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Optimal design of weakly curved compressed bars with Maxwell type creep effects

R. WOJDANOWSKA (KRAKÓW)

THE OPTIMAL shape of the clamped and two-hinged bars compressed by axial force, having an initial deflection line and subjected to creep buckling is determined. A linear Maxwell type creep is considered and as an optimization criterion the minimum logarithmic rate of creep is assumed. The exact solution under the restriction to certain particular initial deflection lines of the bar is obtained. An analogy to elastic solutions is demonstrated. The optimal dimensionless section $\bar{F}(x)$ for the cases of buckling from a convergence plane, buckling in a convergence plane and geometrically similar sections is presented graphically.

1. Introductory remarks

IN THE CASE of structures working at elevated temperatures or structures made of materials exhibiting rheological properties already at room temperature it is required, in optimal structural design, to take into account the rheological effects. Classification of the optimum design problems in rheology and some simple examples of such design are given in ŻYCZKOWSKI's paper [16].

A certain group of auxiliary conditions of optimal structural design in creep conditions is connected with creep buckling. Some problems of this type were considered in [12] and [15] in relation with the Rabotnov-Shesterikov creep buckling theory for a bar being originally straight. A different approach, based on the Kempner-Hoff's buckling theory [5, 7] and an assumption of the small initial curvature of the compressed bars are found in the paper [11] concerning optimum design of simple lattice structures. A comparison of different creep buckling theories may be found in the review papers of HULT [5], HOFF [4] and ŻYCZKOWSKI [13].

In this paper we assume the buckling theory of weakly curved bars but we restrict our considerations to the linear Maxwell type creep. Such a problem for an axially compressed prismatic bar was formulated in 1946 by FREUDENTHAL [2] and RŽANICYN [10]. They confirmed the fact that the growth of deflections with time has an exponential character. In our paper we deal with the optimization of the shape of such a bar (optimal distribution of the cross-section along the axis). We are seeking a minimum volume of the bar at a fixed compressive force and fixed logarithmic creep rate. We restrict our considerations to the cases when a rate of creep is constant (independent of spatial and time variables) and therefore the choice of a corresponding norm for the rate of creep is meaningless. The optimum design problem considered belongs to the variational calculus problems; we obtain the exact solutions under the assumption of certain particular initial deflection lines of the bar.

We show that the problem formulated in close to the optimum design problem of straight elastic bars, originated by Lagrange and examined in detail by CLAUSEN, ČENCOV [1] and NIKOLAI [8]. KELLER [6] considered statically indeterminate cases of the bar and supports. However, as OLHOFF and RASMUSEEN [9] have pointed out, Kellys' solution was erroneous since it did not take into account the different possible forms of the stability loss and they presented a correct solution based on the so-called "bimodal optimization". In our paper we deal with statically determined supports of bars only and in this case therefore, the bimodal optimization does not apply.

2. Formulation of the problem

Consider first a creep buckling problem for non-prismatic, two-hinged and clamped at one end bars of small curvature subjected to the action of axial force. Next we formulate in detail the problem of optimization. In this problem we use the linear Maxwell type law

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\lambda} \tag{2.1}$$

in which ε, σ, E and λ denote, respectively, the strain, stress, Young's modulus and material constant characterizing the viscous properties of the material. A dot above the variable denotes differentiation with respect to time.

After introducing dimensionless time $\tau = \frac{E}{\lambda} t$, restricting to small deflections and assuming $\dot{\varkappa} = -\dot{w}''$, $M = Pw$ (P is compressing force), the differential equation for the deflection line of the bar obeying the linear creep law (2.1), borrowed from [14], assumes the form

$$P\dot{w} + Pw = -\dot{w}''EJ. \tag{2.2}$$

In Eq. (2.2) M and $\varkappa$ denote the bending moment and curvature, respectively, $J = J(x)$ is the moment of inertia of the cross-section of the bar at a point x, $\dot{M}$ and $\dot{\varkappa}$ denote derivatives with respect to dimensionless time τ.

We have the following boundary conditions: for a bar clamped at one end,

$$w(l, \tau) = 0, \quad w'(0, \tau) = 0, \tag{2.3}$$

and for a two-hinged bar,

$$w(l, \tau) = 0, \quad w(-l, \tau) = 0. \tag{2.4}$$

The initial condition assumes the form

$$w(x, 0) = w_+(x) \tag{2.5}$$

in which the function $w_+(x)$ describes the instant (elastic) deflection of the bar.

We confine our study to such solutions of the partial differential equation (2.2) which we may obtain, using the method of separation of variables,

$$w(x, \tau) = w_1(x) \cdot w_2(\tau) \tag{2.6}$$

i.e for creep buckling, retaining geometrical similarity of the deflection line in the particular moments of time. In this way we obtain a relatively simple exact solution: yet, it is restricted to the certain precisely determined initial deflection lines. After introducing Eq. (2.6) into Eq. (2.2) we obtain a system of two equations

$$P+P\frac{w_2(\tau)}{\dot{w}_2(\tau)} = k^2, \qquad EJ(x)\frac{w_1''(x)}{w_1(x)} = -k^2. \tag{2.7}$$

The first equation of this system allows, after integration, for the determination of the function $w_2 = w_2(\tau)$ with accuracy to the constant

$$w_2(\tau) = C_1 \exp \frac{P}{k^2-P}\tau. \tag{2.8}$$

Equations (2.8) and (2.6) make it possible to evaluate the strain rate at each point x and each moment of time τ; the constant k^2 may be chosen correspondingly large so as to give the minimum strain rate. The quantity $P/(k^2-P)$ will be called a logarithmic creep rate:

$$\frac{d}{d\tau}(\ln w) = \frac{d}{d\tau}[\ln w_1(x)+\ln w_2(\tau)] = \frac{P}{k^2-P}. \tag{2.9}$$

The problem of optimization is formulated as follows: we are looking for a shape of the bar of the smallest volume which, for a given force P, will demonstrate a given logarithmic creep rate, i.e. a given value of k^2. In dual formulation we look for a minimal

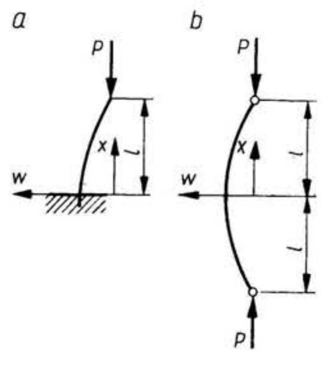

FIG. 1.

rate of creep at a given volume and force P. A construction of the formulae (2.8) and (2.9) shows that the maximal value of k^2 guarantees a minimum creep rate at each point x and each moment of the time τ, so we obtain here the absolute minimum, constant in time and independent of the assumed norm of the creep rate expressed in terms of the variable x [16]. Thus we seek the minimum of the functional

$$V = \int_0^l F(x)dx = \int_0^l k_1 I^q(x)dx \tag{2.10}$$

at fixed values of k^2 and P.

In the above formula k_1 is a cross-section coefficient, the exponent q determines the nature of the convergence of the bar. The formula (2.10) defines the volume of the clamped bar, in the case of a two-hinged bar the volume is determined by means of an analogous integral in the limits of integration from $(-l)$ to (l).

The exponent q (Fig. 2) equals: $q = 1$ for a plane-convergent bar of a constant height of the cross-section (buckling from the convergence plane), $q = \frac{1}{2}$ for a uniformly convergent bar (sections geometrically similar), $q = \frac{1}{3}$ for a plane-convergent bar of a constant width of the cross-section (buckling in a plane of convergence).

The function $I(x)$ must satisfy the second differential equation of the system (2.7); however, the minimization of the functional with an auxiliary condition is not necessary

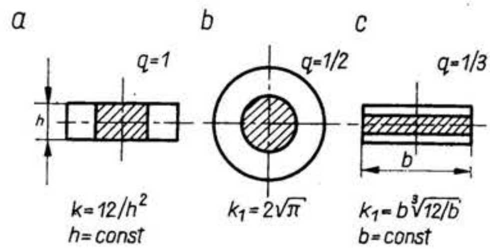

FIG. 2.

here, we can simply evaluate $I(x)$ from the second equation of the system (2.7) and insert it into Eq. (2.10)

$$V = \int_0^l k_1\left(\frac{k^2}{E}\right)\left(\frac{-w_1}{w_1''}\right)^q dx = \int_0^l \varphi\left(\frac{w_1}{w_1''}\right)dx. \tag{2.11}$$

We then obtain the functional which is analogous to the functional describing the optimal shape of the compressed elastic bar.

In quest of its minimum we use the ČENCOV method [1]. The Euler-Lagrange equations conditioning the minimum of the functional (2.11) may then be written in the simple form

$$v w_1'' - w_1 v'' = 0. \tag{2.12}$$

For conservative loads, as GAJEWSKI and ŻYCZKOWSKI have shown [3], the integral of Eq. (2.12) assumes the form

$$-q\nu w_1^{q-1}\cdot(w_1'')^{-(q+1)} = C, \tag{2.13}$$

where C denotes the integration constant and $\nu = k_1(k^2/E)^q$. Depending on the manner of supporting the ends of the bar, the boundary conditions (2.3) or (2.4) should be added to Eq. (2.13).

3. Particular solutions

Integration of Eq. (2.13) with the boundary conditions (2.3) and the initial condition (2.5) leads to the solution

$$w = f_+\left(1-\frac{x^2}{l^2}\right)\exp\frac{P}{k^2-P}\tau \quad \text{for } q = 1 \tag{3.1}$$

in which $f_+ = w_+(0)$ denotes an elastic deflection at the point $x = 0$,

$$x = \frac{2l}{\pi}s^{1/3}\sqrt{1-s^{2/3}}+\frac{2l}{\pi}\arcsin\sqrt{1-s^{2/3}} \quad \text{for } q = \frac{1}{2} \tag{3.2}$$

in which the parameter s equals

$$s = \left(\frac{3\pi}{4l\sqrt{3\left(\frac{\nu}{2C}\right)^{2/3}}}\right)^{3/2} w_1 \tag{3.3}$$

and

$$x = \frac{1}{2}(1+\sqrt{s_1})^{1/2}(2-\sqrt{s_1}) \quad \text{for } q = \frac{1}{3} \tag{3.4}$$

with

$$s_1 = \left[\frac{4\left(-\frac{\nu}{3C}\right)^{3/4}}{B}\right]^2 w_1, \tag{3.5}$$

where B denotes the integration constant.

For the above values of the exponent q, one may evaluate from the second equation of the system (2.7) the moment of inertia $I(x)$.

Using the relations occuring for $\tau = 0$, from Eq. (2.8) we have $w_2(0) = C_1$, and from Eqs. (2.6) and (3.1) (e.g. for $q = 1$) one may evaluate $w(x, 0) = w_1(x)w_2(0) = C_1 w_1(x) = w_+$.

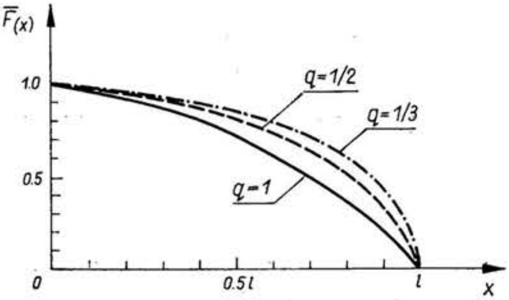

FIG. 3.

Introducing $w_1(x) = w_+(x)/C_1$ into $I(x)$ one may further determine $F(x) = k_1 I^q(x)$ (for $q = 1: F(x) = k_1 I(x)$) and introduce a dimensionless cross-section $\bar{F}(x)$.

The optimal magnitude of the dimensionless cross-section $\bar{F}(x)$ for the particular values of the exponent q is presented graphically in Fig. 3.

4. Determinations of the corresponding initial deflection line

The initial deflection line and the line of the elastic deflection are related by the equation

$$(4.1) \qquad \varkappa_+ - \varkappa_- = \frac{M_+}{EI}.$$

The above equation results from the superposition of the elastic deflection on the initial shape. In this equation $M_+ = Pw_+$, $\varkappa_- = \varkappa_-(x) = -w_-''(x)$ denotes an initial curvature of the bar before loading ($\tau = \_0$) which, after loading, becomes $\varkappa_+ = \varkappa_+(x)$ (curvature resulting from elastic deflection of the bar). The magnitude of the deflection function at the point $x = 0$ (elastic bending deflection) is denoted by $f_+ = w_+(0)$ and initial deflection by $f_- = w_-(0)$. The initial deflection f_- is assumed to be known. The examplary diagrams of loading and deflection in terms of time are presented in Figs. 4 and 5.

Integration of Eq. (4.1) using the relation $I = -\dfrac{k^2}{E}\dfrac{w_+}{w_+''}$ evaluated from Eq. (2.7) after loading, leads to the solution $w_- = [(k^2-P)/k^2]w_+$ where, in the evaluation of the integration constants, the boundary conditions for the clamped at one end bar (2.3) were

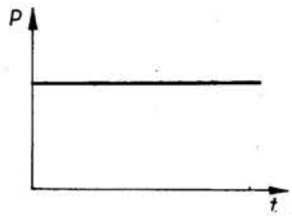

FIG. 4.

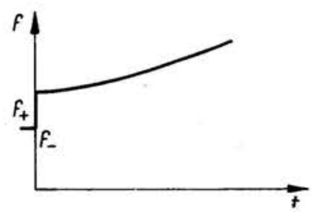

FIG. 5.

used, and the function $w_+ = w_+(x)$ is the deflection line of the bar after loading under the initial condition (2.5). The deflection f_+ is expressed in terms of the initial deflection f_- by means of the relation: $f_+ = f_- k^2/(k^2-P)$.

Below, as an example, we present the initial deflection line for a case $q = 1$:

$$(4.2) \qquad w_- = f_+ \frac{k^2-P}{k^2}\left(1-\frac{x^2}{l^2}\right) = f_-\left(1-\frac{x^2}{l^2}\right)$$

The optimal solution obtained is restricted then to such an initial deflection line.

5. Final remarks

In this paper the optimal shape of an axially compressed bar of small initial curvature subjected to creep buckling is determined. The considerations were restricted to the certain particular function defining the initial deflection line. Under this assumption an analogy with the elastic solutions was demonstrated. Solutions for the elastic range are also optimal for linear creep of the Maxwell type, economy of the material is identical.

Assuming other initial deflection lines leads to numerical complications and the conclusions presented here are no longer valid.

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TECHNICAL UNIVERSITY OF CRAKOV.

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