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On the formulation of plane problems of elasticity in terms of dislocation layers

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IN THIS PAPER two-dimensional boundary value problems of elastostatics are formulated in terms of dislocation layers with the help of the Somigliana integral. These formulations, when expressed in terms of boundary data, lead to the problem of solving boundary integral equations. As for illustrations we consider the traction and displacement problems for a circular region, contact and crack problems. Closed form solutions are obtained in each case.

W niniejszej pracy sformułowano dwuwymiarowe zagadnienia elastostatyki za pomocą warstwy dyslokacji przy użyciu całki Somigliany. Sformułowania te, uwzględniające dane warunki brzegowe, prowadzą do zagadnienia rozwiązania równań całkowych. Dla ilustracji rozważono zagadnienia w naprężeniach i przemieszczeniach dla obszaru kołowego, zagadnienia kontaktowego i problemu szczeliny. W każdym przypadku otrzymano rozwiązanie w postaci zamkniętej.

В настоящей работе сформулированы двумерные задачи эластостатики в функции слоя дислокаций при помощи интеграла Сомильяна. Эти формулировки, учитывающие заданные граничные условия, приводят к задаче решения краево-интегральных уравнений. Для иллюстрации рассмотрены задачи в напряжениях и перемещениях для круговой области, контактной задачи и задачи для щели. В каждом случае получено решение в замкнутом виде.

1. Introduction

A DISTRIBUTION of dislocations or point forces often provides a powerful means of solving boundary value problems in elasticity, where the problems are generally formulated in terms of singular integral equations. ESHELBY [1] appears to have first exploited the idea in connection with inclusion problems by showing that the displacements due to the creation of inclusions may be expressed in terms of displacements produced by a distribution of point forces or, alternatively, in terms of displacements generated by the introduction of a Somigliana dislocation over the inclusion boundary. He has considered only static problems and the corresponding dynamic problems have been considered by WILLIS [2]. The dislocation approach has found further application in elastic and elasto-plastic crack problems; see, for example, BILBY and ESHELBY [3]. Though some particular problems were solved by this approach, there was no systematic exploitation of this approach in boundary value problems until recently. LOUAT [4] only asserted that the dislocation method may be exploited in solving boundary value problems in elastostatics and he did not consider any specific problem to justify his assertion. Resently LARDNER [5] has justified this assertion and exploited the dislocation method more effectively to solve two-dimensional traction and displacement problems of general regions and of half-spaces, crack and contact problems.

A minor shortcoming of this approach is that it is based on heuristic physical arguments which may not appeal to mathematicians. Eshelby's formulation of inclusion problems has been provided with a formal mathematical justification by MAITI and MAKAN [6]. An attempt has been made in the present paper to provide a mathematical justification of the dislocation approach given by LARDNER [5] in solving two dimensional boundary value problems. In doing so we have, in fact, developed an alternative method to solve plane problems of elastostatics, which appears to be more general than that of Lardner.

The present approach essentially hinges on the Somigliana integral [7] which expresses the displacement field inside a stressed body in terms of boundary tractions and displacements. Applying Green's approach [8] to Somigliana's it is possible to derive a "modified Somigliana integral" which expresses the displacements inside a region in terms of tractions and displacements distributed along the common boundary of the region and its complement. Interpreted physically, this formulation indicates that a displacement field inside a region is due to either point forces or dislocations or both distributed along the boundary of the region. This illuminating physical concept, though accepted heuristically for a long time, has been provided first with a sound mathematical footing by MAITI and MAKAN [6]. This formulation has an analogue in potential theory where a harmonic function is expressed in terms of a single layer potential or a double layer potential or both (Green's formula). We have considered here the displacement field in terms of a "modified Somigliana integral" corresponding to the distribution of dislocations along the boundary, whence displacements and the stresses can be computed easily. These displacements and stresses, when expressed in terms of boundary data, reduce to integral equations in terms of unknown dislocation densities distributed along the boundary. To solve these boundary integral equations analytically is, in general, out of the question. However, it has been shown that in some cases these equations are amenable to analytical treatment.

First we have considered the first and second boundary value problems in a circular region where these integral equations are of Hilbert type. Then we have considered some mixed problems of half-spaces, e.g. contact and crack problems, where these integral equations reduce to those of Cauchy, Carleman, Föppl and "air-foil" type. These equations are well known and their solutions can be derived in closed form either by a complex variable technique, see GAKHOV [9], or by the Hilbert transform technique; see, for example, TITCHMARSH [10] and TRICOMI [11]. In the present study we have adopted the latter approach and the solutions of the problems obtained thereby are in perfect agreement with those obtained by other methods.

Singular integrals appear in many places in the paper and are to be understood in the sense of the Cauchy principal value. In all the problems discussed the material medium has been assumed to be isotropic and homogeneous.

2. Somigliana integrals

Let D be a plane region bounded by a smooth contour S. The normal $\mathbf{n}$ will be assumed to be directed outwards from S. The vector variable $\mathbf{x}$ will specify the position of a point in the entire plane (except on S). The Cartesian coordinates of $\mathbf{x}$ will be denoted by (x_1, x_2) and the corresponding primed variables will be used for the points on S. Let $U_{ij}(\mathbf{x}-\mathbf{x}')$

be the i-th component of the displacement at $\mathbf{x}$ produced by a point force (supposed to act in an infinite medium) of magnitude $-4\pi\mu(\lambda+2\mu)/(\lambda+3\mu)$ applied at $\mathbf{x}'$ in the j-th direction, then

$$U_{ij}(\mathbf{x}-\mathbf{x}') = \delta_{ij}\log r + Mr_{,i}r_{,j}, \quad i,j = 1,2, \tag{2.1}$$

where $r^2 = (x_1-x_1')^2+(x_2-x_2')^2$, δ_{ij} is a Kronecker delta and $M = -(\lambda+\mu)/(\lambda+3\mu)$. Here λ, μ are Lamé constants and the subscripts preceded by a comma indicate differentiation with respect to the corresponding Cartesian coordinates. The corresponding tractions $T_{ij}(\mathbf{x}-\mathbf{x}')$ at $\mathbf{x}'$ on S are given by

$$T_{ij}(\mathbf{x}-\mathbf{x}') = \{K\delta_{ij}-4\mu Mr_{,i}r_{,j}\}\frac{d}{dn}(\log r)+K\{(\log r)_{,i}n_j-(\log r)_{,j}n_i\}, \tag{2.2}$$

where $K = 2\mu^2/(\lambda+3\mu)$; the normal components n_i and the normal derivative d/dn are with respect to $\mathbf{x}' \in S$. Then, as quoted by RIZZO [12], the following identities hold:

$$\alpha\int_S \{u_i(\mathbf{x}')T_{ij}(\mathbf{x}-\mathbf{x}')-t_i(\mathbf{x}')U_{ij}(\mathbf{x}-\mathbf{x}')\}ds = u_j(\mathbf{x}); \quad \mathbf{x}\in D, \tag{2.3}$$

$$= 0; \quad \mathbf{x}\in D_e, \tag{2.4}$$

where D_e is the region exterior to D, t_i are the boundary tractions corresponding to the displacements u_i and $\alpha = (\lambda+3\mu)/(4\pi\mu)(\lambda+2\mu)$. The representation (2.3) of the displacement field $\mathbf{u}$ is, in fact, the plane counterpart of Somigliana integral [7].

If the displacements u_i and the stresses σ_{ij} are such that

$$u_i \sim O(r^{-1}), \quad \sigma_{ij} \sim O(r^{-2}) \tag{2.5}$$

at a large distance r and u_i', t_i' denote the displacements and tractions arising out of the region D_e (having the same elastic constants as those of D), then it is possible to show that

$$\alpha\int_S \{u_i'(\mathbf{x}')T_{ij}(\mathbf{x}-\mathbf{x}')-t_i'(\mathbf{x}')U_{ij}(\mathbf{x}-\mathbf{x}')\}ds = u_j'(\mathbf{x}); \quad \mathbf{x}\in D_e, \tag{2.6}$$

$$= 0; \quad \mathbf{x}\in D. \tag{2.7}$$

Adding (2.3) and (2.7) and choosing the direction of the normal outwards from S we obtain

$$u_j(\mathbf{x}) = \alpha\int_S \{(u_i-u_i')T_{ij}-(t_i-t_i')U_{ij}\}ds \tag{2.8}$$

for $\mathbf{x}\in D$, which is indeed the plane counterpart of the displacement field (3.5) derived by MAITI and MAKAN [6] and also valid for $\mathbf{x}\in D_e$. The integral representation of the displacement field (2.8) may be called a "modified Somigliana integral", from which it follows that for the existence of a non-trivial displacement field in a region there must be a discontinuity either in the displacement or in traction or in both across its boundary. This implies physically that a displacement field inside a region is due to a distribution of dislocations or of point forces or of both along the boundary.

If the stresses and displacements have the same behaviour as in (2.5) at a large distance, then corresponding to (2.8) the displacements u_i in the upper half-plane $y > 0$ are given by

$$u_j(\mathbf{x}) = \alpha\int_{-\infty}^{\infty} \{(u_i-u_i')T_{ij}-(t_i-t_i')U_{ij}\}dx, \tag{2.9}$$

where u_i' and t_i' are respectively the boundary displacements and tractions arising out of the stress field in the lower half-plane $y < 0$. This is the "modified Somigliana integral" for the upper half-plane corresponding to (2.8) for general regions and has the same physical significance as that of (2.8). The same representation is also valid for the lower half-plane.

We now derive the displacement fields due to point forces and dislocations distributed continuously along the boundary. If we set $u_i = u_i'$ on S or if there is no relative displacement of the interface (implying again $u_i - u_i' = 0$ on S), then (2.8) assumes the form

(2.10) $$u_j(\mathbf{x}) = -\alpha \int_S p_i(\mathbf{x}') U_{ij}(\mathbf{x}-\mathbf{x}') ds,$$

where $p_i = t_i - t_i'$. The displacement field (2.10) is as if due to point forces distributed continuously along S. Next, if we set $t_i - t_i' = 0$ on S, i.e. if the common boundary is equilibrated, then the equation (2.8) reduces to

(2.11) $$u_j(\mathbf{x}) = \alpha \int_S (u_i - u_i') T_{ij}(\mathbf{x}-\mathbf{x}') ds.$$

This is exactly the case when there is a layer of edge dislocations distributed continuously along S and we obtain from (2.11)

(2.12) $$u_j(\mathbf{x}) = \alpha \int_S b_i(\mathbf{x}') T_{ij}(\mathbf{x}-\mathbf{x}') ds,$$

where **b** is the Burgers vector given by $b_i = u_i - u_i'$. Thus it is obvious that once the boundary sources are known the displacement field may be computed either from (2.10) or from (2.12). In a close analogy with single layer and double layer potential representations of a harmonic function the representations (2.10) and (2.12) may be termed single layer and double layer vector potentials.

For a half-space we derive the displacements

(2.13) $$u_j(\mathbf{x}) = -\alpha \int_{-\infty}^{\infty} p_i(\mathbf{x}') U_{ij}(\mathbf{x}-\mathbf{x}') dx,$$

(2.14) $$u_j(\mathbf{x}) = \alpha \int_{-\infty}^{\infty} b_i(\mathbf{x}') T_{ij}(\mathbf{x}-\mathbf{x}') dx$$

corresponding to (2.10) and (2.12), where p_i and b_i are respectively the distributions of point forces and dislocations along the line $y = 0$. However, it may be mentioned that the vector potentials (2.10) and (2.12) were introduced by KUPRADZE [13] on direct grounds without a recourse to the present analysis or any other approach. For the purpose of the present paper we restrict ourselves to the representations (2.12) and (2.14).

3. Integral equation formulations in general regions

Denote the Cartesian coordinates of **x** and **x**′ by (x, y) and (x', y'). Let s be the arc coordinate of **x**′ and θ be the angle between the directed line from **x** to **x**′ and the positive direction of the x-axis. If $r = |\mathbf{x}-\mathbf{x}'|$, then

(3.1) $$\partial r/\partial x' = (x'-x)/r = \cos\theta, \quad \partial r/\partial y' = (y'-y)/r = \sin\theta$$

and also it follows from the Cauchy-Riemann equations that

$$d(\log r)/dn = d\theta/ds. \tag{3.2}$$

Changing the suffixes 1, 2 to x, y and substituting from (3.1) and (3.2) into (2.12) we obtain

$$u_x(x,y) = \alpha\Bigg[\int\limits_S b_x(s)\left\{(K-2\mu M)\frac{d\theta}{ds} - \mu M\frac{d}{ds}(\sin 2\theta)\right\}ds + \int\limits_S b_y(s)\left\{K\frac{d}{ds}(\log r) - 2\mu M\frac{d}{ds}(\sin^2\theta)\right\}ds\Bigg], \tag{3.3}$$

$$u_y(x,y) = \alpha\Bigg[-\int\limits_S b_x(s)\left\{K\frac{d}{ds}(\log r) + 2\mu M\frac{d}{ds}(\sin^2\theta)\right\}ds + \int\limits_S b_y(s)\left\{(K-2\mu M)\frac{d\theta}{ds} + \mu M\frac{d}{ds}(\sin 2\theta)\right\}ds\Bigg]. \tag{3.4}$$

Integrating by parts and after a little manipulation we obtain from (3.3) and (3.4)

$$u_x(x,y) = -\int\limits_S f_x(s)\left\{\frac{\theta}{2\pi} + \frac{\sin 2\theta}{8\pi(1-\nu)}\right\}ds - \int\limits_S f_y(s)\left\{\frac{1-2\nu}{4\pi(1-\nu)}\log r + \frac{\sin^2\theta}{4\pi(1-\nu)}\right\}ds, \tag{3.5}$$

$$u_y(x,y) = \int\limits_S f_x(s)\left\{\frac{1-2\nu}{4\pi(1-\nu)}\log r - \frac{\sin^2\theta}{4\pi(1-\nu)}\right\}ds + \int\limits_S f_y(s)\left\{-\frac{\theta}{2\pi} + \frac{\sin 2\theta}{8\pi(1-\nu)}\right\}ds, \tag{3.6}$$

where ν is the Poisson's ratio and

$$f_x(s) = db_x/ds, \quad f_y(s) = db_y/ds. \tag{3.7}$$

Returning to Cartesian coordinates we obtain, apart from rigid body displacements, from (3.5) and (3.6)

$$u_x(x,y) = -\int\limits_S f_x(s)\left[\frac{1}{2\pi}\mathrm{tg}^{-1}\frac{y-y'}{x-x'} + \frac{1}{4\pi(1-\nu)}\frac{(x-x')(y-y')}{(x-x')^2+(y-y')^2}\right]ds - \int\limits_S f_y(s)\left[\frac{1-2\nu}{8\pi(1-\nu)}\log\{(x-x')^2+(y-y')^2\} + \frac{1}{4\pi(1-\nu)}\frac{(y-y')^2}{(x-x')^2+(y-y')^2}\right]ds, \tag{3.8}$$

$$u_y(x,y) = \int\limits_S f_x(s)\left[\frac{1-2\nu}{8\pi(1-\nu)}\log\{(x-x')^2+(y-y')^2\} + \frac{1}{4\pi(1-\nu)}\frac{(x-x')^2}{(x-x')^2+(y-y')^2}\right]ds + \int\limits_S f_y(s)\left[\frac{1}{2\pi}\mathrm{tg}^{-1}\frac{x-x'}{y-y'} + \frac{1}{4\pi(1-\nu)}\frac{(x-x')(y-y')}{(x-x')^2+(y-y')^2}\right]ds. \tag{3.9}$$

The displacements (3.8) and (3.9) correspond exactly to those obtained by the superposition of two displacement fields due to layers of edge dislocations of densities $-f_x$ and f_y

distributed along S as shown respectively in Figs. 1 and 2, see ESHELBY [14], HIRTH and LOTHE [15].

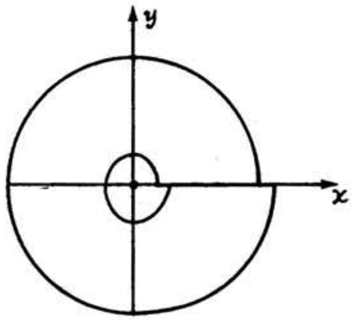

FIG. 1.

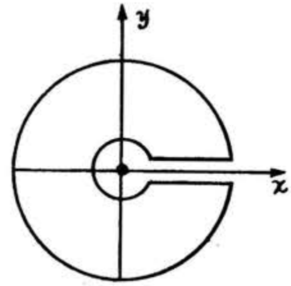

FIG. 2.

The stresses σ_{ij}, computed from (3.8) and (3.9), are given by

$$\text{(3.10)} \quad \sigma_{xx}(x, y) = \frac{\mu}{2\pi(1-\nu)}\left[\int\limits_S f_x(s)\frac{(y-y')\{3(x-x')^2+(y-y')^2\}}{\{(x-x')^2+(y-y')^2\}^2}ds + \int\limits_S f_y(s)\frac{(x-x')\{(y-y')^2-(x-x')^2\}}{\{(x-x')^2+(y-y')^2\}^2}ds\right],$$

$$\text{(3.11)} \quad \sigma_{yy}(x, y) = \frac{\mu}{2\pi(1-\nu)}\left[\int\limits_S f_x(s)\frac{(y-y')\{(y-y')^2-(x-x')^2\}}{\{(x-x')^2+(y-y')^2\}^2}ds - \int\limits_S f_y(s)\frac{(x-x')\{(x-x')^2+3(y-y')^2\}}{\{(x-x')^2+(y-y')^2\}^2}ds\right],$$

$$\text{(3.12)} \quad \sigma_{xy}(x, y) = \frac{\mu}{2\pi(1-\nu)}\left[\int\limits_S f_x(s)\frac{(x-x')\{(y-y')^2-(x-x')^2\}}{\{(x-x')^2+(y-y')^2\}^2}ds + \int\limits_S f_y(s)\frac{(y-y')\{(y-y')^2-(x-x')^2\}}{\{(x-x')^2+(y-y')^2\}^2}ds\right].$$

From the above expressions we derive

$$\text{(3.13)} \quad \sigma_{xx}+\sigma_{yy} = \frac{\mu}{\pi(1-\nu)}\left[\int\limits_S f_x(s)\frac{(y-y')ds}{(x-x')^2+(y-y')^2} - \int\limits_S f_y(s)\frac{(x-x')ds}{(x-x')^2+(y-y')^2}\right],$$

$$\text{(3.14)} \quad \sigma_{xx}-\sigma_{yy}-2i\sigma_{xy}$$

$$= \frac{\mu}{\pi(1-\nu)}\left[\int\limits_S f_x(s)(x-x')\frac{2(x-x')(y-y')+i\{(x-x')^2-(y-y')^2\}}{\{(x-x')^2+(y-y')^2\}^2}ds + \int\limits_S f_y(s)(y-y')\frac{2(x-x')(y-y')+i\{x-x')^2-(y-y')^2\}}{\{(x-x')^2+(y-y')^2\}^2}ds\right],$$

where $i^2 = -1$. Introducing complex variables $z = x+iy$ and $z' = z'+iy'$ we obtain from (3.13) and (3.14)

$$\sigma_{xx}+\sigma_{yy} = \frac{-\mu}{\pi(1-\nu)}\left[\mathrm{Re}\int_S \frac{f_y(s)-if_x(s)}{z-z'}ds\right], \tag{3.15}$$

$$\sigma_{xx}-\sigma_{yy}-2i\sigma_{xy} = \frac{\mu i}{\pi(1-\nu)}\left[\int_S f_x(s)\frac{(x-x')ds}{(z-z')^2}+\int_S f_y(s)\frac{(y-y')ds}{(z-z')^2}\right], \tag{3.16}$$

where Re stands for the real part of the integral. We now derive the boundary values of $\sigma_{xx}+\sigma_{yy}$ and $\sigma_{xx}-\sigma_{yy}-2i\sigma_{xy}$ as $z \to z_0$ on S. Denoting the point z_0 by the arc coordinate s_0 we obtain

$$(\sigma_{xx}+\sigma_{yy})_{s_0} = \frac{-\mu}{\pi(1-\nu)}\left[\mathrm{Re}\int_S \frac{f_y(s)-if_x(s)}{z-z'}ds-\pi\{f_x(s_0)\cos\theta(s_0)+f_y(s_0)\sin\theta(s_0)\}\right], \tag{3.17}$$

$$(\sigma_{xx}-\sigma_{yy}-2i\sigma_{xy})_{s_0} = \frac{\mu i}{\pi(1-\nu)}\left[\int_S f_x(s)\frac{(x_0-x')ds}{(z_0-z')^2}+\int_S f_y(s)\frac{(y_0-y')ds}{(z_0-z')^2}-\pi i\{f_x(s_0)\cos\theta(s_0)+f_y(s_0)\sin\theta(s_0)\}e^{-2i\theta(s_0)}\right], \tag{3.18}$$

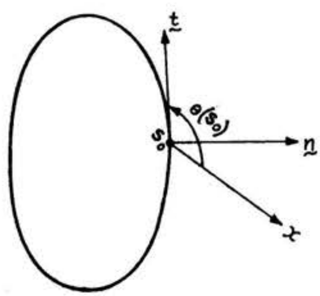

FIG. 3.

where θ (s_0) is the angle between the tangent at s_0 and the x-axis (Fig. 3) and the integrals are to be interpreted as Cauchy's principal values. If the normal and tangential tractions at s_0 be denoted by $N(s_0)$ and $T(s_0)$ respectively, then we know that

$$N(s_0)+iT(s_0) = \frac{1}{2}(\sigma_{xx}+\sigma_{yy})_{s_0}-\frac{1}{2}e^{2i\theta(s_0)}(\sigma_{xx}-\sigma_{yy}-2i\sigma_{xy})_{s_0},$$

whence we obtain from (3.17) and (3.18)

$$N(s_0)-iT(s_0) = \frac{-\mu}{2\pi(1-\nu)}\left[\mathrm{Re}\frac{f_y(s)-if_x(s)}{z_0-z'}ds+ie^{2i\theta(s_0)}\int_S \frac{(x_0-x')f_x(s)+(y_0-y')f_y(s)}{(z_0-z')^2}ds\right]. \tag{3.19}$$

Separating real and imaginary parts from (3.19) we obtain a pair of coupled boundary integral equations in f_x and f_y.

Alternatively, we derive directly from (2.12), in the limit as $\mathbf{x} \to \mathbf{x}_0$ on S,

$$u_j(\mathbf{x}_0) = \frac{1}{2} b_j(\mathbf{x}_0) + \alpha \int_S b_i(\mathbf{x}') T_{ij}(\mathbf{x}_0 - \mathbf{x}') ds, \tag{3.20}$$

which also gives two coupled boundary integral equations in b_x and b_y. We note here that the first boundary value problems may be formulated in terms of integral equations derived from (3.19) whereas the integral equations (3.20) are suitable for formulating second boundary value problems. Further it may be noted that the integral equations (3.19) and (3.20) are not, in general, amenable to analytical treatment. However it will be shown in Sect. 4 that these equations reduce to Hilbert type in the case of a circular boundary, where analytical solutions can be derived easily.

4. First and second boundary value problem for a circle

Let us consider a circular region D of radius a where the normal traction N and shearing traction T are prescribed along its boundary. Setting $z' = ae^{i\alpha}$, $z_0 = ae^{i\beta}$ and $s = a\alpha$ we derive from (3.19)

$$N(\beta) - iT(\beta) = \frac{\mu}{4\pi(1-\nu)} \left[\int_0^{2\pi} G(\alpha) \operatorname{ctg} \frac{\beta-\alpha}{2} d\alpha - 2\int_0^{2\pi} F(\alpha) d\alpha + i \int_0^{2\pi} F(\alpha) \operatorname{ctg} \frac{\beta-\alpha}{2} d\alpha \right], \tag{4.1}$$

where

$$\begin{aligned} G(\alpha) &= f_x \cos\alpha + f_y \sin\alpha, \\ F(\alpha) &= f_x \sin\alpha - f_y \cos\alpha. \end{aligned} \tag{4.2}$$

In the derivation (4.1) it has been assumed that

$$\int_0^{2\pi} f_x(s) ds = \int_0^{2\pi} f_y(s) ds = 0, \tag{4.3}$$

which are required for the single-valuedness of displacements both in D and D_e. Separating real and imaginary parts from (4.1) we derive

$$N(\beta) = \frac{-\mu}{4\pi(1-\nu)} \left[\int_0^{2\pi} G(\alpha) \operatorname{ctg} \frac{\alpha-\beta}{2} d\alpha + 2\int_0^{2\pi} F(\alpha) d\alpha \right], \tag{4.4}$$

$$T(\beta) = \frac{\mu}{4\pi(1-\nu)} \int_0^{2\pi} F(\alpha) \operatorname{ctg} \frac{\alpha-\beta}{2} d\alpha, \tag{4.5}$$

which are Hilbert integral equations, the solutions of which are given by

$$F(\alpha) = F_0 - \frac{1-\nu}{\pi\mu} \int_0^{2\pi} T(\beta) \operatorname{ctg} \frac{\beta-\alpha}{2} d\beta, \tag{4.6}$$

$$G(\alpha) = G_0 + \frac{1-\nu}{\pi\mu} \int_0^{2\pi} N(\beta) \operatorname{ctg} \frac{\beta-\alpha}{2} d\beta, \tag{4.7}$$

where F_0 and G_0 are constants to be determined. To determine F_0 we integrate both sides of (4.6) and obtain

$$\int_0^{2\pi} F(\alpha)\,d\alpha = 2\pi F_0$$

which, when substituted in (4.4), yields

$$F_0 = -(1-\nu)(2\pi\mu)^{-1}\int_0^{2\pi} N(\beta)\,d\beta. \tag{4.8}$$

The constant G_0 cannot be determined in this manner and, in fact, is indeterminate, but it can be taken to be zero since it corresponds to a rigid body displacement. The functions $F(\alpha)$ and $G(\alpha)$, when substituted form (4.6) and (4.7) into (4.2), determine the boundary functions f_x and f_y. This shows that the first boundary value problem can be solved for a circular region.

We now consider the second boundary value problem. Setting $\mathbf{x}_0 = (a\cos\beta, a\sin\beta)$ and $\mathbf{x}' = (a\cos\gamma, a\sin\gamma)$ we derive from (3.20)

$$g_1(\beta) = \frac{1}{2}b_x(\beta) + \frac{\alpha K}{2}\int_0^{2\pi} b_y(\gamma)\,\mathrm{ctg}\frac{\gamma-\beta}{2}\,d\gamma + \alpha\mu M(A\cos\beta - B\sin\beta) + C, \tag{4.9}$$

$$g_2(\beta) = \frac{1}{2}b_y(\beta) - \frac{\alpha K}{2}\int_0^{2\pi} b_x(\gamma)\,\mathrm{ctg}\frac{\gamma-\beta}{2}\,d\gamma + \alpha\mu M(A\sin\beta + B\cos\beta) + D, \tag{4.10}$$

where $g_1(\beta)$, $g_2(\beta)$ are prescribed boundary displacements and A, B, C, D, are constants to be determined. These integral equations are obviously of Hilbert type. By applying Hilbert's inversion formula to (4.9) and (4.10) we obtain after a little manipulation

$$\frac{3-4\nu}{4(1-\nu)^2}b_x(\gamma) = 2g_1(\gamma) - \frac{1-2\nu}{2\pi(1-\nu)}\int_0^{2\pi} g_2(\beta)\,\mathrm{ctg}\frac{\beta-\gamma}{2}\,d\beta + \frac{1}{8\pi(1-\nu)^2}(A\cos\gamma - B\sin\gamma) - \frac{5-12\nu+8\nu^2}{2(1-\nu)^2}C, \tag{4.11}$$

$$\frac{3-4\nu}{4(1-\nu)^2}b_y(\gamma) = 2g_2(\gamma) + \frac{1-2\nu}{2\pi(1-\nu)}\int_0^{2\pi} g_1(\beta)\,\mathrm{ctg}\frac{\beta-\gamma}{2}\,d\beta + \frac{1}{8\pi(1-\nu)^2}(A\sin\gamma + B\cos\gamma) - \frac{5-12\nu+8\nu^2}{2(1-\nu)^2}D. \tag{4.12}$$

The constants A, B, C and D may be determined from (4.9) and (4.10), and are given by the following:

$$A = \frac{2(1-\nu)}{1-2\nu}\int_0^{2\pi}\{g_1(\beta)\cos\beta + g_2(\beta)\sin\beta\}\,d\beta, \tag{4.13}$$

(4.14) $$B = \frac{2(1-\nu)}{1-2\nu}\int_0^{2\pi}\{g_2(\beta)\cos\beta - g_1(\beta)\sin\beta\}\,d\beta,$$

(4.15) $$C = \frac{1}{4\pi}\int_0^{2\pi} g_1(\beta)\,d\beta, \qquad D = \frac{1}{4\pi}\int_0^{2\pi} g_2(\beta)\,d\beta.$$

With these constants the functions b_x and b_y are completely determined from (4.11) and (4.12) and hence the second boundary value problem can be solved easily. However, the constants C and D may be taken to zero as they contribute only to the rigid body displacement.

5. Integral equation formulations in half-planes

Consider the upper half-plane $y > 0$. Setting $\mathbf{x}' = (x, 0)$ and $\mathbf{n} = (0, -1)$ we derive from (2.14)

(5.1) $$u_x(x, y) = \alpha\int_{\infty}^{-\infty} b_x(x')\left[\frac{Ky}{(x-x')^2+y^2} - 4\mu M\frac{y(x-x')^2}{\{(x-x')^2+y^2\}^2}\right]dx' - \alpha\int_{\infty}^{-\infty} b_y(x')\left[\frac{K(x-x')}{(x-x')^2+y^2} + 4\mu M\frac{y^2(x-x')}{\{(x-x')^2+y^2\}^2}\right]dx',$$

(5.2) $$u_y(x, y) = \alpha\int_{\infty}^{-\infty} b_x(x')\left[\frac{K(x-x')}{(x-x')^2+y^2} - 4\mu M\frac{y^2(x-x')}{\{(x-x')^2+y^2\}^2}\right]dx' + \alpha\int_{\infty}^{-\infty} b_y(x')\left[\frac{Ky}{(x-x')^2+y^2} - 4\mu M\frac{y^3}{\{(x-x')^2+y^2\}^2}\right]dx',$$

whence, corresponding to (3.8) and (3.9), we obtain

(5.3) $$u_x(x, y) = -\int_{-\infty}^{\infty} f_x(x')\left[\frac{1}{2\pi}\mathrm{tg}^{-1}\left(\frac{y}{x-x'}\right) + \frac{1}{4\pi(1-\nu)}\frac{y(x-x')}{(x-x')^2+y^2}\right]dx' - \int_{-\infty}^{\infty} f_y(x')\left[\frac{1-2\nu}{8\pi(1-\nu)}\log\{(x-x')^2+y^2\} + \frac{1}{4\pi(1-\nu)}\frac{y^2}{(x-x')^2+y^2}\right]dx',$$

(5.4) $$u_y(x, y) = \int_{-\infty}^{\infty} f_x(x')\left[\frac{1-2\nu}{8\pi(1-\nu)}\log\{(x-x')^2+y^2\} + \frac{1}{4\pi(1-\nu)}\frac{(x-x')^2}{(x-x')^2+y^2}\right]dx' + \int_{-\infty}^{\infty} f_y(x')\left[\frac{1}{2\pi}\mathrm{tg}^{-1}\left(\frac{x-x'}{y}\right) + \frac{1}{4\pi(1-\nu)}\frac{y(x-x')}{(x-x')^2+y^2}\right]dx',$$

where $f_x = db_x/dx'$, etc. The corresponding stresses are given by

$$(5.5)\quad \sigma_{xx}(x,y) = \frac{\mu}{2\pi(1-\nu)}\Bigg[\int_{-\infty}^{\infty} f_x(x')\frac{y\{3(x-x')^2+y^2\}}{\{(x-x')^2+y^2\}^2}dx' + \int_{-\infty}^{\infty} f_y(x')\frac{(x-x')\{y^2-(x-x')^2\}}{\{(x-x')^2+y^2\}^2}dx'\Bigg],$$

$$(5.6)\quad \sigma_{yy}(x,y) = \frac{\mu}{2\pi(1-\nu)}\Bigg[\int_{-\infty}^{\infty} f_x(x')\frac{y\{y^2-(x-x')^2\}}{\{(x-x')^2+y^2\}^2}dx' - \int_{-\infty}^{\infty} f_y(x')\frac{(x-x')\{(x-x')^2+3y^2\}}{\{(x-x')^2+y^2\}^2}dx'\Bigg],$$

$$(5.7)\quad \sigma_{xy}(x,y) = \frac{\mu}{2\pi(1-\nu)}\Bigg[\int_{-\infty}^{\infty} f_x(x')\frac{(x-x')\{y^2-(x-x')^2\}}{\{(x-x')^2+y^2\}^2}dx' + \int_{-\infty}^{\infty} f_y(x')\frac{y\{y^2-(x-x')^2\}}{\{(x-x')^2+y^2\}^2}dx'\Bigg].$$

The boundary values of the displacements and stresses are obtained, in the limit as $y \to 0$, in the following forms:

$$(5.8)\quad u_x(x,0) = -\frac{1}{2}\int_{-\infty}^{\infty} f_x(x')dx' - \frac{1-2\nu}{4\pi(1-\nu)}\int_{-\infty}^{\infty} f_y(x')\log|x-x'|dx',$$

$$(5.9)\quad u_y(x,0) = \frac{1-2\nu}{4\pi(1-\nu)}\int_{-\infty}^{\infty} f_x(x')\log|x-x'|dx' + \frac{1}{4\pi(1-\nu)}\int_{-\infty}^{\infty} f_x(x')dx' - \frac{1}{2}\int_{x}^{\infty} f_y(x')dx',$$

$$(5.10)\quad \sigma_{yy}(x,0) = \frac{\mu}{2\pi(1-\nu)}\int_{-\infty}^{\infty}\frac{f_y(x')dx'}{x'-x},$$

$$(5.11)\quad \sigma_{xy}(x,0) = \frac{\mu}{2\pi(1-\nu)}\int_{-\infty}^{\infty}\frac{f_x(x')dx'}{x'-x},$$

which are standard results in dislocation theory. Further, we derive from (5.8) and (5.9)

$$(5.12)\quad \frac{du_x(x,0)}{dx} = \frac{1}{2}f_x(x) + \frac{1-2\nu}{4\pi(1-\nu)}\int_{-\infty}^{\infty}\frac{f_y(x')dx'}{x'-x},$$

$$(5.13)\quad \frac{du_y(x,0)}{dx} = \frac{1}{2}f_y(x) - \frac{1-2\nu}{4\pi(1-\nu)}\int_{-\infty}^{\infty}\frac{f_x(x')dx'}{x'-x}.$$

The above boundary quantities provide the basis for formulating boundary value problems in terms of integral equations.

5.1. First boundary value problems

Consider the upper half-plane $y > 0$, where the normal traction and shearing traction are specified along the boundary $y = 0$, i.e.

$$\sigma_{yy}(x, 0) = p(x), \tag{5.14}$$

$$\sigma_{xy}(x, 0) = s(x), \tag{5.15}$$

where $p(x)$ and $s(x)$ are prescribed. Substituting from (5.14) and (5.15) into (5.10) and (5.11) we obtain two Cauchy integral equations

$$\frac{\mu}{2\pi(1-\nu)} \int_{-\infty}^{\infty} \frac{f_y(x')\,dx'}{x'-x} = p(x), \tag{5.16}$$

$$\frac{\mu}{2\pi(1-\nu)} \int_{-\infty}^{\infty} \frac{f_x(x')\,dx'}{x'-x} = s(x), \tag{5.17}$$

the solutions of which are given by

$$f_x(x) = -\frac{2(1-\nu)}{\pi\mu} \int_{-\infty}^{\infty} \frac{s(x')\,dx'}{x'-x}, \tag{5.18}$$

$$f_y(x) = -\frac{2(1-\nu)}{\pi\mu} \int_{-\infty}^{\infty} \frac{p(x')\,dx'}{x'-x}, \tag{5.19}$$

see TRICOMI [11]. Thus the functions $f_x(x)$ and $f_y(x)$ are known for all x and hence the first boundary value problem can be solved easily. It is also easy to derive the following boundary relations:

$$\frac{du_x}{dx} = \frac{1-2\nu}{2\mu} s(x) - \frac{1-\nu}{\pi\mu} \int_{-\infty}^{\infty} \frac{s(x')\,dx'}{x'-x}, \tag{5.20}$$

$$\frac{du_y}{dx} = -\frac{1-2\nu}{2\mu} s(x) - \frac{1-\nu}{\pi\mu} \int_{-\infty}^{\infty} \frac{p(x')\,dx'}{x'-x}. \tag{5.21}$$

These important relations are due to MUSKHELISHVILI [16], who has utilized complex variable formulations to derive them. It has been shown here that these relations can also be established from dislocation considerations.

Now substituting from (5.18) and (5.19) into (5.3) and (5.4) we derive the displacements, apart from rigid body displacements,

$$u_x(x, y) = \frac{1}{2\pi\mu} \Bigg[\int_{-\infty}^{\infty} \left\{ (1-\nu)\log\{(x-x')^2+y^2\} + \frac{y^2}{(x-x')^2+y^2} \right\} s(x')\,dx' - \int_{-\infty}^{\infty} \left\{ (1-2\nu)\,\mathrm{tg}^{-1}\left(\frac{y}{x-x'}\right) + \frac{y(x-x')}{(x-x')^2+y^2} \right\} p(x')\,dx' \Bigg], \tag{5.22}$$

$$(5.23)\quad u_y(x,y) = \frac{1}{2\pi\mu}\Bigg[\int_{-\infty}^{\infty}\left\{(1-2\nu)\,\mathrm{tg}^{-1}\left(\frac{y}{x-x'}\right) - \frac{y(x-x')}{(x-x')^2+y^2}\right\}s(x')\,dx' + \int_{-\infty}^{\infty}\left\{(1-\nu)\log\{(x-x')^2+y^2\} + \frac{(x-x')^2}{(x-x')^2+y^2}\right\}p(x')\,dx'\Bigg],$$

whence, in the limit as $y \to 0$, we derive

$$(5.24)\quad u_x(x,0) = \frac{1-\nu}{\pi\mu}\int_{-\infty}^{\infty} s(x')\log|x-x'|\,dx' - \frac{1-2\nu}{2\mu}\int_{x}^{\infty} p(x')\,dx',$$

$$(5.25)\quad u_y(x,0) = \frac{1-2\nu}{2\mu}\int_{x}^{\infty} s(x')\,dx' + \frac{1-\nu}{\pi\mu}\int_{-\infty}^{\infty} p(x')\log|x-x'|\,dx' + \frac{1}{2\pi\mu}\int_{-\infty}^{\infty} p(x')\,dx'.$$

These boundary displacements were first derived by MUSKHELISHVILI [16] and are utilized in formulating some boundary value problems. From (5.22) and (5.23) the stresses σ_{ij} are derived as follows:

$$(5.26)\quad \sigma_{xx}(x,y) = \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\frac{(x-x')^3 s(x')\,dx'}{\{(x-x')^2+y^2\}^2} + \int_{-\infty}^{\infty}\frac{y(x-x')^2 p(x')\,dx'}{\{(x-x')^2+y^2\}^2}\right],$$

$$(5.27)\quad \sigma_{yy}(x,y) = \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\frac{y^2(x-x')s(x')\,dx'}{\{(x-x')^2+y^2\}^2} + \int_{-\infty}^{\infty}\frac{y^3 p(x')\,dx'}{\{(x-x')^2+y^2\}^2}\right],$$

$$(5.28)\quad \sigma_{xy}(x,y) = \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\frac{y(x-x')^2 s(x')\,dx'}{\{(x-x')^2+y^2\}^2} + \int_{-\infty}^{\infty}\frac{y^2(x-x')p(x')\,dx'}{\{(x-x')^2+y^2\}^2}\right].$$

5.2. Second boundary value problems

Suppose that the displacements are prescribed along the line $y = 0$, i.e.

$$(5.29)\quad u_x(x,0) = U(x), \qquad u_y(x,0) = V(x).$$

Then, substituting from (5.29) into (5.12) and (5.13), we obtain the coupled integral equations

$$(5.30)\quad U'(x) = \frac{1}{2}f_x(x) + \frac{1-2\nu}{4\pi(1-\nu)}\int_{-\infty}^{\infty}\frac{f_y(x')\,dx'}{x'-x},$$

$$(5.31)\quad V'(x) = \frac{1}{2}f_y(x) - \frac{1-2\nu}{4\pi(1-\nu)}\int_{-\infty}^{\infty}\frac{f_x(x')\,dx'}{x'-x}$$

in f_x and f_y whence, by applying Hilbert transform, we derive the solutions in the following forms:

$$\frac{3-4\nu}{8(1-\nu)^2} f_x(x) = U'(x) - \frac{1-2\nu}{2\pi(1-\nu)} \int_{-\infty}^{\infty} \frac{V'(x')dx'}{x'-x}, \tag{5.32}$$

$$\frac{3-4\nu}{8(1-\nu)^2} f_y(x) = V'(x) + \frac{1-2\nu}{2\pi(1-\nu)} \int_{-\infty}^{\infty} \frac{U'(x')dx'}{x'-x}. \tag{5.33}$$

Since $U(x)$ and $V(x)$ are specified, the unknown functions f_x and f_y can be determined completely from (5.32) and (5.33). Further, these will lead to the determination of the displacements and stresses in a half-plane.

5.3. Mixed boundary problems

We now discuss some mixed problems of a half-plane, which can be easily dealt with the present method.

Case (i). Suppose that u_x, u_y are prescribed over a region S_x of the x-axis and σ_{xy}, σ_{yy} are prescribed over the complementary region $\bar{S}_x$. Then the unknown functions f_x and f_y can be determined.

Since u_x, u_y are known in S_x and σ_{xy}, σ_{yy} are known in $\bar{S}_x$, then (5.10), (5.11), (5.12) and (5.13) provide a pair of integral equations to be solved for f_x and f_y in S_x. Now f_x is known in S_x and σ_{xy} is known in $\bar{S}_x$. Then it is possible to determine f_x in $\bar{S}_x$ from (5.11). Similarly, the knowledge of f_y in S_x and σ_{yy} in $\bar{S}_x$ leads to the determination of f_y in $\bar{S}_x$ from (5.10). Thus f_x and f_y can be determined for all x. This case has a direct bearing on a crack problem discussed recently by LOWENGRUB [17] and will form the basis of a separate publication.

Case (ii). Suppose that σ_{xy} is prescribed over a region S_x of the x-axis and that u_x is prescribed over the complementary region $\bar{S}_x$. Further, σ_{yy} is prescribed over S_y and u_y is prescribed over $\bar{S}_y$. From the given boundary conditions we observe the following:

a) over the region $S_x \cap S_y$, σ_{xy} and σ_{yy} are known;

b) over the region $\bar{S}_x \cap S_y$, σ_{yy} and u_x are known and hence f_x is known;

c) over the region $S_x \cap \bar{S}_y$, σ_{xy} and u_y are known and hence f_y is known;

d) over the region $\bar{S}_x \cap \bar{S}_y$, u_x and u_y are known and hence (5.10), (5.11), (5.12) and (5.13) provide a pair of integral equations to be solved for f_x and f_y in $\bar{S}_x \cap \bar{S}_y$. Now f_x is known in $\bar{S}_x$ and σ_{xy} is known in S_x. Then f_x can be determined in S_x from (5.11). Similarly, f_y can be determined in S_y from (5.10), since σ_{yy} is known in S_y and f_y is known in $\bar{S}_y$. Thus f_x and f_y are determined for all x.

It may be noted that a large number of physical problems follow from Case (ii) as particular cases as will be seen in subsequent sections.

6. Traction and displacemet problems

Problem 1

Consider the boundary value problem for the upper half-plane $y > 0$, where the stresses and displacements satisfy the boundary conditions

$$u_x(x,0) = U(x), \quad \sigma_{yy}(x,0) = 0. \tag{6.1}$$

The second condition, when applied to (5.10) and (5.12), yields

$$f_x(x) = 2U'(x), \quad f_y(x) = 0 \tag{6.2}$$

for all x. With these values of f_x and f_y the displacements and stresses can be computed easily from (5.3), (5.4), (5.5), (5.6) and (5.7) in the upper half-plane. Further, we may derive from (6.2) the components of Burgers vector given by

$$b_x(x) = 2U(x), \quad b_y(x) = 0 \tag{6.3}$$

which, when substituted in (5.1) and (5.2), give the displacement field.

If, in addition to the boundary conditions (6.1), the shear traction $\sigma_{xy}(x,0)$ is also prescribed, i.e. $\sigma_{xy}(x,0) = s(x)$ along the line $y = 0$, then from (5.18) and (6.2) we get

$$f_x(x) = -\frac{2(1-\nu)}{\pi\mu}\int_{-\infty}^{\infty}\frac{s(x')dx'}{x'-x}, \quad f_y(x) = 0. \tag{6.4}$$

Problem 2

Consider the boundary value problem for the upper half-plane $y > 0$, when the stresses and displacements satisfy the boundary conditions

$$u_y(x,0) = V(x), \quad \sigma_{xy}(x,0) = 0. \tag{6.5}$$

In this case we obtain from (5.11) and (5.13)

$$f_x(x) = 0, \quad f_y(x) = 2V'(x), \tag{6.6}$$

whence we may derive

$$b_x(x) = 0, \quad b_y(x) = 2V(x). \tag{6.7}$$

Now it is possible, as before, to derive the stress and displacements easily. If, in addition to the boundary conditions (6.5), the normal traction is specified, i.e. $\sigma_{yy}(x,0) = p(x)$ then from (5.19) and (6.6) we obtain the functions f_x and f_y in the following forms:

$$f_x(x) = 0, \quad f_y(x) = -\frac{2(1-\nu)}{\pi\mu}\int_{-\infty}^{\infty}\frac{p(x')dx'}{x'-x} \tag{6.8}$$

In deriving (6.3) and (6.7) we have omitted the constants of integration as they will give rise to rigid body displacements. The above problems have been discussed by LARDNER [5] who has derived the dislocation densities (6.2) and (6.6) from physical considerations.

7. Contact problems

In this Section we show how the formulations of Sect. 5 can be exploited in contact problems. We consider here first the problem of indentation in a half-plane both in absence of friction and in presence of friction. Then the generalized plane problem of Hertz is considered.

7.1. Indentation in absence of friction

Consider the indentation of the half-plane $y > 0$ by a rigid punch which occupies the region $|x| < a$, $y = 0$. The normal component of the displacement is known in this region from the profile of the punch. Since the punch is smooth, $\sigma_{xy}(x, 0)$ is zero for $|x| < a$, whereas the regions $|x| > a$ are traction-free. Then the appropriate boundary conditions are as follows:

$$\begin{aligned} &u_y(x, 0) = U(x), \quad |x| < a,\\ &\sigma_{yy}(x, 0) = 0, \qquad |x| > a, \qquad (7.1)\\ &\sigma_{xy}(x, 0) = 0, \quad -\infty < x < \infty. \end{aligned}$$

This problem is a particular case of a mixed problem discussed in Sect. 5. Since $\sigma_{xy} = 0$ for all x, then (5.11) and (5.13) immediately yield

$$f_x = 0, \quad f_y = 2du_y/dx \tag{7.2}$$

for all x, whence by applying second boundary condition to (5.19), we obtain for

$$U'(x) = -\frac{1-\nu}{\pi\mu}\int_{-a}^{a}\frac{\sigma_{yy}(x', 0)dx'}{x'-x}, \tag{7.3}$$

which is an "air-foil" integral equation in $\sigma_{yy}(x, 0)$. The solution of this equation is well known, see TRICOMI [11] and is given by

$$\sigma_{yy}(x, 0) = \frac{\mu}{\pi(1-\nu)}\int_{-a}^{a}\left(\frac{a^2-x'^2}{a^2-x^2}\right)^{'2}\frac{U'(x')dx'}{x'-x}+\frac{C}{\pi(a^2-x^2)^{1/2}} \tag{7.4}$$

for $|x| < a$, where C is a constant yet to be determined. Once C is known, $\sigma_{yy}(x, 0)$ is known for $|x| < a$ and f_y is also known for all x. The stress and displacement field may now be computed easily. The expression for $\sigma_{yy}(x, 0)$, given by (7.4), agrees with those obtained by LARDNER [5] and GALIN [18]. The constant C may be determined from the applied pressure P given by

$$P = -\int_{-a}^{a}\sigma_{yy}(x, 0)dx, \tag{7.5}$$

whence we derive $C = -P$. The case of a shear punch, i.e. when the tangential displacement u_x is prescribed over the region $|x| < a$, can be similarly dealt with.

7.2. Indentation in the presence of friction

Now, in the above case, we assume that the punch is in a state of limiting equilibrium under the action of a tangential force equal to the product of the coefficient of friction (assumed constant) and the pressure applied to the point of the boundary in contact with the punch. As before, in the region $|x| < a$ the normal displacement u_y is known from the profile of punch, while the regions $|x| > a$ are traction-free. The appropriate boundary conditions are

$$(7.6)\qquad \begin{aligned} &u_y(x,0) = U(x), && |x| < a,\\ &\sigma_{xy}(x,0) = t\sigma_{yy}(x,0), && |x| < a,\\ &\sigma_{xy}(x,0) = \sigma_{yy}(x,0) = 0, && |x| > a, \end{aligned}$$

where t is the coefficient friction. The last condition, when applied to (5.18) and (5.19), yield

$$(7.7)\qquad f_x(x) = -\frac{2(1-\nu)}{\pi\mu}\int_{-a}^{a}\frac{\sigma_{xy}(x',0)dx'}{x'-x},$$

$$(7.8)\qquad f_y(x) = -\frac{2(1-\nu)}{\pi\mu}\int_{-a}^{a}\frac{\sigma_{yy}(x',0)dx'}{x'-x}.$$

Then it is obvious that the dislocation densities f_x and f_y are known for all x if we can determine either σ_{xy} or σ_{yy} for $|x| < a$. Setting $\sigma_{yy}(x,0) = p(x)$ for $|x| < a$, we derive from (5.11), (5.13), (7.8) and the second boundary condition

$$(7.9)\qquad p(x)+\frac{2(1-\nu)}{\pi t(1-2\nu)}\int_{-a}^{a}\frac{p(x')dx'}{x'-x} = -\frac{2\mu}{t(1-2\nu)}U'(x).$$

Introducing a constant α such that

$$(7.10)\qquad \operatorname{tg}\pi\alpha = \frac{t(1-2\nu)}{2(1-\nu)},\qquad 0 \leqslant \alpha < \frac{1}{2},$$

we rewrite (7.9) as

$$(7.11)\qquad p(x)-\lambda\int_{-a}^{a}\frac{p(x')dx'}{x'-x} = f(x),$$

where

$$(7.12)\qquad -\lambda\pi = \operatorname{ctg}\pi\alpha,\quad f(x) = -\frac{\mu}{1-\nu}\operatorname{ctg}\pi\alpha U'(x).$$

Thus the main problem leads to the solution of the integral equation (7.11) which is of Carleman type. The solution of this equation is well known, see TRICOMI [11] and is given by

$$(7.13)\quad p(x) = \frac{f(x)}{1+\lambda^2\pi^2}+\frac{Ce^{\tau(x)}}{(a-x)(1+\lambda^2\pi^2)^{1/2}}+\frac{\lambda e^{\tau(x)}}{(a-x)(1+\lambda^2\pi^2)}\int_{-a}^{a}\frac{(a-x')e^{-\tau(x')}f(x')dx'}{x'-x},$$

where

$$C = |\lambda| \int_{-a}^{a} p(x)\,dx, \tag{7.14}$$

$$\tau(x) = \frac{1}{\pi} \int_{-a}^{a} \frac{\theta dx'}{x'-x}, \quad \mathrm{tg}\theta = \lambda\pi. \tag{7.15}$$

Since $\mathrm{tg}\theta = \lambda\pi = -\mathrm{ctg}\pi\alpha = \mathrm{tg}(\pi/2+\pi\alpha)$, $\theta = \pi\left(\alpha+\frac{1}{2}\right)$ and hence

$$\tau(x) = \left(\frac{a-x}{a+x}\right)^{\alpha+\frac{1}{2}} \tag{7.16}$$

Now substituting from (7.12) and (7.16) into (7.13) and noting that

$$P = -\int_{-a}^{a} p(x)\,dx, \tag{7.17}$$

we obtain

$$p(x) = -\frac{\mu}{1-\nu}\cos\pi\alpha\sin\pi\alpha U'(x) - \frac{P\cos\pi\alpha}{\pi(a+x)^{\frac{1}{2}+\alpha}(a-x)^{\frac{1}{2}-\alpha}} + \frac{\mu\cos^2\pi\alpha}{\pi(1-\nu)(a+x)^{\frac{1}{2}+\alpha}(a-x)^{\frac{1}{2}-\alpha}} \int_{-a}^{a} \frac{(a-x')^{\frac{1}{2}+\alpha}(a-x')^{\frac{1}{2}-\alpha}U'(x')\,dx'}{x-x'}. \tag{7.18}$$

When $\alpha = 0$, i.e. when there is no friction this solution agrees with that given by (7.4) with $C = -P$.

7.3. Generalized plane problem of Hertz

Consider two elastic bodies B and B^1 (approximated as the two half-planes) which are in contact along the region $|x| < a$, $y = 0$ (Fig. 4). The external force exerted by B to B^1

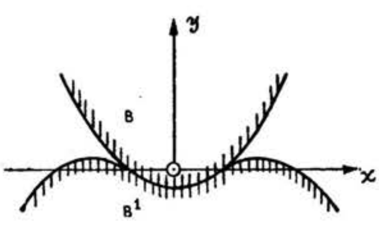

FIG. 4.

is known, whereas the region in contact is not known in advance. The relative displacement of the boundaries is known from the given equations of the boundaries. It will be assumed that there is no friction and that the regions $|x| > a$ are traction-free. The physical quantities and elastic constants for B^1 will be provided with superscript 1 in order to distinguish them from those for B. Then the boundary conditions are given by the following:

$$(7.19)\qquad \begin{aligned} &u_y(x,0)-u_y^1(x,0) = U(x), && |x|<a,\\ &\sigma_{yy}(x,0)=\sigma_{yy}^1(x,0)=0, && |x|>a,\\ &\sigma_{xy}(x,0)=\sigma_{xy}^1(x,0)=0, && -\infty<x<\infty. \end{aligned}$$

The last condition implies that $f_x(x)=f_x^1(x)=0$ for all x. Setting $\sigma_{yy}(x,0)=p(x)$ and noting that $\sigma_{yy}(x,0)=-\sigma_{yy}^1(x,0)$ for $|x|<a$, we derive from (5.19)

$$(7.20)\qquad f_y(x) = -\frac{2(1-\nu)}{\pi\mu}\int_{-a}^{a}\frac{p(x')dx'}{x'-x},$$

$$(7.21)\qquad f_y^1(x) = \frac{2(1-\nu^1)}{\pi\mu^1}\int_{-a}^{a}\frac{p(x')dx'}{x'-x}.$$

From (7.20) and (7.21) we obtain

$$(7.22)\qquad f_y(x)-f_y^1(x) = -\frac{2m}{\pi}\int_{-a}^{a}\frac{p(x')dx'}{x'-x},$$

where

$$(7.23)\qquad m = \frac{1-\nu}{\mu}+\frac{1-\nu^1}{\mu^1}.$$

From (5.13) and the first boundary condition it is to derive

$$(7.24)\qquad f_y(x)-f_y^1(x) = 2U'(x),$$

whence we obtain the integral equation

$$(7.25)\qquad \frac{1}{\pi}\int_{-a}^{a}\frac{p(x')dx'}{x'-x} = -\frac{1}{m}U'(x),$$

the solution of which is given by

$$(7.26)\qquad p(x) = \frac{1}{m\pi}\int_{-a}^{a}\left(\frac{a^2-x'^2}{a^2-x^2}\right)^{1/2}\frac{U'(x')dx'}{x'-x}+\frac{B}{\pi(a^2-x^2)^{1/2}},$$

where B is a constant. The constant B may be determined from the external force P given by

$$P = \int_{-a}^{a}p(x)dx,$$

whence we get $B=P$. Thus the unknown functions f_x and f_y may be determined from (7.20) and (7.21) for all x.

Now (7.26) may be written as

$$(7.27)\qquad p(x) = \frac{Ax+B+c}{\pi\sqrt{Q(x)}}+\frac{\sqrt{Q(x)}}{m\pi}\int_{-a}^{a}\frac{U'(x')dx'}{(x'-x)\sqrt{Q(x')}},$$

where

$$(7.28)\qquad Q(x) = a^2-x^2,$$

$$A = -\frac{1}{m}\int_{-a}^{a}\frac{U'(x')dx'}{\sqrt{Q(x')}}, \tag{7.29}$$

$$C = -\frac{1}{m}\int_{-a}^{a}\frac{x'U'(x')dx'}{\sqrt{Q(x')}}. \tag{7.30}$$

For the bounded pressure $p(x)$ at $x = \pm a$, we must have

$$A = 0, \quad B+C = 0. \tag{7.31}$$

The second condition in (7.31) yields

$$\int_{-a}^{a}\frac{x'U'(x')dx'}{(a^2-x'^2)^{1/2}} = mP, \tag{7.32}$$

which determines the unknown quantity a.

8. Crack problems

The present method can be exploited effectively to solve crack problems and we consider some of them here as illustrations.

8.1. Griffith crack opened by a thin symmetric wedge

Consider the entire plane which contains a crack occupying the region $|x| < a$, $y = 0$. It is assumed that the crack is opened by a thin symmetric wedge which makes contact with the crack surface in the region $|x| \leqslant s < a$ (Fig. 5). In general s is unknown; it may

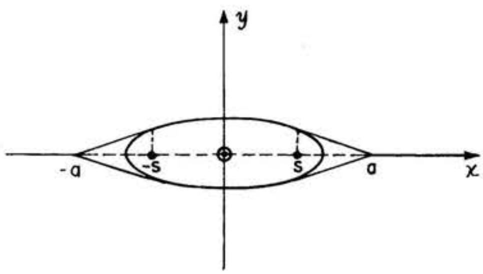

FIG. 5.

be determined from the shape of the wedge and the known pressure distribution $p(x)$ in the region $s < |x| < a$. If the equation of the upper surface of the wedge is given by $y = f(x)$, then the stress distribution in the vicinity of the crack may be determined by solving the following mixed problem of the upper half-plane:

$$\begin{aligned} u_y(x,0) &= p(x), && |x| < s,\\ u_y(x,0) &= 0, && |x| > a,\\ \sigma_{yy}(x,0) &= -p(x), && s < |x| < a,\\ \sigma_{xy}(x,0) &= 0, && -\infty < x < \infty. \end{aligned} \tag{8.1}$$

Since $\sigma_{xy}(x, 0) = 0$, then it follows from (5.11) that $f_x(x) = 0$ for all x. Also it follows from (5.13) that $f_y(x) = 2\, du_y/dx$, whence we obtain from (5.10) that

$$(8.2)\qquad \sigma_{yy}(x, 0) = \frac{\mu}{\pi(1-\nu)} \int_{-a}^{a} \frac{du_y(x', 0)}{dx'} \frac{dx'}{x'-x}$$

$$= \frac{2\mu}{\pi(1-\nu)} \left[\int_{a}^{s} \frac{du_y(x', 0)}{dx'} \frac{x'dx'}{x'^2-x^2} + \int_{s}^{a} \frac{du_y(x', 0)}{dx'} \frac{x'dx'}{x'^2-x^2} \right]$$

which follows from symmetry. Setting $v(x) = du_y/dx$ for $s < x < a$ and taking account of the first boundary condition we rewrite (8.2) as

$$(8.3)\qquad \sigma_{yy}(x, 0) = \frac{2\mu}{\pi(1-\nu)} \left[\int_{0}^{s} \frac{x'f'(x')dx'}{x'^2-x^2} + \int_{s}^{a} \frac{x'v(x')dx'}{x'^2-x^2} \right]$$

whence, by applying the third boundary condition, we derive the integral equation

$$(8.4)\qquad \frac{1}{\pi} \int_{s}^{a} \frac{2x'v(x')dx'}{x'^2-x^2} = -g(x),$$

where

$$(8.5)\qquad g(x) = \frac{1-\nu}{\mu} p(x) + \frac{1}{\pi} \int_{0}^{s} \frac{2x'f'(x')dx'}{x'^2-x^2}.$$

If we determine $v(x)$, then du_y/dx is known for $s < x < a$ and $\sigma_{yy}(x, 0)$ is completely determined. From (5.9) we obtain

$$(8.6)\qquad u_y(x, 0) = -\int_{x}^{\infty} \frac{du_y(x', 0)}{dx'} dx'$$

whence, by applying first and second boundary conditions, we get

$$(8.7)\qquad -f(x) = \int_{x}^{s} \frac{du_y(x', 0)}{dx'} dx' + \int_{s}^{a} \frac{du_y(x', 0)}{dx'} dx'.$$

Now, in the limit as $x \to s_-$, we derive from (8.7)

$$(8.8)\qquad -f(s_-) = \int_{s}^{a} v(x')dx'.$$

The integral equation (8.4) is of Föppl type and the solution may be obtained in the following manner. Define a transform T of a function ϕ such that

$$(8.9)\qquad T[\phi] = \frac{1}{\pi} \int_{s}^{a} \frac{2x'\phi(x')dx'}{x'^2-x^2}.$$

Then setting

$$(8.10)\qquad \phi_1(x) = v(x), \quad \phi_2(x) = \{(x^2-s^2)(a^2-x^2)\}^{1/2}$$

in the convolution theorem [11]

$$T\{\phi_1 T[\phi_2]+\phi_2 T[\phi_1]\} = T[\phi_1]T[\phi_2]-\phi_1\phi_2, \tag{8.11}$$

we obtain the solution of the integral equation (8.4) in two alternative forms:

$$v(x) = \frac{2}{\pi}\left(\frac{x^2-s^2}{a^2-x^2}\right)^{1/2}\int_s^a\left(\frac{a^2-x'^2}{x'^2-s^2}\right)^{1/2}\frac{x'g(x')\,dx'}{x'^2-x^2}+\frac{a_1}{\{(x^2-s^2)(a^2-x^2)\}^{1/2}}, \tag{8.12}$$

$$v(x) = \frac{2}{\pi}\left(\frac{a^2-x^2}{x^2-s^2}\right)^{1/2}\int_s^a\left(\frac{x'^2-s^2}{a^2-x'^2}\right)^{1/2}\frac{x'g(x')\,dx'}{x'^2-x^2}+\frac{a_2}{\{(x^2-s^2)(a^2-x^2)\}^{1/2}}, \tag{8.13}$$

where a_1 and a_2 are constants to be determined from (8.8). These constants are given by

$$a_1 = -\frac{a}{F_1}\left[f(s_-)+\frac{2}{\pi}\int_s^a\int_s^a\left\{\frac{(x^2-s^2)(a^2-x'^2)}{(a^2-x^2)(x'^2-s^2)}\right\}^{1/2}\frac{x'g(x')\,dx\,dx'}{x'^2-x^2}\right], \tag{8.14}$$

$$a_2 = -\frac{a}{F_1}\left[f(s_-)+\frac{2}{\pi}\int_s^a\int_s^a\left\{\frac{(a^2-x^2)(x'^2-s^2)}{(x^2-s^2)(a^2-x'^2)}\right\}^{1/2}\frac{x'g(x')\,dx\,dx'}{x'^2-x^2}\right], \tag{8.15}$$

where F_1 is the complete integral of first kind $F\left[\frac{\pi}{2}, a^{-1}(a^2-s^2)^{1/2}\right]$.

Substituting from (8.12) into (8.3) we obtain

$$\sigma_{yy}(x,0) = \frac{2\mu}{\pi(1-\nu)}\left[\int_0^s\frac{x'f'(x')\,dx'}{x'^2-x^2}-\left(\frac{x^2-s^2}{x^2-a^2}\right)^{1/2}\int_s^a\left(\frac{a^2-x'^2}{x'^2-s^2}\right)^{1/2}\frac{x'g(x')\,dx'}{x'^2-x^2}\right.$$
$$\left.-\frac{\pi a_1}{2\{(x^2-a^2)(x^2-s^2)\}^{1/2}}\right] \tag{8.16}$$

for $|x| > a$. Similarly, substituting from (8.13) into (8.3) we obtain

$$\sigma_{yy}(x,0) = \frac{2\mu}{\pi(1-\nu)}\left[\int_0^s\frac{x'f'(x')\,dx'}{x'^2-x^2}+\left(\frac{a^2-x^2}{s^2-x^2}\right)^{1/2}\int_s^a\left(\frac{x'^2-s^2}{a^2-x'^2}\right)^{1/2}\frac{x'g(x')\,dx'}{x'^2-x^2}\right.$$
$$\left.+\frac{\pi a_2}{2\{(a^2-x^2)(s^2-x^2)\}^{1/2}}\right] \tag{8.17}$$

for $|x| < s$. Substituting from (8.5) into (8.17) we obtain after a little manipulation

$$\sigma_{yy}(x,0) = \frac{2\mu}{\pi(1-\nu)}\left(\frac{a^2-x^2}{s^2-x^2}\right)^{1/2}\left[\int_0^s\left(\frac{s^2-x'^2}{a^2-x'^2}\right)^{1/2}\frac{x'f'(x')\,dx'}{x'^2-x^2}\right.$$
$$\left.+\frac{1-\nu}{\mu}\int_s^a\left(\frac{x'^2-s^2}{a^2-x'^2}\right)^{1/2}\frac{x'p(x')\,dx'}{x'^2-x^2}+\frac{\pi a_2}{2(a^2-x^2)}\right] \tag{8.18}$$

for $|x| < s$. If the wedge is smooth at $x = \pm s$, then $\sigma_{yy}(s-, 0)$ must be finite, whence we derive

$$(8.19) \quad \int_0^s \frac{x'f'(x')dx'}{\{(a^2-x'^2)(s^2-x'^2)\}^{1/2}} - \frac{1-\nu}{\mu}\int_s^a \frac{x'p(x')dx'}{\{(a^2-x'^2)(x'^2-s^2)\}^{1/2}} = \frac{\pi a_2}{2(a^2-s^2)},$$

which determines s. If the wedge is not smooth, then s must be prescribed.

This problem has been solved recently by TWEED [19] by using triple integral equation techniques.

8.2. The problem of crack extension in an infinite body

Consider an infinite body containing a crack which occupies the region $|x| < a$, $y = 0$. If the body is subjected to an applied stress, then the crack extends in general. The crack extension condition is determined from the energy of the system. For an infinite body containing an internal crack, which is under a non-uniform internal pressure, it is necessary to know the displacement of crack surface for the determination of energy of the system. We consider this problem when the crack surface is subject to a non-uniform pressure $p(x)$ and there is no shearing traction. In this case the boundary conditions are given by

$$(8.20) \quad \begin{aligned} u_y(x, 0) &= 0, & |x| &> a, \\ \sigma_{yy}(x, 0) &= p(x), & |x| &< a, \\ \sigma_{xy}(x, 0) &= 0, & -\infty &< x < \infty, \end{aligned}$$

where $p(x)$ is an even function of x.

Since $\sigma_{xy}(x, 0) = 0$, then, as before, $f_x(x) = 0$ for all x. Also it follow from (5.13) that $f_y(x) = 2U'x$, if we set $u_y(x, 0) = U(x)$. The first and second boundary conditions, when applied to (5.10), yield the "air-foil" integral equation

$$(8.21) \quad \frac{\mu}{\pi(1-\nu)}\int_{-a}^{a} \frac{U'(x')dx'}{x'-x} = p(x),$$

the solution of which is given by

$$(8.22) \quad U'(x) = \frac{A}{\pi(a^2-x^2)^{1/2}} - \frac{1-\nu}{\pi\mu}\int_{-a}^{a}\left(\frac{a^2-x'^2}{a^2-x^2}\right)^{1/2}\frac{p(x')dx'}{x'-x},$$

where A is an arbitrary constant. If A is determined, then $f_y(x)$ is determined for all x. For (8.22) we get

$$(8.23) \quad A = \int_{-a}^{a} U'(x)dx = 0,$$

because of symmetry of the crack surface. The physical implication is that the net resultant dislocation content of crack is zero, see BILBY and ESHELBY [3]. It may also be shown that the presence of the constant leads to an infinite total torque in the half-plane $y > 0$ and therefore it must be zero.

In this case the stress component $\sigma_{yy}(x, 0)$ is given by

(8.24)
$$\sigma_{yy}(x, 0) = \frac{1}{\pi} \int_{-a}^{a} \left(\frac{a^2 - x'^2}{x^2 - a^2} \right)^{1/2} \frac{f(x')dx'}{x' - x}$$

for $|x| > a$. At a point very near to crack tip we observe that

(8.25)
$$\sigma_{yy}(x, 0) \sim -2\left(\frac{a}{\pi}\right)^{1/2} \frac{1}{(2\pi|x-a|)^{1/2}} \int_{0}^{a} \frac{f(x')dx'}{(a^2 - x'^2)^{1/2}}$$

whence we obtain the stress intensity factor k_c given by

(8.26)
$$k_c = -2\left(\frac{a}{\pi}\right)^{1/2} \int_{0}^{a} \frac{f(x')dx'}{(a^2 - x'^2)^{1/2}}.$$

If we substitute (8.26) in the crack extension condition of IRWIN [20], we obtain

(8.27)
$$\left[\int_{0}^{a} \frac{p(x')dx'}{(a^2 - x'^2)^{1/2}}\right]^2 = \frac{\pi\gamma E}{2a(1-\nu^2)},$$

where γ is the surface energy and E is the Young's modulus. The condition (8.27) is due to BARENBLATT [21] and is derived also by SMITH [22].

It may happen that the non-uniform pressure distribution arises as a consequence of normal displacement being specified over a part of the crack surface, the remaining part being subject to a prescribed uniform surface pressure. The relevant boundary conditions are given by

(8.28)
$$\begin{aligned} u_y(x, 0) &= t_0, & |x| &< h, \\ u_y(x, 0) &= 0, & |x| &> a, \\ \sigma_{yy}(x, 0) &= -p_0, & h < |x| &< a, \\ \sigma_{xy}(x, 0) &= 0, & -\infty < x &< \infty. \end{aligned}$$

These boundary conditions are the same those given by (8.1) and hence this situation can be dealt with similarly.

The problem of an external crack [23], when the crack occupies the region $|x| > a$, $y = 0$ and is opened by the application of pressure to crack surface, can be solved by this approach. The crack problem under unsymmetrical loadings, discussed by LARDNER [5], can also be formulated with the help of (5.24), (5.25) and the corresponding boundary displacements arising out of the lower half-plane. Both of these problems lead to the solutions of "air-foil" integral equations and need no special treatment.

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