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Optimal control in the design of material continua

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SZEFER in a recent publication [35] outlined for the first time the application of the distributed parameter optimal control theory to material continua. Two control system model variants were considered and associated optimality conditions were derived. This paper develops on the pioneering work of Szefer and has four main thrusts. (i) An alternative derivation of the optimality conditions (Bellman's partial differential functional equation) for a class of problems relating to the model variant II is given. (ii) Acknowledgment is made for the first time of the existence of singular optimal controls in the solutions of the example design problems of Szefer. (iii) An additional system model variant possibility for material continua is detailed. (iv) Related control theory applications to continuum mechanics and structural mechanics are reported.

Szefer w publikacji [35] naszkicował po raz pierwszy zastosowanie parametru rozłożonego w teorii optymalnego sterowania. Rozważył dwa warianty modelu układu sterowania i wyprowadził stowarzyszone warunki optymalności. Praca niniejsza rozwija ideę pionierskiej pracy Szefera w czterech głównych kierunkach: (i) alternatywnego wyprowadzenia warunków optymalności funkcjonalnego równania różniczkowego cząstkowego Bellmana dla klasy zagadnień dotyczących II wariantu modelu, (ii) potwierdzenia po raz pierwszy istnienia osobliwego optymalnego sterowania w rozwiązaniach przykładów Szefera dotyczących projektowania, (iii) szczegółowego zbadania możliwości dodatkowego wariantu modelu układu dla materialnych ośrodków ciągłych, (iv) pokazania zastosowania pokrewnej teorii sterowania do mechaniki ośrodka ciągłego i mechaniki konstrukcji.

Шефер в публикации [35] описал впервые применение параметра распределения в теории оптимального управления. Он рассмотрел два варианта модели распределения управления и вывел ассоциированные условия оптимальности. Настоящая работа развивает идею новаторской работы Шефера в четырех главных направлениях: (i) альтернативного вывода условий оптимальности (функционального дифференциального уравнения в частных производных Беллемана) для класса задач, касающихся II варианта модели; (ii) подтверждения впервые существования особого оптимального управления в решениях примеров Шефера, касающихся проектирования, (iii) подробного исследования возможности дополнительного варианта модели системы для материальных сплошных сред; (iv) указания применения близкой теории управления в механике сплошной среды и в механике конструкций.

1. Introduction

SZEFER [35], writing in this journal, pioneered the application of the distributed parameter optimal control theory to material continua. In so doing, new formulations, interpretations and solution techniques were exposed for use in modelling and optimization in continuum mechanics. Two control system model types (denoted variants I and II) were considered in the study and associated optimization problems were defined. Optimality conditions were derived using the principle of optimality of Bellman, and the conditions were applied to the optimization of plates on elastic foundations, three-dimensional elastic bodies and viscoelastic beams.

This paper develops on the work of Szefer. In particular, an alternative and perhaps more general derivation of the optimality conditions for a class of distributed parameter optimal control problems relating to the model variant II is proposed. Using Bellman's principle of optimality, the parameters defining the imbedded subsystem problems are the state defined on a one-parameter family of surfaces and the parameter of this family of surfaces. The variation is taken in the surfaces' parameter. The resulting equation expressing optimality is a distributed parameter system generalisation of Bellman's equation for lumped systems [8], (equivalently the Hamilton-Jacobi partial differential equation of the variational calculus [21]). This derivation was released in a research report [12] independently of the work in [35], and is presented below for its contrast with the published derivation in this last reference. Both works, [35 and 12], were directed to the same purpose, namely the application of distributed parameter optimal control techniques to material continua. An indication is also given, analogous to the lumped parameter arguments of [33, 27 and 19], of how these derived optimality conditions may be transformed to the lower order conditions of Pontryagin's maximum principle where the canonic equations assume a form similar to the generalised Euler-Lagrange equations outlined in [29], which was for the special two-dimensional case. An example of the use of the maximum principle is given.

It is also noted that, in the examples given in [35], the formulations are candidate singular optimal control problems [7]; that is where, for example, the necessary conditions of Pontryagin's principle are satisfied in a trivial sense and supplementary conditions have to be examined to determine optimality. The occurrence of singular control problems in problems of structural optimization generally has only recently been observed [18, 13, 14, 17]. A discussion on the singular control theory is given and its application on the quoted examples demonstrated. The results indicate that solutions other than jump discontinuous optimal controls exist.

For completeness, an additional model type, denoted variant III, is outlined and optimality conditions are referenced. An example is given of how continua may be modelled in this form. The three model variants are the only forms available in the control theory literature for distributed parameter systems [18]. Variants I and II reduce to a standard lumped parameter form for the finite dimensional case, while variant III does not admit such a reduction. Variant III emphasizes the cross derivative terms in the continuum equations.

Finally, in a literature survey of the subject area, incorporating material that has appeared subsequent to [35], several relevant works should be mentioned. In particular, the survey of ROBINSON [32] and the text of LIONS [28], complement the very detailed survey of the field of optimal control of distributed parameter systems given in [35]. With regard to the application of dynamic programming concepts in the study of optimality in the same field, the early thrust proceeded by a time-incrementing type procedure. ANGEL and BELLMAN [4, 1, 2, 3], however, suggested that minimization problems over regions be formulated through the device of minimizing over subregions. (They were specifically concerned with the Dirichlet functional resulting from the potential equation. DISTEFANO [20] on the biharmonic equation uses a related device.) The choice of the subregion dictates the form of the final results. If an infinitesimal is chosen, as in [35] and in the following, a differential equation results. Dynamic programming as applied to structures is reviewed in [30], while some extentions to "non-serial" systems is given in [36] and probabilistic problems in [15].

The distributed parameter optimal control theory has been successfully applied to structural problems in [5, 6, 12 and 18], while [31, 12 and 16] discuss the modelling of continua according to a distributed parameter format.

The notation adopted by SZEFER [35] is used throughout in the following.

2. Optimality conditions — variant II

The following derivation is offered as equivalent yet alternative and perhaps more general to that given in [35]. Comments in particular are given on transformation of coordinates, allowable variations, limits of applicability and assumptions.

Consider a domain defined in E^n with a coordinate vector $x = (x_1, x_2, \ldots, x_n)^T$, and a system described over a closed region Ω, in this domain, with piecewise smooth boundary

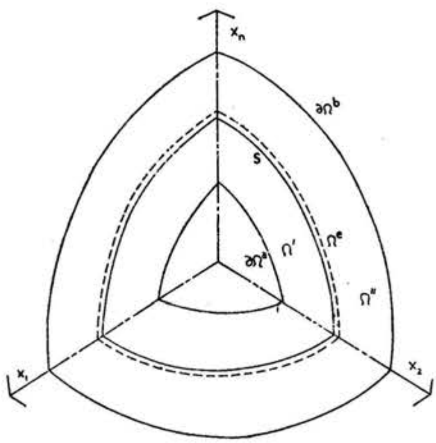

FIG. 1.

surfaces $\partial\Omega^a$ and $\partial\Omega^b$ (Fig. 1, one portion shown only). The system equations will be taken to be of the form (variant II)

$$\frac{\partial U(x)}{\partial x_i} = f^i[x, U(x), \ldots, \partial U(x), \ldots, m(x)], \quad i = 1, 2, \ldots, n, \tag{2.1}$$

where $U(x) = (U_1, \ldots, U_m)^T$ denotes the state and $m(x) = (m_1, \ldots, m_l)^T$ the control at any $x \in \Omega . f^i$ are in general nonlinear vector valued functions, and ∂U denotes a collection of partial derivatives of U each with respect to $x_h (h = 1, 2, \ldots; h \neq i)$.

Alternative admissible controls will be taken to be evaluated according to the optimality criterion

$$I_0 = \int_{\Omega} F[x, U(x), \ldots, \partial U(x), \ldots, m(x)]\, d\Omega . \tag{2.2}$$

The functional equation approach of dynamic programming imbeds this minimization problem within a family of problems with "initial" states and locations of these initial states over Ω as parameters.

Consider the region Ω divided into two subregions Ω' and Ω'' separated by a closed surface S (Fig. 1) belonging to a one-parameter family of surfaces,

$$\Phi(x_1, \ldots, x_n, c) = 0, \tag{2.3}$$

where c is the parameter of the family. S can be reduced to the boundary surfaces $\partial\Omega^a$ and $\partial\Omega^b$ by a continuous deformation. The appendix shows that such a family can be constructed using, for example, a spherical polar coordinate system for the three-dimensional case. On S, the areal measurement $s = s(x_1, \ldots, x_n)$ and the parameter $c = c(x_1, \ldots, x_n)$ which can be solved for $x_1, \ldots, x_n$ to yield $x_1 = x_1(s, c), \ldots, x_n = x_n(s, c)$; that is $U(x) \to U(s, c), m(x) \to m(s, c)$ on S. Define

$$I_0^*[U, m, c] = \int_{\Omega''} F[\gamma, U(\gamma), \ldots, \partial U(\gamma), \ldots, m(\gamma)]d\gamma. \tag{2.2a}$$

That is I_0^* is the criterion evaluated over the region Ω'' from the state U at S to the state at $\partial\Omega^b$ determined by the (admissible) control $\{m(\gamma); \gamma \in \Omega''\}$. Here $m(\gamma)$ is arbitrary and independent of $U(\gamma)$. Suppose now the optimal control $\hat{m}$ is used. At each state U, $\hat{m}$ is determined by U and so $\hat{m} = \hat{m}(U)$. Then

$$\hat{I}_0[U, c] = I_0^*[U, \hat{m}, c] = \min_m I_0^*[U, m, c]. \tag{2.2b}$$

The arguments of $\hat{I}_0$, namely U and c, in this sense may be regarded as parameters defining a family of problems. The integral defining I_0^* may be expressed as the sum of two terms corresponding to an incremental portion over the region Ω^e between two nearby members of the family of surfaces given by $\Phi(x_1, \ldots, x_n, c) = 0$ and $\Phi(x_1, \ldots, x_n, c+\delta c) = 0$ and the residual portion $\Omega^r = \Omega'' - \Omega^e$.

If a change of variables is made in the variables of integration from $(x_1, \ldots, x_n)$ to (s, c), then

$$I_0[U, c] = \min_m \Big[\int_c^{c+\delta c} \int_S F[s, c, U(s, c), \ldots, \partial U(s, c), \ldots, m(s, c)]\,|J(s, c)|\,ds dc + \int_{\Omega^r} F[\nu, U(\nu), \ldots, \partial U(\nu), \ldots, m(\nu)]d\nu + 0(\delta c)\Big], \tag{2.4}$$

where $|J(s, c)|$ denotes the Jacobian; $\nu \in \Omega^r$.

This Jacobian may be evaluated when Φ is not defined explicitly by considering a incremental change in the surface Φ and a transformation of $(x_1, \ldots, x_n)$ coordinates to (t, s) coordinates where equal differential volumes are preserved [12]. t denotes the outward surface normal. For such a transformation

$$|J(s, c)| = \frac{\left|\dfrac{\partial\Phi}{\partial c}\right|}{\left[\left[\dfrac{\partial\Phi}{\partial x_1}\right]^2 + \ldots + \left[\dfrac{\partial\Phi}{\partial x_n}\right]^2\right]}.$$

For small δc and omitting terms $0(\delta c)$ of small order higher than δc

$$\hat{I}_0[U, c] = \min_m \Big[\delta c \int_S F[s, c, U, \ldots, \partial U, \ldots, m]|J(s, c)|\,ds + \hat{I}_0[U', c']\Big] \tag{2.5}$$

using the principle of optimality, and where

$$c' = c+\delta c,$$
$$U' = U(s, c') = U(s, c+\delta c),$$
$$\hat{I}_0[U', c'] = \hat{I}_0[U(s, c+\delta c), c+\delta c].$$

The assumption is now made that $\hat{I}_0$ has partial derivatives with respect to the state U_j and parameter c, and that the derivatives exist. If this assumption holds, $\hat{I}_0[U', c']$ may be expanded in the neighbourhood of $\hat{I}_0[U, c]$. After substituting in Eq. (2.5), cancelling $\hat{I}_0[U, c]$, dividing by δc and letting $\delta c \to 0$, then

$$(2.6)\qquad -\frac{\partial \hat{I}_0[U, c]}{\partial c} = \min_m \int_S \left[F[s, c, U, \ldots, \partial U, \ldots, m]|J(s, c)| + \sum_{j=1}^{n} \frac{\partial \hat{I}_0[U, c]}{\partial U_j}\frac{\partial U_j}{\partial c}\right] ds.$$

This result holds for all c. The optimal solution must satisfy Eq. (2.6) as well as Eq. (2.1). This yields a complete set of equations to determine $\hat{I}_0[U, c]$ being minimized with respect to the "initial" state U. Although the notation does not distinguish, $\dfrac{\partial \hat{I}_0[U, c]}{\partial U_j}$ implies a functional or variational partial derivative [37].

The expression (2.6) is an extended form of Bellman's equation applicable for the distributed parameter problem that was formulated. It is both a necessary and sufficient condition for optimality. The solution of the (Hamilton-Jacobi-) Bellman equation is very difficult in general. Methods for solution are available, for example, in works on classical mechanics and the control theory [21, 32]. The optimal control follows from the solution of Eq. (2.6) for $\hat{I}_0$.

3. Transformation of the optimality conditions

As a result of the difficulties involved in solving the Hamilton-Jacobi-Bellman equation, it is often found more convenient to transform this equation into the lower order equations (Hamilton's canonical equations analogue) occurring in the maximum principle and the Hamiltonian form of the calculus of variations [27]. In so doing absolute minimality is replaced by relative minimality. The transformation may be done by showing that the gradient vector of $\hat{I}_0$ is related to the costate vector required in the canonical equations [19, 9, 32, 27, 21].

Recalling that $c = c(x_1, \ldots, x_n)$, Eq. (2.6) becomes (dropping arguments)

$$(3.1)\qquad -\sum_{i=1}^{n} \frac{\partial \hat{I}_0}{\partial x_i}\frac{\partial x_i}{\partial c} = \min_m \int_S \left\{F|J(s, c)| + \sum_{j=1}^{m}\sum_{i=1}^{n} \frac{\partial \hat{I}_0}{\partial U_j}\frac{\partial U_j}{\partial x_i}\frac{\partial x_i}{\partial c}\right\} ds$$

and further reduces, if the boundaries are aligned with the $x_1, \ldots, x_n$ axes, to

$$(3.2)\qquad -\sum_{i=1}^{n} \frac{\partial \hat{I}_0}{\partial x_i} = \min_m \int_S \left\{F + \sum_{j=1}^{m}\sum_{i=1}^{n} \frac{\partial \hat{I}_0}{\partial U_j}\frac{\partial U_j}{\partial x_i}\right\} ds.$$

The result (3.2) is subject to a certain qualification, however, a qualification resulting from setting the Jacobian and the $\frac{\partial x_i}{\partial c}$ terms equal to unity in Eq. (3.1). This simplification is only possible if the increments δx_i are the same in each of the coordinate directions implying that the inner and outer boundaries $\partial\Omega^a$ and $\partial\Omega^b$ are concentric hypercubes. Where the δx_i differ, ratio terms of the increments according to the particular problem would have to be incorporated. Equation (3.1) remains applicable in all cases. The result (3.2) is nevertheless applicable for all planar regions (and n-dimensional regions with $n-1$ or n interval limits the same) with outer boundaries only, by introducing a suitable imaginary inner boundary. For example, the inner boundary in the two-dimensional case would correspond to a line parallel to the long side of the rectangle; boundary conditions on this inner boundary would be continuity conditions on the state across the boundary.

Exchange the minimization problem for a maximization problem according to max $(-E) = -\min(E)$ as follows:

$$\sum_{i=1}^{n} \frac{\partial \hat{I}_0}{\partial x_i} = \max_{m} \int_{S} \left\{F(-1) + \sum_{j=1}^{m} \sum_{i=1}^{n} \psi_j^i f_j^i\right\} ds, \tag{3.3}$$

where the vectors $\psi^i(x)$, $i = 1, 2, \ldots, n$ with m components defined by

$$\psi^i = \left[-\left|\frac{\partial \hat{I}_0}{\partial U_1}\right|^i, \ldots, -\left|\frac{\partial \hat{I}_0}{\partial U_m}\right|^i \right]^T, \quad i = 1, \ldots, n \tag{3.4}$$

have been introduced. The superscript i denotes that the state is associated with the system equation i, $i = 1, \ldots, n$.

If a Hamiltonian is defined as

$$H(x, U, \ldots, \partial U, \ldots, \psi^i, m) \triangleq -F(x, U, \ldots, \partial U, \ldots, m) + \sum_{j=1}^{m} \sum_{i=1}^{n} \psi_j^i(x) f_j^i(x, U, \ldots, \partial U, \ldots, m) \tag{3.5}$$

then Eq. (3.3) becomes

$$\sum_{i=1}^{n} \frac{\partial \hat{I}_0}{\partial x_i} = \max_{m} \int_{S} H ds. \tag{3.6}$$

Considering the arbitrariness of the location of the surface S, then Eq. (3.6) is a statement of the maximum principle of Pontryagin.

The result (3.6) implies that the control m is chosen over the domain Ω such that the Hamiltonian as given by Eq. (3.5) is maximized everywhere. H is a function of m through F and f^i, $i = 1, \ldots, n$. The vectors ψ^i, $i = 1, \ldots, n$ are obtained from the partial derivatives of $\hat{I}_0$ with respect to the state U as given in Eq. (3.4). This may be a difficult task, first finding $\hat{I}_0$, and so an alternative means of deriving ψ^i would be desirable. This is achieved by deriving a set of costate equations in ψ^i, which can be shown [29, 12] to be of the general form

$$(3.7)\qquad \sum_{i=1}^{n} \frac{\partial \psi_j^i}{\partial x_i} = -\frac{\partial H}{\partial U_j} - (-1)\partial\left[\frac{\partial H}{\partial[\partial U_j]}\right] - ,..., \quad j = 1, ..., m,$$

where the second and subsequent terms on the right hand side of Eq. (3.7) arise from the state derivatives on the right hand side of the system equations (2.1).

4. Singular optimal control

Considering the lumped parameter equivalent of the maximum principle for explanation purpose, an extremal of the optimal control problem is said to be singular if the identities

$$(4.1)\qquad H_m[U, m, \psi, x] \equiv 0, \quad \det|H_{mm}[U, m, \psi, x)]| \equiv 0$$

hold over part or all of the x interval. The subscript notation implies differentiation here. For the case of the control appearing linearly in the Hamiltonian, the above situation occurs. This is the case of the example problems in [35] and is the most commonly appearing singular control formulation. The problem is said to be singular of the order p if the $2p$'th derivative of $\partial H/\partial m_k$ with respect to x is the first to contain the control variable m_k explicitly with a coefficient which is non-zero. The control variable m_k is referred to as a singular control [7]. A totally singular control function satisfies $H_m = 0$ for all $x \in \Omega$. A partially singular control function satisfies $H_m = 0$ only over subintervals of Ω called the singularity intervals. Pontryagin's principle in such situations is seen to be trivially satisfied and supplementary conditions have to be invoked to determine optimality. For the case of vector control, the generalized Legendre-Clebsch condition is [7]

$$\frac{\partial}{\partial m}\left[\frac{d^q}{dx^q} H_m\right] = 0,$$

(4.2) q odd, and

$$(-1)^p \frac{\partial}{\partial m}\left[\frac{d^{2p}}{dx^{2p}} H_m\right] \leqslant 0, \quad \forall x \in \Omega.$$

Other necessary conditions and sufficient conditions may be similarly quoted [7].

Consider the example of [35] on the optimization of a plate on an elastic foundation. For the lumped parameter equivalent problem and using the same notation, a Hamiltonian may be defined

$$(4.3)\qquad H \triangleq -u_1^2 + \psi_1 u_2 + \psi_2 u_3 + \psi_3 u_4 + \psi_4\left[\frac{q}{D} - \frac{k}{D} u_1\right],$$

where

$$u_1 \triangleq \omega, \quad u_2 \triangleq \frac{d\omega}{dx}, \quad u_3 \triangleq \frac{d^2\omega}{dx^2}, \quad u_4 \triangleq \frac{d^3\omega}{dx^3},$$

and $\psi_1, ..., \psi_4$ are costate variables.

According to the maximum principle, the extremal control yields the maximum of the Hamiltonian. For the above Hamiltonian, the solution can be seen to be governed by the values taken by the coefficient of the linearly appearing control term. Formally

$$\hat{k}(U, \psi, x) = \begin{cases} M_1 & \text{if} \quad \phi(U, \psi, x) < 0, \\ M_2 & \text{if} \quad \phi(U, \psi, x) > 0, \end{cases} \tag{4.4}$$

where M_1 and M_2 are the upper and lower bounds on admissible values of the foundation coefficient $k(x)$. $\phi = \frac{-\psi_4 u_1}{D}$ is the coefficient of k in H and is often called a switching function since the control changes sign everytime $\phi(x)$ changes sign. This in principle creates a well-defined piecewise continuous ("bang-bang") control $\hat{k}(x)$, the assumption being that the coefficient ϕ becomes zero at only isolated values of $x \in \Omega$. However, the coefficient may vanish over a finite subinterval of Ω. The corresponding control is termed singular. A singular control may comprise a portion of the optimal solution. The non-singular portions (corresponding to the direct solution of the state and costate equations with $\hat{k} = M_1$ or M_2) of the extremal control are defined by the boundary conditions and certain continuity properties with the singular portions. It is remarked that the presence of singular controls need not necessarily imply that the optimal solution contains singular controls. This has to be shown.

An examination of ϕ is necessary therefore to show the existence or non-existence of a singular control in the optimum solution. The singular solution may be found from the property that ϕ remains zero on the singular arc, or equivalently from the vanishing of the derivatives of ϕ

$$\frac{d\phi}{dx} = \frac{d^2\phi}{dx^2} = \frac{d^3\phi}{dx^3} = \ldots = 0. \tag{4.5}$$

That is, k is determined such that $\phi = 0$ over the particular interval of interest. Each derivative is applied successively until an expression containing the control is obtained. Use is made of the state and costate equations to express the singular control in terms of the state and costate variables.

Differentiating, $\frac{d\phi}{dx}$, $\frac{d^2\phi}{dx^2}$, and $\frac{d^3\phi}{dx^3}$ are all independent of the control. However,

$$\frac{d^4\phi}{dx^4} = -\frac{1}{D}\left[-2u_1^2 - \frac{2u_1\psi_4 k}{D} - 4\psi_1 u_2 + 6\psi_2 u_3 - 4\psi_3 u_4 + \frac{\psi_4 q}{D}\right],$$

which gives

$$k = \frac{D}{2u_1\psi_4}\left[-2u_1^2 - 4\psi_1 u_2 + 6\psi_2 u_3 - 4\psi_3 u_4 + \frac{\psi_4 q}{D}\right].$$

The generalized Legendre-Clebsch condition, $p = 2$

$$(-1)^2\left[-\frac{1}{D}\left(-\frac{2u_1\psi_4}{D}\right)\right] = \frac{2u_1\psi_4}{D^2} \leqslant 0 \tag{4.6}$$

from Eq. (4.5). This implies that there is no jump discontinuity of the foundation coefficient. In fact the solution may be termed "bang-singular-bang" rather than "bang-bang". It is noted in the above that $p = 2$ in the generalized Legendre-Clebsch condition. For p

even, jump discontinuities in the control, when transferring from a singular subarc to a non-singular subarc, are ruled out [7]. Indeed the singular subarc joins the non-singular subarc with onset of saturation.

Similar exercises may be performed on the other examples in [35] to determine whether the singular solution is in fact part of the optimal solution. The above computations were given for the lumped parameter case on which essentially all of the control literature has been concentrated. The difficulty in translating the lumped parameter conditions to the distributed parameter case may be estimated when a survey of the distributed parameter field only uncovers two references [34, 18]. For the present it seems that the problem of singular solutions in the design of material continua can only be handled by alluding to the equivalent finite dimensional problem.

5. A variant III possibility

A system model alternative to variants I and II appears in the control literature [22, 23, 10] and may equally well be used in the continuum mechanics. The model, denoted here as variant III for description purposes, relies on the symmetry of the governing continua constitutive equations by emphasizing cross derivative terms in the following manner:

$$\frac{\partial^2 U(x)}{\partial x_1 \partial x_2} = f[x, U(x), \ldots, \partial U(x), \ldots, m(x)], \tag{5.1}$$

where $x = (x_1, x_2)^T$, $\partial U(x)$ denotes derivatives of U with respect to either x_1 or x_2, and f is in general a nonlinear vector-valued function of the arguments shown.

The model is applicable in descriptions over two-dimensional planar regions. The left hand sides are now (compare with variants I and II) second-order derivatives corresponding to the isolation of the cross derivative terms from the remaining derivative terms. No cross derivatives of state appear on the right hand sides. The form, unlike variants I and II, is not reducible to a standard lumped parameter version on transferring to a one-dimensional x. The form of variant III, Eq. (5.1), may be extended to domains of dimensionality greater than two, by isolating the highest cross derivative on the left hand side.

To illustrate the type of decomposition involved in formulating variant III, consider the equation for an elastic plate on a Winkler foundation [35, Eq. (4.1)]

$$D\Delta\Delta\omega = q(x_1, x_2) - k(x_1, x)_2\omega, \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

Set $U_1 = \omega$, and $U_2 = D\dfrac{\partial^2 \omega}{\partial x_1 \partial x_2}$ as states, and $m = k$ as the control.

Differentiating

$$\frac{\partial^2 U_1}{\partial x_1 \partial x_2} = \frac{U_2}{D},$$

$$\frac{\partial^2 U_2}{\partial x_1 \partial x_2} = \frac{1}{2}\left[q - mU_1 - D\frac{\partial^4 U_1}{\partial x_1^4} - D\frac{\partial^4 U_1}{\partial x_2^4}\right].$$

This is now in the general form of variant III. It is remarked that, as with all state equation formulations, the set of state variables is non-unique.

Historically systems modelled according to a variant III format were the first distributed parameter systems for which a maximum principle was obtained. Its introduction was the beginning of the transfer from integral equation systems as pioneered by BUTKOVSKI and LERNER [11] to the more general differential equation systems. A. I. EGOROV's initial investigations with a variant III form were on quasi-linear partial differential equations [22], proving sufficiency of the optimization for the linear case. This was generalized to sets of equations of the second order [23] and special conditions were obtained for hyperbolic, parabolic and elliptic equations [24, 25, 26]. In all cases the basic mode of derivation of the necessary conditions for optimality followed Rozonoer's method [32]. For a summary of A. I. Egorov's work, see BUTKOVSKI [10], where results for special controls are given. CARMICHAEL [12] summarizes the necessary conditions for optimality for both general and rectangular domains.

6. Concluding remarks

The Hamilton-Jacobi-Bellman functional equation may be obtained for a class of distributed parameter systems, variant II, through the device of a one-parameter family of surfaces. The underlying restrictive assumptions relate to smoothness and continuity conditions on $\hat{I}_0$ and are no greater than for the conventional lumped parameter functional equation. The functional equation may be reinterpreted as a distributed parameter version of Pontryagin's maximum principle.

Model variants I, II and III, together with their associated optimality conditions, will cover a very broad class of modelling and optimization problems likely to be encountered in continuum mechanics. The choice of the class of system with which to model any continuum will vary with the characteristics of the particular continuum. The model variants II and III are suitable for systems behaving similarly in each of their independent variable (x) directions. Where behaviour differs in one particular direction, the model variant I would probably be preferred.

Singular controls can be shown to form part of the optimal solution in continua design problems. The optimal control in such cases is not necessarily jump discontinuous. Characteristically the singular controls occur where the control appears linearly in the model and/or the criterion. Additional necessary conditions are then required to establish the optimality or non-optimality of the singular solution.

The advantages and rationale of formulating problems in continuum mechanics within a control system context is admirably stated by SZEFER [35] and needs no further reinforcement.

Appendix. A one-parameter family of surfaces

A one-parameter family of surfaces is used above in a qualitative manner. This appendix shows that such a family can be constructed, for example, using a spherical polar coordinate system in the three-dimensional case.

Consider a spherical coordinate system (ϱ, θ, ϕ) [Fig. 2].

Set $OP = \varrho_1, \quad OQ = \varrho, \quad OR = \varrho_2,$

$AC = \Gamma_0, \quad AB = c\Gamma_0,$

$PR = \Gamma, \quad PQ = c\Gamma.$

When $c = 0$, S coincides with $\partial\Omega^a$, $c = 1$, S coincides with $\partial\Omega^b$. Now

$$\varrho = OQ = \varrho_1 + c\Gamma = \varrho_1 + c(\varrho_2 - \varrho_1),$$

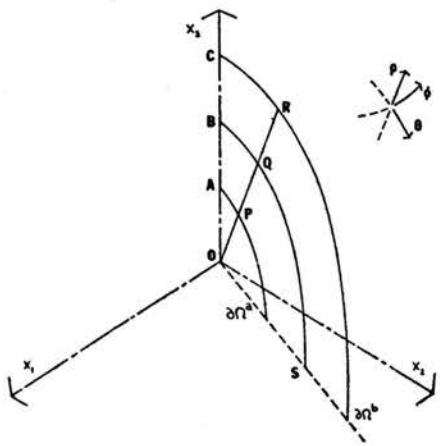

FIG. 2.

where ϱ, ϱ_1 and ϱ_2 are all functions of θ, ϕ. Then c defines a family of surfaces which moves from $\partial\Omega^a$ to $\partial\Omega^b$ as c goes from 0 to 1.

The transformation from the (x_1, x_2, x_3) coordinate system to the (ϱ, θ, ϕ) system is

$$x_1 = \varrho\sin\theta\cos\phi, \quad x_2 = \varrho\sin\theta\sin\phi, \quad x_3 = \varrho\cos\theta$$

for which the Jacobian $|J(\varrho, \theta, \phi)| = \varrho^2\sin\theta$.

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