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Elasticity of cracked medium

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An elastic medium is considered with stable random frictionless cracks that can open or close depending on the load. The material is shown to obey equations of nonlinear elasticity which are homogeneous but not additive, defining a new class of materials, called pseudo-linear, more general than the linear ones. Constitutive relations are similar to linear ones where elastic constants are replaced by suitable functions. The basic properties of these functions are examined. The cases of isotropy and orthotropy are analysed in more detail.

Rozważa się ośrodek sprężysty z ustalonym losowym układem rys bez tarcia, które mogą zamykać się i otwierać zależnie od obciążenia. Pokazano, że materiał taki opisuje się równaniami nieliniowej sprężystości, które są jednorodne lecz nie addytywne; definiują one nową klasę materiałów, ogólniejszych od liniowych, zwanych pseudoliniowymi. Związki konstytutywne mają postać podobną do liniowych, lecz stałe sprężystości zastąpione są odpowiednimi funkcjami. Zbadano podstawowe własności tych funkcji. Zanalizowano bliżej przypadki izotropii i ortotropii.

Рассматривается упругая среда с установленной случайной системой трещин без трения, которые могут закрываться и открываться в зависимости от нагрузки. Показано, что такой материал описывается уравнениями нелинейной упругости, которые однородны, но не аддитивны; определяют они новый класс материалов, более общий чем линейные, называемый псевдолинейными. Определяющие уравнения имеют вид аналогичный линейным, но упругие постоянные заменены соответствующими функциями. Исследованы основные свойства этих функций. Анализируются подробнее случаи изотропии и ортотропии.

1. Preliminaries

At the present stage of research on the mechanics of cracks two types of approach prevail in the literature:

mathematical analysis of a lingle crack, crack expansion and fields of stress generated in the neighbourhood;

investigation of the bulk behaviour and constitutive equations of materials with numerous random micro-cracks (concrete, rock, brittle materials).

The latter problem, analysed in the present paper, has been dealt so far in a double way, the point of issue being that of, say, concrete structures designers (empirical approach with much simplified theoretical interpretation) or of theoretical mechanics of continua (mainly theories using the concept of internal state parameters). The first of the mentioned approaches is supported by extensive empirical work restricted, however, in general to beam and rod structures. Moreover, it lacks a cogent generalization for three-dimensional problems. The second one seems to be still far from a comprehensive solution for such a complicated material as concrete. Here, apart from different types of defects, there interact many other physical phenomena; on the other hand, the size of inhomogeneities calls for a nonlocal theoretical description and occasions many experimental difficulties (scattering of results, need for statistical treatment).

In the present paper, applying the method of abstraction, we confine ourselves to surface defects called cracks and derive constitutive equations for stable crack systems. We use the concept of the representative volume element (RVE), defined as a volume element sufficiently large as related to the mean crack size, including many random cracks uniformly scattered (homogeneous stochastic distribution) and, at the same time, small enough for a homogeneity of macro-fields to be assumed. More precisely, in a deterministic homogeneous reference medium loaded by the same body and boundary forces, the fields in the domain of RVE would be homogeneous.

The element exhibits macro- stress $\boldsymbol{\sigma}$, strain $\boldsymbol{\epsilon}$ and displacement $\mathbf{u}$,

$$\boldsymbol{\sigma} = [\acute{\boldsymbol{\sigma}}], \quad \mathbf{u} = [\acute{\mathbf{u}}], \quad \boldsymbol{\epsilon} = \operatorname{sym}\operatorname{grad}\mathbf{u}. \tag{1.1}$$

The notation is invariant, the supercripts refer to random (micro) quantities and [ ] denotes volume averaging. Note that the grad operator cannot be immediately applied to the random field of $\acute{\mathbf{u}}$ (showing discontinuities). The formulae (1.1) are quoted for illustration only since we do not make use of them in the sequel. The following relation holds for homogeneous macro-fields:

$$\mathbf{u} = \boldsymbol{\epsilon}(\mathbf{x}-\mathbf{x}_0), \tag{1.2}$$

where $\mathbf{u}$ is the displacement at the point $\mathbf{x}$ and $\mathbf{x}_0$ is the reference (rest) point; the product is understood as a vector space transformation.

Observe that we can make use of the concept of RVE in the double context: volume averaging or deterministic (phenomenological) description. Volume averaging is fundamentally different from the stochastic one, although in many cases both lead to the same results (property of ergodicity). We lose the nonlocal effect and must restrict ourselves to macro-homogeneous fields. Observe that the RVE is not a precise concept since the limit transition to unbounded medium, i.e. from micro- to macro-quantities, is not uniqely defined. In the second context (phenomenological description) averaging does not appear at all (we employ only macro-quantities); still, RVE remains useful making it possible to take account of "structural" phenomena. The relevant quantities are mostly interpreted as internal constitutive parameters. In particular, in the latter context we need not define a crack (as a discontinuity surface of the displacement vector) since the respective quantities do not appear in the formulae.

The parameters of this type have been assumed in various forms, as "porosity" parameters [3, 4], damage parameters [1, 2], tensor parameter of crack density [7], stress concentration or strain energy concentration factors [5, 6] and the like. All those concepts seem not to account sufficiently for the geometry of cracks, not describable in such a simple manner (by a single quantity).

The basic feature of cracks depends on the fact the these can open or close according to the direction and sense of loads. The principal stress trajectories pass round the cracks and separate volume "flaws" with approximately vanishing stresses; consequently, the medium could be replaced by an equivalent inhomogeneous (multiphase) one (Fig. 1). However, when the tension turns into compression, the forced inhomogeneity disappears, that is, the latter depends on the state of stress and this yields an essential generalization towards the classical theory of composite bodies. Further generalization is needed for materially inhomogeneous and cracked media (and this is the general case). Now we must have at our disposal the descriptions (1) of the geometry of cracks, (2) of the elastic tensor field; if we want to analyse the development and propagation of cracks, we need, moreover, (3) the distribution of strength properties. In the present paper we are not concerned with the

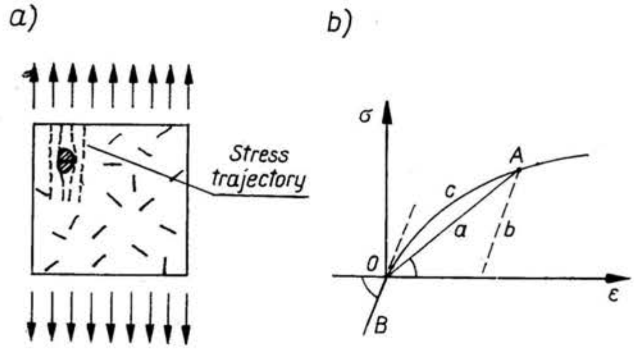

FIG. 1. a — stable cracks (unloading), b — plain material elastic-plastic, c — increasing cracks.

latter problem and this will prove to be equivalent to restriction to the elastic range. However, a special kind of nonlinear elasticity (called pseudo-linear) will be involved. Thus the problem will be seen to reduce to the investigation of pseudo-linear solids.

2. Pseudo-linear elasticity

Consider, within the infinitesimal theory, the one-to-one mapping

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\epsilon}), \quad \boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\boldsymbol{\sigma}), \tag{2.1}$$

characteristic of elastic behaviour, and examine whether this may hold for a linear-elastic cracked material. Obviously, this is not the case for cracks that may expand or move since irreversible processes would set in (change of elastic energy into crack surface energy). For the same reason no friction (i.e. energy dissipation) at cracks can be admitted. We reject also a possible infinite friction at rest which would prevent any mutual displacement of opposite surfaces at a closed crack and accompanying dissipation of energy. However, it would make the deformation process path-dependent and the unloaded state not free of eigenstrain- and stress fields. Thus we come at a model of glib cracks, called in the sequel the ideal crack system. The branch $\overline{AO}$ at unloading (Fig. 1b) depicts the possible behaviour of ideal stable cracks that have formed during the loading process (curve $\overline{OA}$). The change of sense of the load which makes the crack close results in the deviation of the line ($\overline{AOB}$). Note, by the way, that the so-called phase II calculation of reinforced concrete girders is based on the assumption of stable cracks. On the other hand, it is clear that the analysis of a stable system should precede the second stage of calculation for expanding cracks.

The essential point is that the stress vector always be normal to the crack surface (crack closed) or zero (crack open). Indeed, consider the (deterministic) boundary problem of elasticity for a cracked (multi-connected) body and assume the cracks form regular surfaces. Let the opposite crack surfaces be loaded by normal (vanishing or not) stress vectors $\hat{\boldsymbol{\sigma}}_{(n)}$ where $\mathbf{n}$ is the unit normal to the crack surface element (stress vectors are denoted by a superscript in order to make them distinctive from tensors and, similarly, for strains). Decompose the displacement vector in a normal to surface component $\mathbf{u}_{(n)}$ and a tangential $\mathbf{u}_{(t)}$. The elementary work yielded by two opposite crack surface elements dS an dS' is

$$\hat{\boldsymbol{\sigma}}_{(n)} \cdot \mathbf{u} dS + \hat{\boldsymbol{\sigma}}'_{(n)} \cdot \mathbf{u}' dS'$$

("primed" quantities refer to opposite elements). In an open crack $\hat{\boldsymbol{\sigma}}_{(n)} = \hat{\boldsymbol{\sigma}}'_{(n)} = 0$ and the work contribution disappears. In a closed crack an elementary work increase in the time dt for $dS = dS'$, $\hat{\boldsymbol{\sigma}}_{(n)} = -\hat{\boldsymbol{\sigma}}'_{(n)}$ (equilibrated forces in contact) amounts to

$$(\hat{\boldsymbol{\sigma}}_{(n)} \cdot d\mathbf{u}_{(n)} + \hat{\boldsymbol{\sigma}} \cdot d\mathbf{u}_{(t)}) dS + (\hat{\boldsymbol{\sigma}}'_{(n)} \cdot d\mathbf{u}'_{(n)} + \hat{\boldsymbol{\sigma}}'_{(n)} \cdot d\mathbf{u}'_{(t)}) dS'.$$

The first terms in the parantheses cancel since $d\mathbf{u}_{(n)} = d\mathbf{u}'_{(n)}$ and the second ones yield $\hat{\boldsymbol{\sigma}}_{(n)} \cdot (d\mathbf{u}_{(t)} - d\mathbf{u}'_{(t)}) dS$, i.e. they also disappear because of orthogonality. It follows that the forces at ideal cracks provide no work contribution in any case. Now, carrying out the standard proof of uniqueness of solution to the boundary problem (crack surfaces belonging to the boundary) we only need integral $\int (\hat{\boldsymbol{\sigma}} \cdot d\mathbf{u}) dS$ to disappear over the whole boundary for a difference of two eventual solutions to the same boundary problem (the stress vector $\hat{\boldsymbol{\sigma}}$ and $d\mathbf{u}$ refer to such a difference). But the integral disappears by the usual argument for the external boundary and by the one above for ideal cracks. This implies the uniqueness for a positive definite elasticity field and usual assumptions about smoothness. Upon formulating the boundary problem for the RVE and homogeneous macro-fields we come at reversible functions (2.1).

The second important property of the relations (2.1) is that these are homogeneous of degreee one

$$k\boldsymbol{\sigma} = \boldsymbol{\sigma}(k\boldsymbol{\epsilon}), \quad k\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(k\boldsymbol{\sigma}) \tag{2.2}$$

for any real non-negative k. For $k = 0$ it simply states that strains disappear at unloading and vice versa. For $k > 0$ the boundary problem of classical elasticity for k-fold $\hat{\boldsymbol{\sigma}}$ and/or $\mathbf{u}$ at boundary and zero body forces has the solution $k\mathbf{u}(\mathbf{x})$, $k\boldsymbol{\epsilon}(\mathbf{x})$, $k\boldsymbol{\sigma}(\mathbf{x})$ provided the above fields are a solution for $k = 1$ and the S_u and $S_{\hat{\sigma}}$ boundary (for prescribed $\mathbf{u}$ and $\boldsymbol{\sigma}$ at the boundary, consecutively) remains fixed. The latter condition is seen to hold in cracks for $k > 0$ since $k\hat{\boldsymbol{\sigma}}_{(n)}$, $\hat{\boldsymbol{\sigma}}_{(n)} \neq 0$ cannot make the crack open and if the crack has been open $\hat{\boldsymbol{\sigma}}_{(n)} = 0$, $k\hat{\boldsymbol{\sigma}}_{(n)} = 0$. In other words, for the relations (2.2), the closed cracks remain closed and the open ones open. Thus the k-fold field, $k > 0$ (leaving the internal constitution unchanged, i.e. open and closed cracks) provides a solution and by the precedent argument the solution is unique; consequently, the relations (2.2) hold.

On the other hand, the relations (2.1) are not additive, that is, in general $\boldsymbol{\sigma}(\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2) \neq$ $\neq \boldsymbol{\sigma}(\boldsymbol{\epsilon}_1) + \boldsymbol{\sigma}(\boldsymbol{\epsilon}_2)$ (unless $\boldsymbol{\epsilon}_2 = (k-1)\boldsymbol{\epsilon}_1$, $k > 0$), and similarly for $\boldsymbol{\epsilon}$, since the superposition of $\boldsymbol{\epsilon}_1$ would change the open- and closed- crack system which is equipollent to changing the internal structure of the material. Thus the operators (2.1) are no more linear since homogeneity and additivity are both needed for linearity; nevertheless, they preserve a certain item of linearity (homogeneity). The homogeneous, yet not additive, relations (2.1) will be called in the sequel *pseudo-linear*. Note that homogeneity is a weaker item than additivity since the latter implies the former for rational $\mathbf{k}$ in finite-dimensional spaces and nothing similar occurs in the "reverse" direction. Pseudo-linear elastic media define a class of materials much wider than the linear elastic ones (the latter form a subclass of those only). Elastic bodies with ideal stable crack systems are seen to be one of the possible representatives of this class. In the sequel we shall examine some important features of these materials.

According to the Euler theorem on homogeneous functions,

$$\boldsymbol{\sigma} = (\operatorname{grad}\boldsymbol{\sigma})\boldsymbol{\epsilon}, \tag{2.3}$$

all quantities and operations referring to the strain space (i.e. we consider the function $\boldsymbol{\sigma}(\boldsymbol{\epsilon})$, not $\boldsymbol{\sigma}(\mathbf{x})$!). By the same theorem the function grad $[\boldsymbol{\sigma}(\boldsymbol{\epsilon})]$ is homogeneous of degree 0, that is, its value does not change for a given direction and sense of the argument $\boldsymbol{\epsilon}$ looked upon as a vector in the 9-dimensional strain space. Observe that the Euler identity is restricted here to all $k > 0$ and $\boldsymbol{\epsilon} \neq 0$, where the function $\boldsymbol{\sigma}(\boldsymbol{\epsilon})$ is supposed to be differentiable; at the point 0 there appears a nonregularity of $\boldsymbol{\sigma}(\boldsymbol{\epsilon})$ and discontinuity of grad $[\boldsymbol{\sigma}(\boldsymbol{\epsilon})]$ (cf. the vertex point O of $\overline{AOB}$ in Fig. 1b). In other words, the function grad $\boldsymbol{\sigma}$ preserves its value on the radius determined by the "directional" vector $\mathbf{e}$, defined by

$$\boldsymbol{\epsilon} = |\boldsymbol{\epsilon}|\mathbf{e}, \quad |\boldsymbol{\epsilon}| = \sqrt{\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}}, \tag{2.4}$$

where $|\boldsymbol{\epsilon}|$ may be interpreted as the length of the vector $\boldsymbol{\epsilon}$. For opposite senses of the unit vector $+\mathbf{e}$, $-\mathbf{e}$, the function assumes, in general, different values (this recalls the behaviour of a solenoidal surface). That is to say, the function in question depends only on the ratio of the components of $\boldsymbol{\epsilon}$ provided the signs of the components do not change. For fixed spatial orientation of $\boldsymbol{\epsilon}$, i.e. of the principal directions of the tensor $\boldsymbol{\epsilon}$, it depends on the ratio $\varepsilon_1 : \varepsilon_2 : \varepsilon_3$ of principal strains, under the above restriction. The sign-dependence, ensured by the constraint $k > 0$, reflects unilateral internal constraints imposed by closing cracks.

Thus, for proportional paths of loading, say $\boldsymbol{\sigma} = \varkappa\boldsymbol{\sigma}_0$, constitutive relations are linear only as long as $\varkappa$ does not change the sign. When we pass in this manner from $\boldsymbol{\sigma}$ to $-\boldsymbol{\sigma}$ we obtain the "broken-line" dependence (cf. Fig. 1b); if we pass round the O point, the line will be "smoothed" in the vicinity and becomes regular. For non-proportional loading paths, that is when we cross the radii $\mathbf{e}$, the relation $\boldsymbol{\sigma}(\boldsymbol{\epsilon})$ is curvilinear; for instance, $\overline{AO}$ (in Fig. 1) becomes a curve at one-directional traction under simultaneous constant (non-zero) lateral compression.

The vectors $\mathbf{e}$ define a unit sphere in the 9-space and we can consider functions on this sphere instead of on the $\boldsymbol{\epsilon}$'s (Fig. 2). Let us define, in analogy to the elastic tensor, the function

$$\operatorname{grad}[\boldsymbol{\sigma}(\boldsymbol{\epsilon})] = \mathbf{C}(\boldsymbol{\epsilon}) \tag{2.5}$$

$\mathbf{C}$ denoting, according to interpretation, a 9×9 matrix-valued function of unit vectors $\mathbf{e}$ (in the 9-space) or a fourth order tensor-valued function of normed second order tensors **e** (i.e. tensors of unit magnitude) in the 3-space. A similar argument holds for the relation $(2.2)_2$, where $\mathbf{s} = \boldsymbol{\sigma}/\sqrt{\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}}$, consequently, we come at the following constitutive relations for pseudo-linear elastic media:

$$\boldsymbol{\sigma} = \mathbf{C}(\mathbf{e})\boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} = \mathbf{S}(\mathbf{s})\boldsymbol{\sigma}, \tag{2.6}$$

where a change of sense of **e**, **s** leads in general to a change of $\mathbf{C}(\mathbf{e})$, $\mathbf{S}(\mathbf{s})$. The latter functions preserve fixed values on the radii in the 9-space, therefore they can be visualized simply

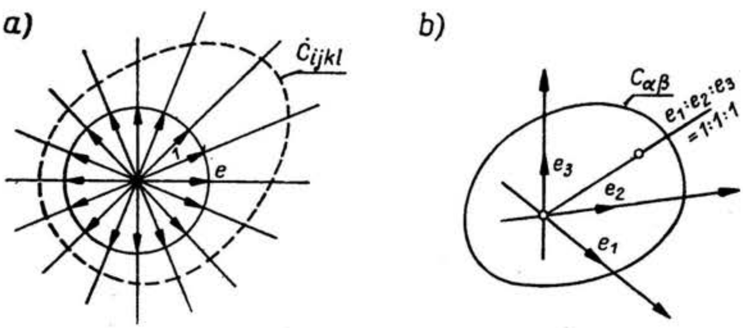

FIG. 2.

in the graph (cf. Fig. 2). If the plotted line depicts -sphere, classical elasticity follows. Thus, pseudo-linearity leads to an outer similarity of formulae to the linear-elastic ones; however, **C** and **S** are now functions of the 9-direction.

The functions under consideration may be shown to possess similar properties as in linear elasticity. For the symmetric tensors $\boldsymbol{\epsilon}$, $\boldsymbol{\sigma}$, evidently,

$$C_{ijkl}(\mathbf{e}) = C_{ijlk}(\mathbf{e}) = C_{jikl}(\mathbf{e}) \tag{2.7}$$

and similarly for **S**. Consequently, the matrices reduce to the 6×6 ones, and the number of different functions reduces from 81 to 36. For nonlinear relations of the type (2.1) not to be in contradiction with the first law of thermodynamics, these should be derivable from the elastic potential $W(\boldsymbol{\epsilon})$ (similarly for $\boldsymbol{\sigma}$),

$$\boldsymbol{\sigma} = \operatorname{grad} W(\boldsymbol{\epsilon}). \tag{2.8}$$

Combining this with Eq. (2.5) we come, analogously to classical elasticity, at the relationship

$$C_{ijkl}(\mathbf{e}) = C_{klij}(\mathbf{e}). \tag{2.9}$$

It follows that the 6×6-matrices are symmetric and the number of functions reduces to 21. Note that we could also derive the above property from the assumption about the local existence of the potential which is preserved under taking volume integrals. Equating the total potential to the work of $\boldsymbol{\sigma}$ on $\boldsymbol{\epsilon}$ we would obtain the relation (2.9). Finally, in order to make the strain energy

$$E = \frac{1}{2}\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\epsilon}\cdot\mathbf{C}(\mathbf{e})\boldsymbol{\epsilon}$$

positive, the matrix $\mathbf{C}(\mathbf{e})$ (and similarly for $\mathbf{S}(\mathbf{s})$) should be positive definite for any **e**.

3. Isotropy, orthotropy

In order to specialize further the form of the functions $\mathbf{C}(\mathbf{e})$, $\mathbf{S}(\mathbf{s})$ we must make some assumptions about the geometry of the crack system. Suppose the latter is isotropic in the meaning that the relations (2.6) are invariant under orthogonal transformation. Note that such a definition does not necessitate any direct description of the crack geometry. Thus we postulate

$$\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T = C(\mathbf{Q}\mathbf{e}\mathbf{Q}^T)\mathbf{Q}\boldsymbol{\epsilon}\mathbf{Q}^T, \tag{3.1}$$

where $\mathbf{Q}$ denotes the orthogonal transformation, i.e. an element of the full group of orthogonal transformations (in the sequel we confine ouerselves to the relation $(2.6)_1$; clearly, a similar argument holds for $(2.6)_2$).

By the known representation theorem for isotropic tensor-valued tensor functions of the type (2.1) where $\boldsymbol{\epsilon}$, $\boldsymbol{\sigma}$ are second-order symmetric tensors in the 3-space, every proper vector of $\boldsymbol{\epsilon}$ is also a proper vector of $\boldsymbol{\sigma}$. It follows that the relations (2.1) reduce to the ones between principal values (i.e. proper numbers) of tensors and, using the relations (2.6), we obtain

$$\sigma_\alpha = C_{\alpha\beta}(e_\gamma)\varepsilon_\beta, \quad \alpha, \beta, \gamma = 1, 2, 3. \tag{3.2}$$

Here σ_α, ε_α, e_α are principal stresses and strains (denoted by simple indices) and $f(e_\gamma)$ is interpreted as $f(e_1, e_2, e_3)$. Hence, employing coordinates in the basis of the principal directions of $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$, we can look upon $[\sigma_\alpha]$, $[\varepsilon_\alpha]$ as 9- or 6-vectors with only three first components different from 0 and $[C_{\alpha\beta}]$ as the respective 3×3-submatrix of a 9×9- or 6×6-elastic stiffness matrix. In the said basis the number of non-zero functions $C_{ijkl}(e_\gamma)$ amounts to 9 and these are functions of the type C_{iikk} (no sum). In view of the relation (2.9), 6 of them are independent and later in the sequel we shall see to what extent they are actually different.

Thus, for isotropy, the elastic tensor function in the relations (2.6) takes the form

$$\mathbf{C}(\mathbf{e}) = \mathbf{C}(e_\gamma, \boldsymbol{\nu}_\gamma) = \sum_{\alpha,\beta} C_{\alpha\beta}(e_\gamma)\boldsymbol{\nu}_\alpha \otimes \boldsymbol{\nu}_\alpha \otimes \boldsymbol{\nu}_\beta \otimes \boldsymbol{\nu}_\beta, \tag{3.3}$$

where $\boldsymbol{\nu}_\gamma$ are unit vectors in the principal directions of $\mathbf{e}$ (consequently, functions of $\mathbf{e}$). If $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ are expressed in their common principal frame $\boldsymbol{\nu}_\gamma$, the component representation (3.2) follows. In a general frame of orthogonal unit vectors $\mathbf{n}_i$ where

$$\boldsymbol{\sigma} = \sigma_{ij}\mathbf{n}_i \otimes \mathbf{n}_j, \quad \boldsymbol{\epsilon} = \varepsilon_{kl}\mathbf{n}_k \otimes \mathbf{n}_l,$$

inserting this and Eq. (3.3) in the relation $(2.6)_1$, we obtain the constitutive equation in components:

$$\sigma_{ij} = \sum_{\alpha,\beta} C_{\alpha\beta}(e_\gamma)\nu_{\alpha i}\nu_{\alpha j}\nu_{\beta k}\nu_{\beta l}\varepsilon_{kl} \tag{3.4}$$

(usual summing over k, l) where $\nu_{\gamma i} = \boldsymbol{\nu}_j \cdot \mathbf{n}_i$ are coordinates of the principal unit vectors of $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$, and

$$C_{ijkl}(\mathbf{e}) = \sum_{\alpha,\beta} C_{\alpha\beta}(e_\gamma)\nu_{\alpha i}\nu_{\alpha j}\nu_{\beta k}\nu_{\beta l}. \tag{3.5}$$

We obtain again 21 component functions (in view of Eqs. (2.7) and (2.9)); however, these are all seen to be derived form the 6 basic ones by simple superposition, i.e. multiplication by transformation coefficients and summing up.

In the isotropic case, for constant e_γ's (proportional loading) C_{ijkl} remain constant in a convected coordinate system, co-rotating with $\boldsymbol{\epsilon}$- and $\boldsymbol{\sigma}$-ellipsoids. Thus the system of open and closed cracks follows the rotation of $\boldsymbol{\sigma}$ and is fixed in the respective spatial orientation, that is, the internal constitution adapts itself to loads. If we could "freeze up" the momentary structure, the functions (3.5) would become constant, i.e. they would become connected with a definite spatial orientation and the material would exhibit linear elastic orthotropy. The number of non-zero components (3.5) would amount to 9 in the coordinate system connected with material symmetry planes (symmetry of the open and closed crack system). Shear components of $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ would not disappear, in general, in the system of "original" principal unit vectors, contrary to the "unfrozen" structure.

The functions $C_{\alpha\beta}(e_\gamma)$ are not all different. From the relations (3.2) it follows that these are invariant under orthogonal transformation since they do not depend on the directors $\boldsymbol{\nu}_\alpha$. Let us perform the particular transformation carrying the unit frame of $\boldsymbol{\nu}_\alpha$ in itself, yet cenverting principal axes. Refer the quantities in the relations (3.2) not to the moving frame but to the fixed ("original") one. Then the transformation results in interchanging the respective indices. Denoting, for the sake of simplicity, for a while: e_1, e_2, e_3 by a, b, c, we obtain two following groups of equalities:

$$(3.6)\qquad C_{11}(a,b,c) = C_{11}(a,c,b) = C_{22}(b,a,c) = C_{22}(c,a,b) = C_{33}(b,c,a) = C_{33}(c,b,a),$$

$$(3.6')\qquad C_{12}(a,b,c) = C_{21}(b,a,c) = C_{23}(c,a,b) = C_{32}(c,b,a) = C_{13}(a,c,b) = C_{31}(b,c,a)$$

(apart from $C_{\alpha\beta} = C_{\beta\alpha}$ for not interchanged arguments). One must keep in mind that the indices in C and the locus of the argument in the parentheses are connected with the consecutive fixed vectors $\boldsymbol{\nu}_\alpha$. Thus there are altogether only two different functions, say $C_{11}(e_1, e_2, e_3)$ and $C_{12}(e_1, e_2, e_3)$ and this recalls two elastic constants in linear isotropy.

The above argument suggests graphical representation according to Fig. 2b, in the 3-space, where the functions $C_{11}(\mathbf{e})$, $C_{12}(\mathbf{e})$ depend on the directions $e_1:e_2:e_3 = \varepsilon_1:\varepsilon_2:\varepsilon_3$. The number of independent arguments reduces in fact to 2, say the spherical coordinates on the unit sphere of $\mathbf{e}$. Of course, the arguments may be selected arbitrarily; for example, we could take also the ratios $\varepsilon_2/\varepsilon_1 (= e_2/e_1)$, $\varepsilon_3/\varepsilon_1$ for $\varepsilon_1 \geqslant \varepsilon_2 \geqslant \varepsilon_3$. The symmetry properties of the surface C_{11} follow from Eq. (3.6a), viz. $C_{11}(e_1, e_2, e_3) = C_{11}(e_1, e_3, e_2)$, that is, the surface is symmetric with regard to the plane $e_2 = e_3$, whereas the surface C_{12} is symmetric respective to the plane $e_1 = e_2$ in virtue of $C_{12} = C_{21}$.

A small increase of ε_1 in the relations (3.2) will be produced by the slightest increase of σ_1 under possibly severe concentration of stress, i.e. a large volume of flaws in Fig. 1. This occurs for a possibly great amount of open cracks, that is, for uniform tension. The reasoning points out that the extremal points (if any) of the C_{11} surface are expected to lie on the axis $e_1:e_2:e_3 = 1:1:1$, and, similarly, for C_{12}.

Let us now pass to the anisotropic case. Remark that the distribution of cracks is given in general by a function of spatial orientation (i.e. of the normal to crack vector) and needs not exhibit any symmetry. However, the cracks often will be produced by a definite stress tensor, brought about by a pronounced load (exceeding considerably the following ones). Then the crack system will show the same sort of symmetry as a second order symmetric tensor, i.e. in general orthotropy. Speaking of a crack system symmetry, we mean of course averaged quantities, that is, the property which manifests itself in the invariance of macro--stress and strain at certain orthogonal transformations. At any rate, orthotropy may be looked upon as a first approximation property of the actual crack distribution function (which, possibly, is not exactly known). This motivates a special concern in the orhtotropic symmetry.

Consider a symmetry plane of the crack system and fix the orthonormal basis in such a manner that this be the $x_1 x_2$-plane. Take the following orthogonal transformation:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

i.e. reflection in the $x_1 x_2$-plane. Consequently,

$$\mathbf{Q}\boldsymbol{\epsilon}\mathbf{Q}^T = \mathbf{Q}\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ & \varepsilon_{22} & \varepsilon_{23} \\ & & \varepsilon_{33} \end{bmatrix}\mathbf{Q}^T = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & -\varepsilon_{13} \\ & \varepsilon_{22} & -\varepsilon_{23} \\ & & \varepsilon_{33} \end{bmatrix}$$

and, similarly, for $\boldsymbol{\sigma}$, $\mathbf{e}$. Expressing the relation (3.1) in the 6-vector-matrix form we have

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -\sigma_{23} \\ -\sigma_{31} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C'_{1111} & C'_{1122} & C'_{1133} & C'_{1123} & C'_{1131} & C'_{1112} \\ & C'_{2222} & C'_{2233} & C'_{2223} & C'_{2231} & C'_{2212} \\ & & C'_{3333} & C'_{3323} & C'_{3331} & C'_{3312} \\ & & & C'_{2323} & C'_{2331} & C'_{2512} \\ & & & & C'_{3131} & C'_{3112} \\ & & & & & C'_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ -\varepsilon_{23} \\ -\varepsilon_{31} \\ \varepsilon_{12} \end{bmatrix},$$

where for $C_{ijkl}(\mathbf{QeQ}^T)$ we have put

$$C'_{ijkl} = C_{ijkl}(e_{11}, e_{22}, e_{33}, -e_{23}, -e_{31}, e_{12}).$$

The above vector-matrix equation must hold along with the "original" one $(2.6)_1$ with not scored C_{ijkl} and all terms positive, for each ε_{ij}. Similar equations may be established for reflexions in the $x_3 x_1$-and $x_3 x_2$-planes. Thus we obtain by turns 6 orginal and 6 transformed equations for consecutive σ_{ij} and for each symmetry. After having brought the equations to the same sign of the left hand side, equating in each pair of the corresponding equations the coefficients at respective ε_{ij}, we come at a set of equalities between the C_{ijkl} and C'_{ijkl}. For example, for the $x_1 x_2$-plane symmetry and the pair of equations for σ_{11} we have $C_{1111} = C'_{1111}, \dots, C_{1123} = -C'_{1123}, \dots$, i.e.,

$$C_{1111}(e_{11}, e_{22}, e_{33}, e_{23}, e_{31}, e_{12}) = C_{1111}(e_{11}, e_{22}, e_{33}, -e_{23}, -e_{31}, e_{12}),$$
$$C_{1123}(e_{11}, e_{22}, e_{33}, e_{23}, e_{31}, e_{12}) = -C_{1123}(e_{11}, e_{22}, e_{33}, -e_{23}, -e_{31}, e_{12})$$

for each e_{ij}. From the first equality we infer that C_{1111} can depend only on $|e_{23}|$, $|e_{31}|$ etc. After performing similar calculations for all symmetry planes we arrive at the following results:

$$\begin{aligned} &C_{ijij}(e_{11}, e_{22}, e_{33}, e_{23}, e_{31}, e_{12}) = C_{ijij}(e_{11}, e_{22}, e_{33}, |e_{23}|, |e_{31}|, |e_{12}|), \\ (3.7) \qquad &C_{iikk}(e_{11}, e_{22}, e_{33}, e_{23}, e_{31}, e_{12}) = C_{iikk}(e_{11}, e_{22}, e_{33}, |e_{23}|, |e_{31}|, |e_{12}|), \\ &C_{ikkl} = 0 \text{ — for remaining systems of } ijkl \end{aligned}$$

(no sum over repeated indices). The latter identities follow from $C_{ijkl} = -C_{ijkl}$ for the same arguments (including $|e_{23}|, \ldots$) when we take account of all symmetries. It follows that all diagonal terms and terms of the first 3×3 submatrix do not depend on the sense of shear arguments. On the whole, the elastic matrix takes the form

$$
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\
 & C_{2222} & C_{2233} & 0 & 0 & 0 \\
 & & C_{3333} & 0 & 0 & 0 \\
 & & & C_{2323} & 0 & 0 \\
 & & & & C_{3131} & 0 \\
 & & & & & C_{1212}
\end{bmatrix}
\tag{3.8}
$$

resembling the linear-elastic matrix with 9 independent constants, here replaced by the functions (3.7). However, a simple representation of the type (3.3) is no more possible since the component functions depend on both the principal directions of ϵ (determining open and closed cracks) and of those for orthotropic crack distribution. It would come to light if we expressed the functions (3.7) along with their arguments in an arbitrary frame. A closer investigation, disclosing possibly more refined symmetry properties, would require a description of crack geometry by means of, say, crack distribution functions or respective correlation functions.

References

1. F. Bastenaire, *Étude critique de la notion de dommage appliqueé à une classe ètendue d'essais de fatigue*, Coll. Fatigue, IUTAM, Stockholm, Mai 1955.
2. Cz. Eimer, *Wytrzymałość reologiczna betonu w świetle hipotezy uszkodzenia* (Rheologic strength of concrete in the light of damage hypothesis) [in Polish], Arch. Inż. Ląd., **17**, 15–31, 1971.
3. R. J. Green, *A plasticity theory for porous solids*, Int. J. Mech. Sci., **14**, 215–224, 1972.
4. L. M. Kačanov, *Osnovy mechaniki razrušenia*, Izd. Nauka, Moskwa 1974.
5. M. K. Kassir, G. C. Sih, *Three dimensional crack problems*, Noordhoff Int. Publ. 1975.
6. G. C. Sih, *Some basic problems in fracture mechanics and new concepts*, J. Eng. Fracture Mech., **5**, 365–377, 1973.
7. A. A. Vakulenko, M. L. Kačanov, *Kontinualnaja tieoria sried z treščinami*, MTT, **4**, 1959–1966, 1971.

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