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Finite-amplitude oscillations of a spherical cavity in a nearly-incompressible elastic material

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THIS PAPER analyses the influence of small compressibility on spherically symmetric oscillations of an elastic material. In an ideally incompressible body, spherically symmetric finite amplitude motions outside a traction-free cavity are strictly periodic [1]. Using a simple matching procedure, we show how the radiation of infinitesimal dilatation waves at high speed to large radii modifies these oscillations. A first-order ordinary differential equation for the decay of these oscillations is derived using the method of two time scales. The result is checked against an explicit analytic result for small amplitude disturbances, and indicates that even for rubberlike materials the characteristic decay time is not many multiples of the period of the oscillations.

Niniejsza praca zawiera analizę wpływu małej ściśliwości na drgania o kulistej symetrii występujące w materiale sprężystym. W ciele idealnie nieściśliwym ruchy kuliście symetryczne na zewnątrz wolnej od naprężeń pustki są ściśle periodyczne [1]. Stosując prostą procedurę dopasowania pokazujemy jak promieniowanie infinitezymalnych fal dylatacyjnych przy dużej prędkości na dalekie odległości modyfikuje te drgania. Wykorzystując metodę dwóch skali czasu, wyprowadzono równanie różniczkowe zwyczajne pierwszego rzędu do opisu zanikania tych drgań. Wynik porównano ze znanym rozwiązaniem analitycznym dla zaburzeń o małych amplitudach. Wykazano, że nawet dla materiałów gumopodobnych charakterystyczny czas zaniku nie przekracza kilku okresów drgań.

Настоящая работа содержит анализ влияния малой сжимаемости на колебания, со сферической симметрией, выступающие в упругом материале. В идеально несжимаемом теле сферически симметричные движения вне свободной от напряжений пустоты являются точно периодическими [1]. Применяя простую процедуру согласования, показываем как излучение инфинитезимальных дилатационны волн при большой скорости на далеких расстояниях модифицирует эти колебания. Используя пертурбационный метод двух масштабов времени, выведены дифференциальные обыкновенные уравнения первого порядка для описания затухания этих колебаний. Результат сравнен с известным аналитическим решением для возмущений с малыми амплитудами. Показано, что даже для резиноподобных материалов характеристика времени затухания не является многократностью периода колебаний.

1. Introduction

IN [1], KNOWLES and JAKUB considered radially symmetric oscillations of an incompressible material containing a spherical cavity. As in the work of GUO and SOLECKI [2], who analysed motions of thick-walled spheres, the kinematic constraint of incompressibility combines with spherical symmetry to reduce attention to the discussion of ordinary differential equations. Under very weak conditions on the elastic constitutive law, it is then shown that any motion having constant pressure on the cavity wall is periodic.

In practice no material is strictly incompressible, so that pressure disturbances travel with a large (but finite) wavespeed. Consequently, there is radiation of energy. This radiation must imply some decay of the amplitude of radial oscillations. It is the aim of this paper to show how small compressibility modifies the radial deformations — in particular, showing that at large distances the dominant disturbance is an outgoing dilatational wave of linear elasticity.

By a judicious, yet simple, use of matched asymptotic expansions, a first-order ordinary differential equation governing the amplitude decay is derived. This equation is solved in the special case of small amplitude disturbances, and its prediction that the rate of decay is inversely proportional to cavity radius, and directly proportional to $\left(\frac{1}{2}-\sigma\right)^{1/2}$ where σ is the Poisson's ratio, is checked with the aid of linear elasticity theory.

2. Basic formulation

We consider radially symmetric deformations of an isotropic elastic material which in its reference configuration is unstressed, has density ϱ, and possesses a cavity of radius a. Then, we let $r = r(R, t)$ $(R \geqslant a)$ denote the radius at time t of a shell which in the reference configuration has radius R. The velocity $v = \partial r/\partial t$ is purely radial and the material has one principal direction of strain in the radial direction, with the principal stretch $\lambda \equiv \partial r/\partial R$. All orthogonal directions are also principal, with equal stretches $\lambda_1 \equiv r/R$. If the corresponding principal engineering stresses (force per unit unstrained area) are T and T_1, the momentum equation takes the form

$$\varrho \frac{\partial v}{\partial t} = \frac{\partial T}{\partial R} + \frac{2(T-T_1)}{R}. \tag{2.1}$$

The strain energy density may be expressed in the form $W = W(\Lambda, \Delta)$ where the dilatation $\Delta \equiv \lambda\lambda_1^2$ and the ratio $\Lambda \equiv \lambda_1/\lambda$ completely determine all the strain invariants. With this representation, the principal engineering stresses take the form

$$T = -\frac{\lambda_1}{\lambda^2}\frac{\partial W}{\partial \Lambda} + \lambda_1^2 \frac{\partial W}{\partial \Delta}, \quad T_1 = \frac{1}{2\lambda}\frac{\partial W}{\partial \Lambda} + \lambda\lambda_1 \frac{\partial W}{\partial \Delta}, \tag{2.2}$$

showing that the mean normal Cauchy tension is

$$\frac{1}{3}\left(\frac{T}{\lambda_1^2} + 2\frac{T_1}{\lambda\lambda_1}\right) = \frac{\partial W}{\partial \Delta}.$$

This suggests that the pressure p be defined by $p = -\partial W/\partial \Delta$. Then Eqs. (2.2) agree with the incompressible theory, in which $\partial W/\partial \Delta$ becomes meaningless because $\Delta \equiv 1$ but is replaced in Eqs. (2.2) by the parameter $-p$ which is not functionally related to the deformation. The characteristic of nearly-incompressible materials is that Δ is only marginally affected by the pressure, but large pressure changes must accompany any appreciable change in Δ. Consequently, it is natural to use p as an independent state variable, rather than Δ.

To accomplish this, we introduce the enthalpy $h(\Lambda, p)$ defined, as a Legendre transform of W, by

$$h(\Lambda, p) \equiv W(\Lambda, \Delta) + p\Delta, \quad p \equiv -\partial W/\partial \Delta.$$

This leads to

$$\frac{\partial h}{\partial \Lambda} = \frac{\partial W}{\partial \Lambda} \quad \text{and} \quad \frac{\partial h}{\partial p} = \Delta .$$

Over a wide range of pressure variations within nearly-incompressible materials we may take $\Delta - 1 = -\varepsilon^2\{p-k(\Lambda)\}$, where the small parameter ε^2 is the reciprocal of the bulk modulus, and is taken to be independent of Λ. The term $k(\Lambda)$ (with $k(1) = 0$) is included to allow the dilatation to depend slightly on Λ even when $p = 0$. For these materials, we obtain

$$\frac{\partial h}{\partial p} = \Delta = 1-\varepsilon^2\{p-k(\Lambda)\},$$

so that the enthalpy must have the form

$$h = p+g(\Lambda)-\varepsilon^2\left\{\frac{1}{2}p^2-pk(\Lambda)\right\}.$$

This leads to

$$W = g(\Lambda)+\frac{\varepsilon^2}{2}\left(k(\Lambda)-\frac{\Delta-1}{\varepsilon^2}\right) = g(\Lambda)+\frac{1}{2}\varepsilon^2p^2, \tag{2.3}$$

so that the principal stresses are

$$T = -\frac{\Lambda}{\lambda}\{g'(\Lambda)+\varepsilon^2pk'(\Lambda)\}-\lambda_1^2p, \quad T_1 = \frac{1}{2\lambda}\{g'(\Lambda)+\varepsilon^2pk'(\Lambda)\}-\lambda\lambda_1p, \tag{2.4}$$

where dashes denote ordinary derivatives. Here $g(\Lambda)$ (with $g(1) = 0, g'(1) = 0$) is the strain energy of the incompressible material described by the formal limit $\varepsilon \to 0$.

Using the identity

$$\frac{\partial}{\partial R}(\lambda_1^2) = \frac{\partial}{\partial R}\left(\frac{r^2}{R^2}\right) = -\frac{r^2}{R^2}\frac{2(\Lambda-1)}{R\Lambda}$$

with the expressions (2.4) we obtain

$$\frac{2(T-T_1)}{R} = \frac{2(\Lambda-1)T}{R\Lambda}-\frac{3}{\lambda R}\{g'(\Lambda)+\varepsilon^2pk'(\Lambda)\},$$

which allows Eq (2.1) to be rewritten as

$$\varrho\frac{R^2}{r^2}\frac{\partial v}{\partial t}+\frac{3R^2}{r^3}\{g'(\Lambda)+\varepsilon^2pk'(\Lambda)\} = \frac{\partial}{\partial R}\left(\frac{R^2T}{r^2}\right).$$

When the non-dimensional coordinates $x \equiv r/a$ and $X \equiv R/a$ are introduced, this becomes

$$\varrho a^2\frac{X^2}{x^2}\frac{\partial^2 x}{\partial t^2}+\frac{3X^2}{x^3}\{g'(\Lambda)+\varepsilon^2pk'(\Lambda)\} = \frac{\partial}{\partial X}\left(\frac{X^2T}{x^2}\right). \tag{2.5}$$

This may be used, together with the kinematic relation

$$\frac{x^2}{X^2}\frac{\partial x}{\partial X} = \Delta = 1-\varepsilon^2\{p-k(\Lambda)\}, \tag{2.6}$$

to construct asymptotic expansions in ε describing radially symmetric motions in which no signal radiates inwards from large distances, and with the prescribed pressure

$$-\frac{X^2T}{x^2} = -\frac{T}{\lambda_1^2} = p+\Lambda\Delta^{-1}\{g'(\Lambda)+\varepsilon^2pk'(\Lambda)\} = F(t) \tag{2.7}$$

on the cavity wall $X = 1$ $(R = a)$.

3. Fundamental solutions

Since ε is small, Eq. (2.6) shows that either $x \simeq X$ so that radial displacements are small, or else displacements have finite amplitude but involve virtually no change in volume. In the incompressible limit $\varepsilon = 0$, Eq. (2.6) becomes a kinematic constraint implying that

$$x^3 = X^3+\eta(t),$$

where $\eta(t)$ is any function of time. Thus, for slightly compressible materials it is appropriate to represent position in the form

$$x(X, t) = x^*(X, t)+\varepsilon^2s(X, t; \varepsilon),$$

where

$$x^* \equiv \{X^3+\eta(t)\}^{\frac{1}{3}}. \tag{3.1}$$

Then

$$\frac{v}{a} = \frac{\partial x}{\partial t} = \frac{\dot{\eta}(t)}{3x^{*2}}+\varepsilon^2\frac{\partial s}{\partial t}, \qquad \lambda = \frac{\partial x}{\partial X} = \frac{X^2}{x^{*2}}+\varepsilon^2\frac{\partial s}{\partial X},$$

$$\Lambda = \frac{x^{*3}}{X^3}\left(1+\varepsilon^2\frac{s}{x^*}\right)\left(1+\varepsilon^2\frac{x^{*2}}{X^2}\frac{\partial s}{\partial X}\right)^{-1}, \quad \Delta = \left(1+\varepsilon^2\frac{s}{x^*}\right)^2\left(1+\varepsilon^2\frac{x^{*2}}{X^2}\frac{\partial s}{\partial X}\right)$$

and

$$p-k(\Lambda) = -\frac{x^{*2}}{X^2}\frac{\partial s}{\partial X}-\frac{2s}{x^*}-\varepsilon^2\left(\frac{2x^*s}{X^2}\frac{\partial s}{\partial X}+\frac{s^2}{x^{*2}}\right)-\varepsilon^4\frac{s^2}{X^2}\frac{\partial s}{\partial X},$$

where a superposed dot denotes an ordinary derivative with respect to t. When these expressions are substituted into Eq (2.5), we obtain

$$\varrho a^2X^2\left\{\frac{\ddot{\eta}}{3x^{*4}}-\frac{2}{9}\frac{\dot{\eta}^2}{x^{*7}}\right\}+\frac{3X^2}{x^{*3}}g'\left(\frac{x^{*3}}{X^3}\right) = \frac{\partial}{\partial X}\left(\frac{X^2T}{x^2}\right)+O(\varepsilon^2). \tag{3.2}$$

Thus, correct to $O(\varepsilon)$, the motion corresponds to incompressible deformations which are completely specified in terms of the generalized coordinate $\eta(t)$. Moreover, to this approximation, Eq. (3.2) specifies the variation of pressure p with X (or R) at each instant t, and may readily be integrated with respect to X and then t, as in [1] and [2]. However, this procedure is unnecessary.

For all $\overline{R} \equiv a\overline{X} > a$, the rate-of-working equation is

$$\frac{d}{dt}\left\{\frac{1}{2}\varrho\int\limits_a^{\overline{R}} v^2 4\pi R^2dR\right\}+\frac{d}{dt}\left\{\int\limits_a^{\overline{R}} W4\pi R^2dR\right\} = [4\pi R^2Tv]_a^{\overline{R}},$$

where the right hand side expresses the difference between the rate of working on material in $R > \bar{R}$ and the rate at which the prescribed pressure $F(t)$ in Eq. (2.7) supplies work. In non-dimensional coordinates the equation becomes

$$(3.3)\qquad \frac{d}{dt}\left\{\frac{1}{2}\varrho a^2\int_1^{\bar{X}}\left(\frac{\partial x}{\partial t}\right)^2 X^2 dX\right\}+\frac{d}{dt}\left\{\int_1^{\bar{X}} WX^2 dX\right\}=\left[X^2 T\frac{\partial x}{\partial t}\right]_1^{\bar{X}},$$

and correct to $O(\varepsilon)$ the left hand side equals

$$(3.4)\qquad \varrho a^2\frac{d}{dt}\left[\frac{\dot{\eta}^2}{18}\left\{\frac{1}{(1+\eta)^{\frac{1}{3}}}-\frac{1}{(\bar{X}^3+\eta)^{\frac{1}{3}}}\right\}\right]+\frac{d}{dt}\left\{\frac{\eta}{3}\int_{1+\eta/\bar{X}^3}^{1+\eta}(u-1)^{-2}g(u)\,du\right\}.$$

If we let $\bar{X}\to\infty$ and make the minor notational changes $\eta(t) = 1+y^3(t)$, $g(u) = \varrho a^2 W_0(u)$, this corresponds to the expressions in Eqs. (4.5) and (4.8) of [1], and so the only difficulty in application is the appropriate determination of $X^2T\,\partial x/\partial t$ as $\bar{X}\to\infty$.

4. The outer region

The approximation $x \simeq x^* = \{X^3+\eta(t)\}^{\frac{1}{3}}$ used in deriving Eq. (3.4) exhibits the decay of $x-X$ with radial distance, but treats such disturbances as exactly "in phase" at all radii X. When $X = O(\varepsilon^{-1})$ this treatment cannot be correct, since pressure disturbances travel with speed which is $O(\varepsilon^{-1})$. To allow for this, we examine separately the behaviour of small amplitude disturbances at large X.

We introduce the stretched coordinate $z = \varepsilon X$. Then, since $\{X^3+\eta\}^{\frac{1}{3}} \simeq X+\frac{1}{3}\eta/X^2$ for large X, it is appropriate to look for solutions of Eqs. (2.5) and (2.6) in which $x-X$ is $O(\varepsilon^2)$. Consequently, we set

$$x = X+\varepsilon^2 w(z, t; \varepsilon),\quad p = \varepsilon\hat{p}(z, t; \varepsilon),\quad \text{where } z = \varepsilon X.$$

Then

$$\lambda = 1+\varepsilon^3\partial w/\partial z,\quad \lambda_1 = 1+\varepsilon^3 w/z,\quad \Lambda-1 = O(\varepsilon^3),$$

and since $g'(1) = 0$, we have

$$g'(\Lambda) = O(\varepsilon^3)\quad\text{and}\quad T = -\varepsilon\hat{p}+O(\varepsilon^3),$$

so that Eq (2.5) leads to

$$(4.1)\qquad \varrho a^2\frac{\partial^2 w}{\partial t^2}+\frac{\partial\hat{p}}{\partial z} = O(\varepsilon^3).$$

Similarly, since $k(1) = 0$, we have $k(\Lambda) = O(\varepsilon^3)$ and Eq (2.6) gives

$$(4.2)\qquad \hat{p} = -\left(\frac{\partial w}{\partial z}+\frac{2w}{z}\right)+O(\varepsilon^2).$$

Thus, the formal solutions

$$w = w_0(z, t)+\varepsilon^2 w_1(z, t)+\ldots,\quad \hat{p} = \hat{p}_0(z, t)+\varepsilon^2\hat{p}_1(z, t)+\ldots$$

may be sought. In these, Eq. (4.2) implies that

$$z^2\hat{p}_0 = -\frac{\partial}{\partial z}(z^2 w_0)$$

and since Eq. (4.1) shows that

$$z^2\frac{\partial \hat{p}_0}{\partial z} = -\varrho a^2 \frac{\partial^2}{\partial t^2}(z^2 w_0),$$

we have

$$\varrho a^2 \frac{\partial^2}{\partial t^2}(z^2\hat{p}_0) = -\varrho a^2 \frac{\partial}{\partial z}\left\{\frac{\partial^2}{\partial t^2}(z^2 w_0)\right\} = \frac{\partial}{\partial z}\left(z^2\frac{\partial \hat{p}_0}{\partial z}\right) = z\frac{\partial^2}{\partial z^2}(z\hat{p}_0).$$

Consequently, $z\hat{p}_0$ is a solution of the wave equation, giving thus the general expressions

$$\hat{p}_0(z, t) = \frac{m''(t-z/c)+n''(t+z/c)}{z}, \quad w_0(z, t) = -c^2\frac{\partial}{\partial z}\left\{\frac{m(t-z/c)+n(t+z/c)}{z}\right\},$$

in which the functions $m(t-z/c)$ and $n(t+z/c)$ describe outgoing and incoming spherically symmetric waves respectively and $c^2 = (\varrho a^2)^{-1}$ corresponds to the physical speeds $dR/dt = \pm(\varepsilon^2\varrho)^{-\frac{1}{2}}$. This is hardly surprising since $-\varepsilon\hat{p}$ is the dominant contribution to both T and T_1, and so the material behaves essentially like a fluid with a bulk modulus ε^{-2}. As we require that no disturbances converge from large z, we set $n \equiv 0$ and obtain

$$\hat{p}_0(z, t) = \frac{m''(t-z/c)}{z}, \quad w_0(z, t) = \frac{cm'(t-z/c)}{z} + \frac{c^2 m(t-z/c)}{z^2}. \tag{4.3}$$

The function $m(t-z/c)$ may be related to $\eta(t)$ by standard matching procedures [3]: the leading term $\varepsilon^2 w_0$ in the outer expansion for $x-X$ is written in terms of the inner variable $X = z/\varepsilon$, giving

$$x-X \sim \varepsilon^2 w_0 = \frac{c^2 m(t-\varepsilon X/c)}{X^2} + \frac{\varepsilon c m'(t-\varepsilon X/c)}{X},$$
$$\sim \frac{c^2 m(t)}{X^2}.$$

Similarly, the leading term $x^*(X, t)-X$ of the inner expansion (3.1) is written in terms of the outer variable $z = \varepsilon X$, giving

$$x-X \sim \{X^3+\eta(t)\}^{\frac{1}{3}} - X = X[\{1-\varepsilon^3\eta(t)/z^3\}^{\frac{1}{3}} - 1],$$
$$\sim \frac{z}{\varepsilon}\frac{\varepsilon^3\eta(t)}{3z^3} = \frac{\varepsilon^2\eta(t)}{3z^2} = \frac{\eta(t)}{3X^2}.$$

Then, to ensure that these two expressions agree, we must choose $m(t)$ as

$$m(t) = \frac{\eta(t)}{3c^2}. \tag{4.4}$$

Moreover, with this choice, Eqs. (4.3) show that the leading term $\varepsilon\hat{p}_0(z, t)$ in the outer expression for p has the inner form

$$\varepsilon\hat{p}_0 = \frac{\varepsilon\eta''(t-\varepsilon X/c)}{3c^2(\varepsilon X)} \sim \frac{\eta''(t)}{3c^2X},$$

which agrees with Eq. (3.2) as $X \to \infty$ (and $\Lambda \to 1$) where it becomes

$$\frac{\partial p}{\partial X} \sim -\frac{1}{c^2}\frac{\eta''(t)}{3X^2}.$$

This matching shows that an oscillation which at finite X is described by Eq. (3.1) with an appropriate function $\eta(t)$ gives rise to pressure waves which at finite $z \equiv \varepsilon X$ (i.e. $X = O(\varepsilon^{-1})$) take the form

$$p \sim \frac{\varepsilon\eta''(t-z/c)}{3c^2z} = \frac{\eta''(t-\varepsilon X/c)}{3c^2X},$$

(4.5)

$$x \sim X + \frac{\varepsilon\eta'(t-\varepsilon X/c)}{3cX} + \frac{\eta(t-\varepsilon X/c)}{3X^2}.$$

The expressions (4.5) may be combined with the inner expressions to give the leading terms in composite expansions valid for all X, giving, for example,

$$x \sim \{X^3+\eta(t)\}^{\frac{1}{3}} + \frac{\varepsilon\eta'(zt-\varepsilon X/c)}{3cX} + \frac{\eta(t-\varepsilon X/c)}{3X^2} - \frac{\eta(t)}{3X^2}$$

which is correct to $O(\varepsilon^2)$ in both the inner and the outer regions. Equivalently this expression may be replaced by a simpler, but equally valid, expression

$$x \sim \{X^3+\eta(t-\varepsilon X/c)+\varepsilon(X/c)\eta'(t-\varepsilon X/c)\}^{\frac{1}{3}} \tag{4.6}$$

which exhibits the wavelike nature of the disturbance at all X.

5. The intermediate expansion

To apply the rate-of-working equation (3.3), we must choose $\bar{X}$ such that both sides of the equation may be correctly determined to the required accuracy. This is most simply achieved by writing both the *inner* and the *outer* expansions for x and p in terms of an *intermediate* variable y defined by

$$y = \mu X = \varepsilon^{-1}\mu z \quad \text{with} \quad \mu = o(1) \quad \textit{and} \quad \varepsilon/\mu = o(1).$$

Then, from the expressions (3.1) we obtain

$$x = x^* + \varepsilon^2 s \sim X + \mu^2\frac{\eta(t)}{3y^2}, \quad \frac{\partial x}{\partial t} \sim \mu^2\frac{\dot{\eta}(t)}{3y^2}, \quad \Lambda \sim 1+\mu^3\frac{\eta(t)}{y^3},$$

which agree with the leading term obtained by setting $z = \varepsilon\mu^{-1}y$ in Eqs. (4.5). This illustrates how the regions of validity of the inner and outer expansions overlap, a property which must be true to all orders of the expansion (at least for a suitable choice of μ) in order that KAPLUN's [4] justification of the matching procedure may be applied. Likewise, by choosing $\overline{X} = \mu^{-1}\overline{y}$ for any finite $\overline{y}$ and a suitable μ, both sides of Eq. (3.3) may be expressed in terms of the function $\eta(t)$.

Firstly, we show that with this choice of $\overline{X}$ the expression (3.4) correctly expresses the rate of energy increase correct to $O(\varepsilon)$ for *all* $\varepsilon = o(\mu)$.

Since p is $O(1)$ for finite X and $p \sim \mu\ddot{\eta}(t)/3c^2y = \ddot{\eta}(t)/3c^2X$ for all finite values of $y = \mu X$, then pX is $O(1)$ for all X. Moreover, since substitution of Eq. (3.1) into Eq. (2.6) leads to

$$\frac{\partial}{\partial X}(x^{*2}s) \sim -X^2\{p-k(\Lambda)\} = O(X),$$

we deduce that $s = O(1)$ throughout $1 \leqslant X \leqslant \mu^{-1}\overline{y}$. Consequently,

$$\left(\frac{\partial x}{\partial t}\right)^2 = \left(\frac{\dot{\eta}}{3x^{*2}}\right)^2[1+O(\varepsilon^2x^{*2})]$$

and

$$W = g(\Lambda)+\frac{1}{2}\varepsilon^2p^2 = g(1+\eta/X^3)\{1+O(\varepsilon^2)\}+O(\varepsilon^2p^2).$$

By setting $\eta/X^3 \equiv u-1$ we then have

$$\int\limits_1^{\mu^{-1}\overline{y}} g(1+\eta/X^3)X^2dX = -\frac{1}{3}\eta\int\limits_{1+\eta}^{1+\mu^3\eta/\overline{y}^3}(u-1)^{-2}g(u)\,du,$$

and the left hand side of Eq. (3.3) becomes

$$(5.1)\quad \frac{d}{dt}\left\{\frac{\varrho a^2}{18}\frac{\dot{\eta}^2}{(1+\eta)^{\frac{1}{3}}}+\frac{\eta}{3}\int\limits_1^{1+\eta}(u-1)^{-2}g(u)\,du-\frac{\mu\varrho a^2}{18}\frac{\dot{\eta}^2}{(\overline{y}^3+\mu^3\eta)^{\frac{1}{3}}}\right\}$$
$$-\frac{d}{dt}\left\{\frac{\eta}{3}\int\limits_{1+\eta}^{1+\mu^3\eta/\overline{y}^3}(u-1)^{-2}g(u)\,du+O(\varepsilon^2/\mu)\right\}.$$

Here, the last term is $o(\varepsilon)$ for all $\varepsilon = o(\mu)$, and the remaining terms are those of Eq. (3.4) The first two are those which survive in the limit $\overline{X} \to \infty$, the third is $O(\mu)$ and is an estimate of the kinetic energy in $X > \mu^{-1}\overline{y}$, while the fourth term is $O(\mu^3)$(1).

If we use the first term of the inner expansion to evaluate the right hand side of Eq. (3.3) with $\overline{X} = \mu^{-1}\overline{y}$, the only significant term is one which cancels with the third term in Eq. (5.1). However, the first term in the outer approximation gives more information, and determines the right hand side up to a term of $O(\varepsilon)$. It gives

$$T \sim -p \sim -\mu\frac{\eta''(t-\varepsilon y/\mu c)}{3c^2y} \sim -\mu\frac{\ddot{\eta}(t)}{3c^2y}+\varepsilon\frac{\dddot{\eta}(t)}{3c^3}$$

$$x-X \sim \mu^2\frac{\eta(t-\varepsilon y/\mu c)}{3y^2}+\varepsilon\mu\frac{\eta'(t-\varepsilon y/\mu c)}{3cy} \sim \mu^2\frac{\eta(t)}{3y^2}-\varepsilon^2\frac{\ddot{\eta}(t)}{6c^2},$$

(1) We assume that $g''(1)$ exists, so that $\partial T/\partial\lambda$, $\partial T/\partial\lambda_1$, etc. exist at $\lambda = 1$, $\lambda_1 = 1$ and so $(u-1)^{-2}g(u)$ remains finite as $u \to 1$.

so that

$$X^2 T \frac{\partial x}{\partial t} \sim -\frac{y^2}{\mu^2}\left\{\mu \frac{\ddot{\eta}(t)}{3c^2 y} - \varepsilon \frac{\dddot{\eta}(t)}{3c^3}\right\}\mu^2 \frac{\dot{\eta}(t)}{3y^2}, \tag{5.2}$$

$$\sim -\mu \frac{\dot{\eta}(t)\ddot{\eta}(t)}{9c^2 y} + \varepsilon \frac{\dot{\eta}(t)\dddot{\eta}(t)}{9c^3}.$$

The first term differs from the third term of Eq. (5.1) only by $O(\mu^4)$, and so substitution expressions (5.1) and (5.2) into Eq. (3.3) gives, correct to $O(\varepsilon)$,

$$\frac{d}{dt}\left\{\frac{\varrho a^2}{18} \frac{\dot{\eta}^2}{(1+\eta)^{\frac{1}{3}}} + \frac{\eta}{3}\int_1^{1+\eta} (u-1)^{-2} g(u)\,du\right\} = \varepsilon \frac{\dot{\eta}(t)\dddot{\eta}(t)}{9c^3} + \frac{F(t)\dot{\eta}(t)}{3}. \tag{5.3}$$

Notice that, although the derivation requires that $\mu = o(\varepsilon^{\frac{1}{3}})$, the first term on the right hand side is independent of both $\bar{y}$ and μ, and so accounts for the rate at which energy is radiated to large distances. Also, by using Eq. (2.7) and the identity $x^3(1, t) = 1+\eta(t)$, it may be shown that the second term accounts for the rate of working on the cavity wall.

6. The decay of the oscillations

In the strictly incompressible limit $\varepsilon \to 0$, Eq. (5.3) may be reduced to Eq. (4.2) of KNOWLES and JAKUB [1]. Its solutions are exactly periodic in two important classes of situations — namely, when the internal pressure $F(t)$ is a function only of the cavity radius $(1+\eta)^{\frac{1}{3}}$ (and hence of $\eta(t)$), and during time intervals when $F(t)$ is constant. This second case, with $F(t) = p_0$ as in [1], leads to

$$\frac{\dot{\eta}^2}{(1+\eta)^{\frac{1}{3}}} + K(\eta) = N, \tag{6.1}$$

where N is a constant such that $(18/4\pi\varrho a^5)N = 9c^2 N/2\pi a^3$ measures the total energy, and

$$K(\eta) \equiv 6c^2\left\{\eta \int_1^{1+\eta} (u-1)^{-2} g(u)\,du - \eta p_0\right\}. \tag{6.2}$$

In the first case, with $F(t) = Q'(\eta)$, Eq. (6.1) still holds provided the term ηp_0 in Eq. (6.2) is replaced by $Q(\eta)$. Consequently, we may discuss the two situations together, having due regard for the definition of $K(\eta)$.

The function $K(\eta)$ will have a minimum at some value $\bar{\eta}$ (with $\bar{\eta} = 0$ if $F(t) = 0$ at the reference configuration $\eta = 0$), and under weak restrictions similar to those discussed in [1] will be concave. Consequently, to each value $N(>K(\bar{\eta}))$ there correspond two roots $\eta = \eta_1(N)$, $\eta = \eta_2(N)$ (with $\eta_1 < \bar{\eta} < \eta_2$) of the equation

$$K(\eta) = N.$$

The solution $\eta(t)$ of Eq. (6.1) oscillates periodically between these values, with period

$$T = T(N) \equiv 2\int\limits_{\eta_1}^{\eta_2} \frac{d\eta}{\sqrt{\{(1+\eta)^{\frac{1}{3}}[N-K(\eta)]\}}}. \tag{6.3}$$

These are the nonlinear oscillations investigated in [1].

The dominant effect of compressibility is to modify Eq. (6.1), replacing it by

$$\frac{d}{dt}\left\{\frac{\dot{\eta}^2}{(1+\eta)^{\frac{1}{3}}}+K(\eta)\right\} = \frac{2\varepsilon\dot{\eta}\dddot{\eta}}{c}, \tag{6.4}$$

or, equivalently,

$$\frac{2\ddot{\eta}}{(1+\eta)^{\frac{1}{3}}} - \frac{\dot{\eta}^2}{3(1+\eta)^{\frac{4}{3}}} + K'(\eta) = \frac{2\varepsilon\dddot{\eta}}{c}. \tag{6.4$_1$}$$

Hence, defining N to be the quantity

$$N \equiv (1+\eta)^{-\frac{1}{3}}\dot{\eta}^2 + K(\eta), \tag{6.5}$$

we see that dN/dt is small, so that Eq. (6.4) describes the slow modulation in amplitude of the nonlinear oscillations(²). Appreciable changes in N take place only on the "slow" time scale measured by $\tau = \varepsilon t$. Consequently, solutions to Eq. (6.4) may be found by the method of two time scales, in which we write

$$N = \tilde{N}(\tau)+\varepsilon\hat{N}(t, \tau; \varepsilon), \quad \tau = \varepsilon t.$$

Then N performs small oscillations about the smoothly varying value $\tilde{N}(\tau) = \tilde{N}(\varepsilon t)$, and without loss of generality we may insist that $\hat{N}$ is strictly periodic in the fast variable t, with mean value zero, for all *fixed* τ and ε. Correspondingly we write

$$\eta = \tilde{\eta}(t, \tau)+\varepsilon\hat{\eta}(t, \tau; \varepsilon),$$

so that

$$\dot{\eta} = \frac{\partial\tilde{\eta}}{\partial t} + \varepsilon\left(\frac{\partial\tilde{\eta}}{\partial\tau}+\frac{\partial\hat{\eta}}{\partial t}\right) + O(\varepsilon^2),$$

and while insisting that $\hat{\eta}$ is strictly periodic in t we may choose that

$$(1+\tilde{\eta})^{-\frac{1}{3}}\left(\frac{\partial\tilde{\eta}}{\partial t}\right)^2 + K(\tilde{\eta}) \equiv \tilde{N}(\tau). \tag{6.6}$$

Thus, when τ is regarded as fixed, the oscillations $\eta(t, \tau)$ are the periodic solutions of Eq.

(²) Equation (6.4) does not belong to one of the usual types of equations governing such phenomena — the small parameter multiplies the derivative of the highest order. This suggests that the perturbation $\varepsilon \neq 0$ may be singular. However, the assumptions leading to Eq (6.2) do not allow rapid jumps in $\ddot{\eta}$. More generally, under impulsive loading of the cavity wall as discussed in [1], the deformations (3.1) must be replaced by wavelike disturbances even at finite X, so that Eq (5.3) is inappropriate during the corresponding short time intervals.

(6.1) with the energy $18c^2\tilde{N}(\tau)$ and the period $T(\tilde{N})$ given by Eq. (6.3). Then, substituting into Eq. (6.4), we obtain

$$\varepsilon\frac{d\tilde{N}}{d\tau}+\varepsilon\frac{\partial\hat{N}}{\partial t}+O(\varepsilon^2) = \frac{2\varepsilon}{c}\frac{\partial\tilde{\eta}}{\partial t}\frac{\partial^3\tilde{\eta}}{\partial t^3}+O(\varepsilon^2)$$

in which the first term on the right is strictly periodic in t. To ensure that $\hat{N}$ also is periodic, we must have, correct to $O(\varepsilon)$,

$$\frac{d\tilde{N}}{d\tau} = \text{mean value of}\,\frac{2}{c}\frac{\partial\tilde{\eta}}{\partial t}\frac{\partial^3\tilde{\eta}}{\partial t^3},\tag{6.7}$$

$$= \frac{1}{T(\tilde{N})}\int\limits_0^{T(\tilde{N})}\frac{2}{c}\frac{\partial\tilde{\eta}}{\partial t}\frac{\partial^3\tilde{\eta}}{\partial t^3}dt.$$

Now, rearranging Eq. (6.6) as

$$\left(\frac{\partial\tilde{\eta}}{\partial t}\right)^2 = (1+\tilde{\eta})^{\frac{1}{3}}\{\tilde{N}(\tau)-K(\tilde{\eta})\} \equiv \varphi(\tilde{\eta}),$$

we have

$$2\frac{\partial^2\tilde{\eta}}{\partial t^2} = \varphi'(\tilde{\eta}),\quad 2\frac{\partial^3\tilde{\eta}}{\partial t^3} = \varphi''(\tilde{\eta})\frac{\partial\tilde{\eta}}{\partial t},$$

so that

$$2\frac{\partial\tilde{\eta}}{\partial t}\frac{\partial^3\tilde{\eta}}{\partial t^3} = \varphi''(\tilde{\eta})S(\tilde{\eta}).\tag{6.8}$$

(For simplicity, we have suppressed from φ the explicit dependence on $\tilde{N}(\tau)$.) Consequently, from Eqs. (6.3), (6.7) and (6.8) the equation governing decay of $\tilde{N}$ (and hence of "amplitude" $\eta_2(\tilde{N})-\eta_1(\tilde{N})$) is found to be

$$\frac{d\tilde{N}}{dt}\int\limits_{\eta_1(\tilde{N})}^{\eta_2(\tilde{N})}\varphi^{-\frac{1}{2}}(\eta)d\eta = \frac{1}{c}\int\limits_{\eta_1(\tilde{N})}^{\eta_2(\tilde{N})}\varphi''(\eta)\varphi^{\frac{1}{2}}(\eta)d\eta,\tag{6.9}$$

in which both integrals depend only on $\tilde{N}$, and the limits $\eta_1(\tilde{N})$, $\eta_2(\tilde{N})$ are roots of $\varphi(\eta) = 0$. Equation (6.9) is then a first-order ordinary differential equation for $\tilde{N}$, which is readily soluble numerically once $K(\eta)$ is fixed by the choice of a particular material and of a suitable boundary condition on the cavity wall.

7. Linearization

As a specific example of the predictions of the previous sections we note here the form of the eventual decay of free oscillations as the displacements become small. This allows us also to check the predictions against an explicit solution derived from linear elasticity.

When $x \ll 1$, we have $\Lambda \simeq 1$, so that

$$g(\Lambda) \simeq \frac{1}{2}g''(1)(\Lambda-1)^2,$$

leading to

$$\int_{1}^{1+\eta} (u-1)^{-2} g(u)\,du \simeq \frac{1}{2} g''(1)\eta .$$

Moreover, in free oscillations with $p_0 = 0$ Eq. (6.2) gives

$$K(\eta) = 3c^2 g''(1)\eta^2,$$

and, since η is small, Eq. (6.6) may be approximated as

$$\left(\frac{\partial \tilde{\eta}}{\partial t}\right)^2 + 3c^2 g''(1)\tilde{\eta}^2 = \tilde{N}(\tau). \tag{7.1}$$

This corresponds to $\varphi(\eta) = \alpha^2(\eta_2^2 - \eta^2)$, where $\alpha^2 = 3c^2 g''(1)$ and $\tilde{N} = \alpha^2\eta_2^2$, so that

$$\frac{1}{2} T \equiv \int_{-\eta_2}^{\eta_2} \frac{d\eta}{\alpha\sqrt{\eta_2^2 - \eta^2}} = \frac{\pi}{\alpha}, \quad \int_{-\eta_2}^{\eta_2} \varphi''(\eta)\varphi^{\frac{1}{2}}(\eta)\,d\eta = -\pi\alpha^3\eta_2^2 = -\alpha\pi\tilde{N}$$

and Eq. (6.9) becomes

$$\frac{d\tilde{N}}{d\tau} = -\frac{\alpha^2}{c}\tilde{N}.$$

Hence the decay of $\tilde{N}$ has the form

$$\tilde{N}(\tau) \sim \tilde{N}_0 \exp\{-3cg''(1)\tau\}.$$

Since for a linear elastic material $g''(1)$ and ε^2 are related to the Young's modulus E and Poisson's ratio σ by

$$E = \frac{3}{2}(1+\sigma)g''(1) \quad \text{and} \quad 1 - 2\sigma = \frac{1}{3}\varepsilon^2 E, \tag{7.2}$$

we have $g''(1) = \frac{4}{9}E$ to $O(\varepsilon)$ and $T \sim \pi a(3\varrho/E)^{\frac{1}{2}}$, so that oscillations governed by Eq. (7.1) have the form

$$\tilde{\eta} \sim \tilde{N}_0 \cos\psi(t) \exp\left(-\frac{2E\varepsilon}{3\varrho^{\frac{1}{2}} a} t\right), \quad \frac{d\psi}{dt} \sim \frac{2\pi}{T} \sim \left(\frac{4E}{3\varrho a^2}\right)^{\frac{1}{2}}. \tag{7.3}$$

In this approximation, both the period and the decay time are proportional to the cavity radius a, and the amplitude decays by the same factor e^{-1} during each timespan corresponding to

$$\frac{3\varrho^{\frac{1}{2}} a}{2\varepsilon E T} = \frac{1}{2\pi}\left(\frac{3}{\varepsilon^2 E}\right)^{\frac{1}{2}} = \frac{1}{2\pi(1-2\sigma)^{\frac{1}{2}}} \tag{7.4}$$

periods of oscillation. This number of oscillations is independent not only of the amplitude of the oscillation and of the radius of the undeformed cavity, but also of the Young's modulus of the material. It depends only on the amount by which the Poisson's ratio differs from $\frac{1}{2}$.

To check these results by means of the linear elasticity theory we write displacements in the form $r = R + au$, with u small. The relevant approximations are

$$\Delta - 1 = \frac{\partial u}{\partial X} + 2\frac{u}{X} = \varepsilon^2 p, \quad \Lambda - 1 = \frac{u}{X} - \frac{\partial u}{\partial X},$$

$$W = \frac{1}{2} g''(1)(\Lambda-1)^2 + \frac{1}{2}\varepsilon^2 p^2 = \frac{E}{3(1+\sigma)}\left(\frac{u}{X} - \frac{\partial u}{\partial X}\right)^2 + \frac{E}{6(1-2\sigma)}\left(\frac{\partial u}{\partial X} + 2\frac{u}{X}\right)^2,$$

so that the equation for radially symmetric disturbances becomes (see also [5], p. 286)

$$\{\varepsilon^{-2} + g''(1)\}\frac{\partial}{\partial X}\left(\frac{\partial u}{\partial X} + 2\frac{u}{X}\right) = \varrho a^2 \frac{\partial^2 u}{\partial t^2},$$

where $\varepsilon^{-2} + g''(1) = \frac{1}{3}E\{(1-2\sigma)^{-1} + 2(1+\sigma)^{-1}\}$ may be expressed using Eq. (7.2) in terms of the Lamé constants λ, μ as $\lambda + 2\mu$. This is the wave equation in spherical polar coordinates and so, like Eqs. (4.1) and (4.2), has general solutions composed of outgoing and incoming waves. The disturbance corresponding to a prescribed variation of pressure (2.7) on $X = 1$ and propagating into a region at rest with $u = 0$ must consist only of an outgoing wave and so, like Eq. (4.3), has the form

$$u = \frac{\partial}{\partial X}\left\{\frac{s(t - X/c_0)}{X}\right\} \quad \text{where} \quad c_0^2 = \frac{\varepsilon^{-2} + g''(1)}{\varrho a^2}.$$

If, after some time t_0, the pressure on the cavity wall becomes zero, the boundary condition $T = 0$ becomes

$$\{\varepsilon^{-2} + g''(1)\}\frac{\partial u}{\partial X} + \{2\varepsilon^{-2} - g''(1)\}u = 0 \quad \text{at} \quad X = 1$$

and leads to

$$\frac{s''}{c_0^2} + 2\frac{s'}{c_0} + 2s - k\left(\frac{s'}{c_0} + s\right) = 0, \quad s = s(t - c_0^{-1}), \tag{7.5}$$

where $k = \{2\varepsilon^{-2} - g''(1)\}/\{\varepsilon^{-2} + g''(1)\}$ is expressible in terms of the Lamé constants as $2\lambda/(\lambda + 2\mu)$. The ordinary differential equation (7.5) has solutions $s(t - c_0^{-1}) \propto \exp i\omega(t - c_0^{-1})$ where

$$\frac{\omega^2}{c_0^2} - 2\frac{i\omega}{c_0} - 2 + k\left(\frac{i\omega}{c_0} + 1\right) = 0,$$

giving

$$\frac{\omega}{c_0} = i\left(1 - \frac{1}{2}k\right) \pm \left(1 - \frac{1}{4}k^2\right)^{\frac{1}{2}} = \frac{\pm\sqrt{(1-2\sigma)} + i(1-2\sigma)}{1-\sigma}.$$

By taking the real part of s, we find that the general solution u has the form

$$u = \frac{\partial}{\partial X}\left\{A\frac{\cos\omega_R(t - X/c_0 - t_1)}{X}\exp - \omega_I(t - X/c_0)\right\}, \tag{7.6}$$

where A and t_1 are arbitrary constants, $\omega_R = |\mathrm{Re}\,\omega|$ and $\omega_I = \mathrm{Im}\,\omega$. This represents a spherical wave propagating outwards with the speed $a dX/dt = ac_0 = \varepsilon^{-1}\{3\varrho^{-1}(1-\sigma)/(1+\sigma)\}^{\frac{1}{2}}$

$\simeq \varepsilon^{-1}\varrho^{-\frac{1}{2}}$. The profile, at each fixed X, is a damped harmonic oscillation with angular frequency

$$\omega_R = a^{-1}\left\{\frac{E}{3(1-\sigma^2)}\right\}^{\frac{1}{2}} = \left\{\frac{4E}{3\varrho a^2}\right\}^{\frac{1}{2}}\left\{1+\frac{2}{9}\varepsilon^2 E - \frac{1}{27}\varepsilon^4 E^2\right\}^{-\frac{1}{2}},$$

and with amplitude which decays by the factor e^{-1} during each time interval

$$\omega_I^{-1} = (1-2\sigma)^{-\frac{1}{2}}\omega_R^{-1} = \left(\frac{3}{\varepsilon^2 E}\right)^{\frac{1}{2}}\omega_R^{-1}.$$

These results confirm the predictions in Eqs. (7.3) and (7.4) and, since $\omega_I = \varepsilon\omega_R\left(\frac{1}{3}E\right)^{\frac{1}{2}}$, they show how the decay in amplitude is associated with the longer timescale $\tau = \varepsilon t$.

However, typical values [6] of $1-2\sigma$ for rubberlike materials which frequently are considered to be nearly-incompressible are 1.2—2.8×10^{-4}. These values indicate that (for finite as well as infinitesimal displacements) the amplitude of free oscillations will be halved after approximately $(2\pi)^{-1}(1-2\sigma)^{-\frac{1}{2}}\log_e 2 \simeq 7-10$ periods of oscillation. Thus, the effect is noticeable, even if the fractional decrease in frequency $\frac{1}{9}\varepsilon^2 E = \frac{1}{3}(1-2\sigma)$ is small compared with variations in the expression (4.11) of [1] for the period of the oscillation.

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