phi||_r = ||\Phi\chi_{(0,\infty)}||$$

with $\chi_{(0,\infty)}$ as the characteristic function of $(0, \infty)$ is clearly a semi-norm. The space of past histories is the function space $\mathfrak{B}_r$ obtained by calling the same past histories ${}_r\Phi, {}_r\Psi$ for which $||{}_r\Phi - {}_r\Psi||_r = 0$. Like $\mathfrak{B}$ and V_9 the space $\mathfrak{B}_r$ is a Banach space.

The following fact is the next consequence of the assumption A_0):

c. *The history space $\mathfrak{B}$ is algebraically and topologically the direct sum of V_9 and $\mathfrak{B}_r$, that is,*

$$(3.3) \qquad \mathfrak{B} = V_9 \oplus \mathfrak{B}_r,$$

and the norm $||\cdot||$ on $\mathfrak{B}$ is equivalent([10]) *to the norm $||\cdot||'$ defined by*

$$(3.4) \qquad ||\Phi||' = |\Phi(0)|_9 + ||{}_r\Phi||_r.$$

The assumption A_0) must be completed by the following two (cf. [6]):

A_1) *for each tensor $P \in V_9$ the constant function $P^\dagger$ is in V,*

A_2) *the space $\mathfrak{B}_r$ is separable.*

Here $P^\dagger$ denotes the function which holds the constant value P, i.e. $P^\dagger(s) = P$ for all $s \geqslant 0$. It may be proven that both assumptions A_2) and A_1) are equivalent to the assumption that tame histories with time-derivatives of compact support are dense in $\mathfrak{B}$ (cf. [4, 9]).

The last two assumptions introduced by Coleman and Mizel have the form of the so-called relaxation property:

A_3) *for each Φ in V*

$$(3.5) \qquad \lim_{\sigma\to\infty} ||E^\sigma\Phi - \Phi(0)^\dagger|| = 0.$$

This assumption([11]) will be fundamental in our further consideration.

A_4) *the stress response functional Π is a continuous function over its domain of definition $\mathfrak{C}$.*

At the end of this section we wish to note that the assumptions A_1) — A_2) are obeyed, for example, by each Banach space $\mathscr{L}_p(h)$ formed from those V_9 — valued functions Ψ on $[0, \infty)$ for which

$$(3.6) \qquad ||\Psi||^p = \mu(\{0\})|\Psi(0)|_9^p + \int_0^\infty |\Psi(s)|_9^p h(s)\,ds$$

exists and is finite, provided $1 \leqslant p < \infty$ and $h(s)$ is a fixed, positive monotone-decreasing function for large s, summable (in the Lebesgue sense) on $(0, \infty)$. For the space $\mathscr{L}_\infty$ we have the norm

$$(3.7) \qquad ||\Psi|| = \mu\{0\}|\Psi(0)|_9 + \operatorname*{ess\,sup}_{s\in(0,\infty)}(|\Psi(s)|_9 h(s)),$$

([10]) The equivalence of $||\cdot||'$ and $||\cdot||$ shows that the present value $\Phi(0)$ of Φ has approximately the same importance for (the norm of Φ) as for its entire past history ${}_r\Phi$.

([11]) It expresses the following property: if $p^{t+\sigma}$ is a static continuation of history p^t and the constitutive function $\mathscr{r}$ is continuous over $\mathfrak{C}$, then $\mathscr{r}(p^{t+\sigma})$ approaches its "equilibrium value" $\mathscr{r}(p^\dagger)$ as $\sigma \to \infty$, i.e. the value of $\mathscr{r}$ at the constant history $p^\dagger(s) \equiv p^t(0)$, $s \geqslant 0$.

where ess sup is taken with respect to Lebesgue measure. The function[12] h fulfills the indentity for each interval $(a, b) \in (0, \infty)$, $\mu((a, b)) = \int_a^b d\mu = \int_a^b h(s)ds$.

Next, we can notice that the separability of the history space $\mathfrak{B}_r$ required in A_2) implies the following results:

d. Every element of $\mathfrak{B}_r$ is of absolutely continuous norm; it is equivalent to the dominated convergence property (cf. [6, 8]), i.e. for each ${}_r\Phi \in \mathfrak{B}_r$ and for each sequence ${}_r\Psi^n$ in $\mathfrak{B}_r$ such that for all n $|{}_r\Psi^n(s)|_9 \leqslant |{}_r\Phi|_9$ μ — a.e. and ${}_r\Psi^n(s) \to {}_r\Phi(s)\mu$ — a.e. we have $||{}_r\Psi^n - {}_r\Phi||_r \to 0$.

e. Bounded functions of compact support are dense in $\mathfrak{B}_r$.

f. Continuous functions of compact support are dense in $\mathfrak{B}_r$.

4. Viscoelastic materials

In the Preface we defined three classes of materials. Viscoelastic materials are characterized by the identical subsequent response following tests (A) and (C). Recall that test (A) is described by a constant history with the value 1, i.e. in our notation by the history $1^\dagger$: $[0, \infty) \to V_9^c$,

$$1^\dagger(s) = 1 \text{ for any } s \in [0, \infty). \tag{4.1}$$

Test (C), however, is described by an infinitely long static continuation of any arbitrary history with a terminal value equal to 1. Let Λ be any arbitrary history from V with its terminal (present) value 1, i.e. $\Lambda:[0, \infty) \to V_9^c$, $\Lambda \in \mathsf{V}$ and

$$\Lambda(0) = 1. \tag{4.2}$$

For any $\tau > 0$ the history $E^\tau\Lambda$ is the static continuation of Λ by the amount τ. Test (C) is described by the history $\Lambda^{(\infty)}$ given by

$$\Lambda^{(\infty)} = \lim_{\tau\to\infty} E^\tau\Lambda.$$

Our aim is to compare the subsequent response of material following histories $1^\dagger$ and $\Lambda^{(\infty)}$, i.e. the value of Π at two histories Ψ_1, Ψ_2 such that

$$\Psi_1(s) = \Psi_2(s), \quad \text{when} \quad s \in [0, \sigma_0)$$

for some recent interval $[0, \sigma_0)$. (This interval can be identified with the interval $[0, t)$, which takes place in the Preface). Furthermore, on the interval $[\sigma_0, \infty)$ the history Ψ_1, has to be identical with $1^\dagger$ but the history Ψ_2 with $\Lambda^{(\infty)}$.

As it was mentioned in the Introduction, we consider most general situations in which the terminal value of the history in test (C) is not necessary Λ.

To end this, let us investigate the responses of the material on the sequence of histories $\{\underset{\tau}{\Phi}\}_{\tau=0}^{\infty}$ from $\mathfrak{C}$ such that for each $\tau \in (0, \infty)$

$$\underset{\tau}{\Phi} = \Phi\chi_{[0,\sigma_0)} + \Phi(\sigma_0)\chi_{[\sigma_0,\sigma_0+\tau)} + \Phi\chi_{[\sigma_0+\tau,\infty)}, \tag{4.3}$$
$$\sigma_0 = \text{const.},$$

[12] The function $h(s)$ which takes place in Eqs. (3.6) and (3.7) is the so-called influence function. It is the Radon-Nikodym derivative $d\mu/d\lambda$ where λ represents Lebesgue measure on $(0, \infty)$. In [3] one can find the conditions under which the norm in $\mathscr{L}_p(h)$ has the relaxation property.

where Φ is an arbitrary history from $\mathfrak{C}$ continuous in σ_0. We can see that for different τ the elements of the sequence $\{\underset{\tau}{\Phi}\}_{\tau=0}^{\infty}$ differ by the durations of their constant part $\Phi(\sigma_0)\chi_{[\sigma_0,\sigma_0+\sigma)}$. Let us calculate the limit of the sequence $\{\underset{\tau}{\Phi}\}_{\tau=0}^{\infty}$. At first,

$$\Phi(\sigma_0)\chi_{[\sigma_0,\sigma_0+\tau)}+\Phi\chi_{[\sigma_0+\tau,\infty)} = \big(E^{\tau}(S_{\sigma_0}\Phi)\big)\chi_{[\sigma_0,\infty)}. \tag{4.4}$$

By the assumption A_3) we have

$$\lim_{\tau\to\infty} E^{\tau}\big(S_{\sigma_0}(\Phi)\big) = (S_{\sigma_0}\Phi)(0)^{\dagger}.$$

Hence

$$\lim_{\tau\to\infty}\big(E^{\tau}(S_{\sigma_0}\Phi)\chi_{[\sigma_0,\infty)}\big) = \Phi(\sigma_0)^{\dagger}\chi_{[\sigma_0,\infty)}$$

because of the identity $(S_{\sigma_0}\Phi)(0) = \Phi(\sigma_0)$. Finally, we obtain

$$\lim_{\tau\to\infty}\underset{\tau}{\Phi} = \Phi\chi_{[0,\sigma_0)}+\Phi(\sigma_0)^{\dagger}\chi_{[\sigma_0,\infty)} \equiv \underset{\infty}{\Phi}. \tag{4.5}$$

The continuity of the functional Π implies

$$\lim_{\tau\to\infty}\Pi(\underset{\tau}{\Phi}) = \Pi(\underset{\infty}{\Phi}).$$

Now, we want to compare the values of Π at $\underset{0}{\Phi}$ and $\underset{\infty}{\Phi}$, i.e. $\Pi(\underset{0}{\Phi}), \Pi(\underset{\infty}{\Phi})$, where $\underset{0}{\Phi}$ is defined as follows:

$$\underset{0}{\Phi} = \Phi\chi_{[0,\sigma_0)}+\Phi(\sigma_0)^{\dagger}\chi_{[\sigma_0,\infty)}. \tag{4.6}$$

We can see that $\underset{0}{\Phi}$ describes an experiment on the time interval $(0, \sigma_0)$ which follows the constant history of deformation.

We can see that only in the case

$$||\underset{0}{\Phi}-\underset{\infty}{\Phi}|| = 0, \tag{4.7}$$

we are sure that

$$\Pi(\underset{0}{\Phi}) = \Pi(\underset{\infty}{\Phi}).$$

But Eq. (4.7) takes place because the history Φ is constant on the interval (σ_0, ∞) and equal to Φ on $[0, \sigma_0)$ (cf. (4.5), (4.6))

$$\underset{0}{\Phi}-\underset{\infty}{\Phi} = 0.$$

Hence we can formulate

Proposition. For general materials with the relaxation property the following relation holds:

$$\lim_{\tau\to\infty}\Pi(\underset{\tau}{\Phi}) = \Pi(\underset{0}{\Phi}), \tag{4.8}$$

where the sequence of the histories $\underset{\tau}{\Phi}$ is defined by Eq. (4.3) and the history $\underset{0}{\Phi}$ by Eq. (4.6).

Hence as a result we have:

Theorem 1. *Any material with memory and with the relaxation property may be, at the most, viscoelastic. It cannot be viscoplastic.*

Part II

1. Introduction

HYDROMORPHIC materials were defined in the Preface as simple materials whose responses following experiments (B) as well as (C) are the same as a response following the reference experiment (A).

In this part of the paper we want to derive the general constitutive equation of materials which satisfies these requirements. In order to be more precise we introduce the following definitions.

Two deformation subhistories on the interval $[0, t]$ are said to be equivalent if their relative deformation gradients with respect to the corresponding configurations at $t = 0$, are equal in the entire interval. By a hydromorphic material we mean a viscoelastic material whose subsequent response to two equivalent subhistories following two different constant histories differs by, at the most, a hydrostatic pressure.

2. Natural states

Let $\Pi_\varkappa$ denote the constitutive functional such that for a given history of the deformation gradient

$$F: [0, \infty] \to V_9^+ \tag{2.1}$$

i.e. $F(s) \in V_9^+$ for $s \geqslant 0$, the value $\Pi_\varkappa(F)$ is the deviatoric part of the Cauchy stress tensor (at a material point X at some time t).

In a more suggestive form we can write

$$\text{stress deviator } T_D = \Pi_\varkappa\big(F_{s=0}^{\infty}(s)\big), \tag{2.2}$$

in order to underline that the response at the present time of a material depends on the history of a deformation up to the present time.

Of central importance for the present paper is the concept of a natural state. From time to time one can find this notion in the literature (cf. [11]). To be precise we give its definition. First of all we have to notice that the constitutive functional, its form, depends on a (reference) configuration $\varkappa$ with respect to which the deformations are measured.

DEFINITION 1. *A configuration $\varkappa$ will be called the primitive natural configuration if*

$$\Pi_\varkappa(1^\dagger) = 0, \tag{2.3}$$

where $1^\dagger(s) = 1$, for $s \geqslant 0$, is the constant history with the value one.

It is known that Π depends only on the equivalence class $[\varkappa]$ of the configuration to which $\varkappa$ belongs, where

$$[\varkappa] \equiv \{\bar{\varkappa} : \mathrm{Grad}(\bar{\varkappa} \cdot \varkappa^{-1}) = 1\}. \tag{2.4}$$

According to [11] the set $[\varkappa]$ is called the local configuration at a material point X. In that notation (cf. [11]) the class of equivalence $[\varkappa]$ is denoted by $\nabla\varkappa(X)$

$$[\varkappa] \equiv \nabla\varkappa \equiv \mathbf{K}, \tag{2.5}$$

and we say that the local configuration K is the gradient at X of the (global) configuration $\varkappa$. For each local configuration K and any invertible tensor P (which represents a local deformation) we can define a new local configuration PK by([13])

$$PK \equiv \{\lambda \cdot \varkappa | \operatorname{Grad} \lambda = P \text{ and } \nabla \varkappa = K\}. \tag{2.6}$$

So we can define the primitive natural state as a local configuration K_N such that

$$\Pi_{K_N}(1^\dagger) = 0. \tag{2.7}$$

DEFINITION 2. *A pair* (K, R), *with* K *as a local configuration and* R — *an invertible tensor* (*a deformation gradient*), *will be called the permanent natural state if*

$$\Pi_K(R^\dagger) = 0, \tag{2.8}$$

where $R^\dagger(s) = R$, *for all* $s \in [0, \infty)$.

LEMMA 1. Each permanent natural state assigns a primitive natural state.

LEMMA 2. If K_N and $\hat{K}_N$ are two primitive natural states which differ by a deformation P, i.e. $\hat{K}_N = PK_N$, then

$$\Pi_{K_N}(P^\dagger) = 0. \tag{2.9}$$

The proofs by the application of the previous results. At the end of this section let us consider consequence of the principle of the frame-indifference [11]: for each history of deformation F and each orthogonal tensor function Q on $[0, \infty)$, the following identity

$$Q(0)\Pi_K F Q(0)^T = \Pi_K(QF) \tag{2.10}$$

holds, where the superscript T denotes the transposition.

The following results is a simple consequence of Eq. (2.10).

LEMMA 3. If (K, R) is a permanent natural state such that the deformation R is an orthogonal tensor, then

$$\Pi_K(1^\dagger) = 0.$$

3. Tests starting from natural states

We are interested in the response of the material with memory to subhistories that start from different natural states. Precisely, we would like to investigate the physical situation in which we have to compare the responses of two specimens to the same deformation tests, made from the same material, which were in two different stress-free configurations (i.e. in two different natural states). Here "the same test" means that deformations of both specimens measured with respect to their initial stress-free configurations are the same. Such tests often take place in an engineering laboratory.

Starting from a primitive natural state K_N we consider a deformation history $F(s)$ which was constant on the (last) time interval $[0, \tau]$ with the value P, i.e. $F(s) = P$, for $0 \leqslant s \leqslant \tau$ but on the time interval $[\tau, \infty]$ it was some function of s. Assume that the history F computed with respect to a reference configuration $\varkappa$ is such that the pair (K_N, P)

([13]) Note that all operations are local in the small neighbourhood of a material point X.

forms a permanent natural state with $K_N = \nabla\varkappa$(i.e. K_N is the local configuration determined by the configuration $\varkappa$; cf. (2.6) and (2.7)). The relaxation property of the material with memory implies that for the sufficiently large τ the difference between the histories F and $P^\dagger$ becomes small and, consequently,

(3.1) $$\lim_{\tau\to\infty} ||F-P^\dagger|| = 0, \quad \text{when} \quad F(0) = P,$$

where || || denotes the norm in the history space([14]). (The domain of definition of the constitutive functional Π is a cone in this history space).

The continuity of the functional together with the assumption about (K_N, P) imply

$$\lim_{\tau\to\infty} \Pi_{K_N}(F) = 0.$$

Now, we want to continue the history F with some subhistory([15]) $\bar{f}$, i.e. a function defined on the finite time interval $[0, a]$ with the values of deformation gradients:

(3.2) $$\bar{f}:[0, a] \to V_9^+.$$

We denote the continuation of F with $\bar{f}$ by $F\mathrm{v}\bar{f}$, where

(3.3) $$(F\mathrm{v}\bar{f})(s) = \begin{cases} \bar{f}(s), & 0 \leqslant s \leqslant a, \\ F(s-a), & s > a. \end{cases}$$

On the other hand, we consider the continuation of the constant history $1^\dagger$ with the subhistory $f = \bar{f}P^{-1}$, where

(3.4) $$(1^\dagger\mathrm{v}f)(s) = \begin{cases} f(s), & 0 \leqslant s \leqslant a, \\ 1 & s > a. \end{cases}$$

Now, if an engineer wanted to perform a test $\bar{f}$ on two specimens which are in the stress-free configurations, then if he does not known that the first specimen has in its memory some non-constant history of deformation F but the second specimen all the time has been kept in the reference natural configuration $\varkappa$ with the deformation $1^\dagger$, then the engineer will expect the same responses (the same values of the stress) during the test f. In our notation, he expects the following identity:

(3.5) $$\Pi_{K_N}(F\mathrm{v}fP) = \Pi_{K_N}(1^\dagger\mathrm{v}f), \quad \text{where} \quad F = E^\tau S_\tau F, \quad \tau > 0,$$

for any test f, with $F(0) = P$. Recall that (K_N, P) forms a permanent natural state. Due to the result of Lemma 1 there exists a primitive natural state $\hat{K}_N$ such that $\hat{K}_N = PK_N$. In this state Eq. (3.5) takes the form

(3.6) $$\Pi_{\hat{K}_N}(G\mathrm{v}f) = \Pi_{K_N}(1^\dagger\mathrm{v}f),$$

where

$$G(s) = F(s)P^{-1} \quad \text{for all} \quad s \geqslant 0,$$

$$f(s) = \bar{f}(s)P^{-1} \quad \text{for all} \quad s \in [0, a].$$

([14]) The history space is a Banach function space, see the previous part.

([15]) A subhistory $\bar{f}$ will be called the test.

Additionally, by the definition of F we have

$$G(s) = 1 \quad \text{for} \quad s \in [0, \tau],$$

and for sufficiently large τ the relaxation property implies that $||G-1^{\dagger}||$ becomes so small that we can assume $||G-1^{\dagger}|| = 0$. As the consequence of the last remark and Eq. (3.10) we have

(3.7) $$\Pi_{\hat{K}_N}(1^{\dagger}\mathrm{v}f) = \Pi_{K_N}(1^{\dagger}\mathrm{v}f)$$

or, equivalently,

(3.8) $$\Pi_{K_N}((1^{\dagger}\mathrm{v}f)P) = \Pi_{K_N}(1^{\dagger}\mathrm{v}f)$$

for an arbitrary primitive natural state K_N such that the pair (K_N, P) forms a permanent natural state.

Somebody could ask if it is possible to have the identity (3.8) true for all tensors P from V_9^+. If he wanted to apply it he would be sure that the complete group of symmetry contains all invertible tensors. However, such a situation is impossible from the physical point of view. Moreover, it was proven [12] that the second law of thermodynamics (the Clausius-Duhem inequality) requires(16) that the complete symmetry group be a subgroup of the unimodular group (i.e. the set of all invertible tensors with the determinate one; this group represents all isochoric deformations).

4. Relaxed states

Let R be a given deformation gradient and define a set Λ_R in the following way(17):

(4.1) $$\Lambda_R = \{H\colon \text{there exists } \sigma > 0 \text{ such that } H(s) = R, \quad s \in [0, \sigma]\}.$$

Note that if H belongs to Λ_R, then $H = E^{\sigma}S_{\sigma}H$ for some $\sigma > 0$. It means that the history F introduced in the previous section is an element of the set Λ_P, with σ equal to τ.

DEFINITION 3. *A pair* (K, H), *with* K *a local configuration and* H *a history, will be called the proper relaxed state if* H *belongs to* $\Lambda_{H(0)}$ *and*

(4.2) $$\Pi_K(E^{\sigma}H) = 0 \quad \textit{for each} \quad \sigma \geqslant 0.$$

The conditions of the definition mean that the response of the material on each static continuation of H is that same as on H and the stress at H vanishes.

The next definition corresponds to the case of the relation (4.3):

(4.3) $$\lim_{\sigma\to\infty} \Pi_K(E^{\sigma}H) = 0.$$

DEFINITION 4. *A pair* (K, H) *will be called the almost relaxed state if the relation* (4.3) *holds.*

(16) The proof holds in [12] under some physically reasonable conditions and under the assumption that the free energy function is defined per unit mass (not per unit volume).

(17) For each R the set Λ_R contains all histories which are constant on some recent time interval with the value R. Note that for each history G and number σ the static continuation $E^{\sigma}G$ belongs to $\Lambda_{G(0)}$.

LEMMA 4. Each relaxed state (proper or almost relaxed) determines a permanent natural configuration.

As a consequence of Lemmas 1 and 2 we have the following remark:

COROLLARY. Each relaxed state determines a primitive natural state.

Now we are ready to formulate the basic assumption of the paper.

POSTULATE 1. There exists a primitive natural state K_N such that for each invertible tensor P there exists at least one history of deformation H in Λ_P such that the pair (K_N, H) is a (proper or almost) relaxed state.

LEMMA 5. If the material fulfills Postulate 1 with one primitive natural state K_N, then it fulfills with any primitive or permanent natural state.

The proof is obvious.

THEOREM 1. *If the material fulfills Postulate 1, then each local configuration is a primitive natural state.*

5. Representation theorems

This theorem together with Lemma 5 give us the condition (in the form of Postulate 1) under which the material possesses any number of natural states.

In this section we return to the case of tests which begin at natural states. In what follows we assume that the material under consideration fulfills Postulate 1 and the next postulate which is the precise expression of the identity (3.8) assumed in Sect. 3. Furthermore, hydromorphic materials may be defined as materials with the relaxation property and satisfying Postulate 1 as well as the following one;

POSTULATE 2. The response of the material to a deformation test that starts from a different primitive natural state is the same.

This postulate may be written in the following form, (cf. (3.8)) for each test $f:[0, a] \to V_9^+$ and each invertible tensor P

$$\Pi_{K_N}(1^\dagger \mathrm{v} f) = \Pi_{K_N^*}\big((1^\dagger \mathrm{v} f)P\big). \tag{5.1}$$

Because of the purpose of the present investigation we restrict our intention (and the domain of definition of the functional Π) to the set $\pi_\dagger$ of histories where

$$\pi_\dagger \equiv \{H: H(s+a) = H(a) \quad \text{for} \quad s \geqslant 0\}. \tag{5.2}$$

Let us note that each element of the set above may be written in the form

$$H = f(a)^\dagger \mathrm{v} f,$$

where f maps $[0, a]$ into V_9^+, the set of all deformation gradients.

The first consequence of Postulates 1 and 2 is the independence of the constitutive functional Π_{K_N} of K_N.

LEMMA 6. For the material which fulfills both Postulates there exists the constitutive functional $\Pi_\dagger$ defined on $\pi_\dagger$ such that

$$\Pi_{K_N}(H) = \Pi_\dagger(H) \tag{5.3}$$

for each natural state K_N and history H from $\pi_\dagger$. Additionally, this functional has the property

$$\Pi_\dagger(H) = \Pi(HP) \tag{5.4}$$

for each invertible tensor P.

Theorem 2. *If the material fulfills Postulates 1 and 2 then there exists the functional $\Pi_\dagger$ such that for each natural state* K *and the history H from the set $\pi_\dagger$ given by Eq.* (5.2) *as the domain of definition of $\Pi_\dagger$, we have*

$$\Pi_K(H) = \Pi_\dagger\big(HH(s_0)^{-1}\big), \tag{5.5}$$

where s_0 is an arbitrary number from the interval $[0, a]$.

At the end of this section we discuss consequences of the principle of material objectivity.

Lemma 7. The functional $\Pi_\dagger$ is isotropic.

Theorem 3. *The functional $\Pi_\dagger$ of the material under consideration has the property*

$$\Pi_\dagger(F) = \Pi_\dagger(U_{(0)}) \tag{5.6}$$

for each history of the deformation gradient from $\pi_\dagger$, where $U_{(0)}$ is the relative history of the right stretch tensor corresponding to F.

It often happens that the Cauchy-Green strain tensor C is used instead of the stretch U. In that case, one has to define the new functional $\Pi_\dagger$ by

$$\hat{\Pi}_\dagger(U^2_{(0)}) = \Pi_\dagger(U_{(0)}). \tag{5.7}$$

Then the constitutive equation may be written in the terms of $\Pi_\dagger$ and C as follows:

$$T_D = \hat{\Pi}_\dagger(C_{(0)}). \tag{5.8}$$

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POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH
and
UNIVERSITY OF IOWA
DIVISION OF MATERIALS ENGINEERING.

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