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A method for computing compressible flows past a profile set between permeable walls

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WE ARE concerned with the determination of subcritical, irrotational, steady plane flows of a compressible, inviscid fluid past a lifting profile set between two linear permeable walls, when the speed distribution at infinity is uniform and parallel to the walls. The walls are assumed to be far enough from the profile so that their working condition may be semi-linearized with respect to the conditions at infinity. Therefore, the research of the stream function reduces to the determination of the solution of a nonlinear variational inequality in a weighted Sobolev space. We prove existence and uniqueness as long as the flow is subcritical and the profile is smooth. We show that the circulation of the speed vector along the profile can be fitted by regulating the pressures outside the permeable walls. As it is usually done, we bound the domain by setting the uniform flow as the boundary condition at a finite distance. We give a convergence theorem with an error estimate as the diameter of the bounded domain increases to infinity. At last, we describe an algorithm, the convergence of which is proved, to compute directly the speed distribution in the physical plane. We give the first numerical results which have been computed with a finite element method of order one as the walls are completely permeable.

Praca dotyczy określenia podkrytycznych, bezwirowych, ustalonych płaskich przepływów ściśliwej nielepkiej cieczy za układem nośnych profili znajdujących się pomiędzy dwiema przepuszczalnymi ściankami, gdy rozkład prędkości w nieskończoności jest równomierny i równo legły do ścianek. Założono, że ścianki są wystarczająco daleko od profili, tak że ich oddziaływanie na profile może być semi-zlinearyzowane w stosunku do warunków w nieskończoności. Dlatego zbadanie funkcji prądu sprowadza się do określenia rozwiązania nieliniowej nierówności wariacyjnej w przestrzeni Sobolewa z wagą. Wykazano istnienie i jednoznaczność rozwiązania do chwili, gdy przepływ jest podkrytyczny a profil gładki. Pokazano, że cyrkulację wektora prędkości wzdłuż profilu można dopasować przez regulację ciśnień na zewnątrz przepuszczalnych ścianek. Wyprowadzono zbieżny algorytm celem obliczenia rozkładu prędkości bezpośrednio na płaszczyźnie fizycznej. Przedstawiono pierwsze wyniki numeryczne obliczone metodą elementów skończonych o aproksymacji pierwszego rzędu dla przypadku pełnej nieprzepuszczalności ścianek.

Работа касается определения докритических, безвихревых, установившихся, плоских течений сжимаемой, невязкой жидкости за системой несущих профилей, находящихся между двумя проницаемыми стенками, когда распределение скоростей в бесконечности равномерно и параллельно стенкам. Предположено, что стенки находятся достаточно далеко от профилей, так, что их воздействие на профили может быть полулинеаризовано по отношению к условиям в бесконечности. Поэтому исследование функции тока сводится к определению решения нелинейного вариационного неравенства в пространства Соболева с весом. Показано существование и единственность решения к моменту, когда течение докритическое, а профиль — гладкий. Показано, что циркуляцию вектора скорости вдоль профиля можно согласовать путем регуляции давлений вне проницаемых стенок. Выведен сходящийся алгоритм с целью вычисления распределения скоростей непосредственно на физической плоскости. Представлены первые численные результаты расчитанные методом конечных элементов с аппроксимацией первого порядка для случая полной непроницаемости стенок.

1. Introduction

THIS PAPER is devoted to the determination of subcritical irrotational steady plane flows of a compressible, inviscid fluid past a lifting profile set between two permeable linear walls. The speed distribution at infinity is uniform and parallel to the walls. The working condition of the walls proceeds from the Darcy law. The main interest of the problem is to be related to wind tunnel corrections.

From the numerical point of view the problem has already been studied by several authors, for instance LAVAL [1]. Our aim is to analyse the statement of such a functional frame that leads to a variational formulation in a way which makes it possible to prove numerical approximations based on the classical finite element method in bounded domains.

According to the above device, we have already dealt with a simpler model of the problem in a previous paper [2]. The flow was assumed to be incompressible past a symmetric profile without incidence. The permeable walls were assumed to be far enough from the profile so that their working condition was linearized with respect to the conditions at infinity. Thus we got a linear elliptic equation with oblique derivative boundary conditions along the walls.

In this work we take into account the compressible effects. The linear problem is turned into a nonlinear one. The equation remains elliptic as long as the fluid is subsonic, therefore we have to deal with an additional constraint on the speed distribution. We still assume that the working condition of the walls may be linearized. Noting that the order of approximation remains the same we actually use a so-called [3] semi-linearization to get natural boundary conditions for the elliptic operator. Taking into account the arguments already used for the study of compressible flows past lifting profiles in an infinite atmosphere [4], we mainly show that the problem reduces to determine the solution of a nonlinear variational inequality.

An outline of the paper is as follows. Sections 2 and 3 are devoted to the statement of the problem and its transformation. In Sects. 4 and 5 we give some technical lemmas and mathematical results, proofs of which follow analogous devices as in [4]. Theorem 1 proves the existence and uniqueness of the stream function. Theorem 2 shows that the circulation can be fitted by regulating pressures in the plenum chambers. We construct in Sect. 6 a sequence of approximate problems in bounded open domains by setting the uniform flow as the boundary condition along two vertical lines at a finite distance. Theorem 3 proves the existence and uniqueness of the approximate stream functions. Theorem 4 shows that this sequence converges towards the solution and gives an error estimate as the diameter of the bounded domains increases to infinity. In Section 7 we exhibit an algorithm which allows a direct computing of the speed distribution in the physical plane by solving a sequence of harmonic problems with linear mixed-type boundary conditions. At last, we discuss in Section 8 the first numerical results which have been computed with a finite element method as the walls are completely permeable.

2. Statement of the problem

Let 0 be the axis origin. The x-axis is directed parallel to the speed at infinity $\mathbf{q}_\infty = (u_\infty, 0)$ and two permeable walls $\Gamma_{\pm H}$ are determined by the equations $y = \pm H$ (see Fig. 1). We assume that the profile $\mathscr{P}$ is bounded by a smooth curve Γ (1) and Ω will denote the unbounded region exterior to $\mathscr{P}$ and set between Γ_H and Γ_{-H}.

(1) We assume that Γ is at least a third continuously differentiable simple closed curve.

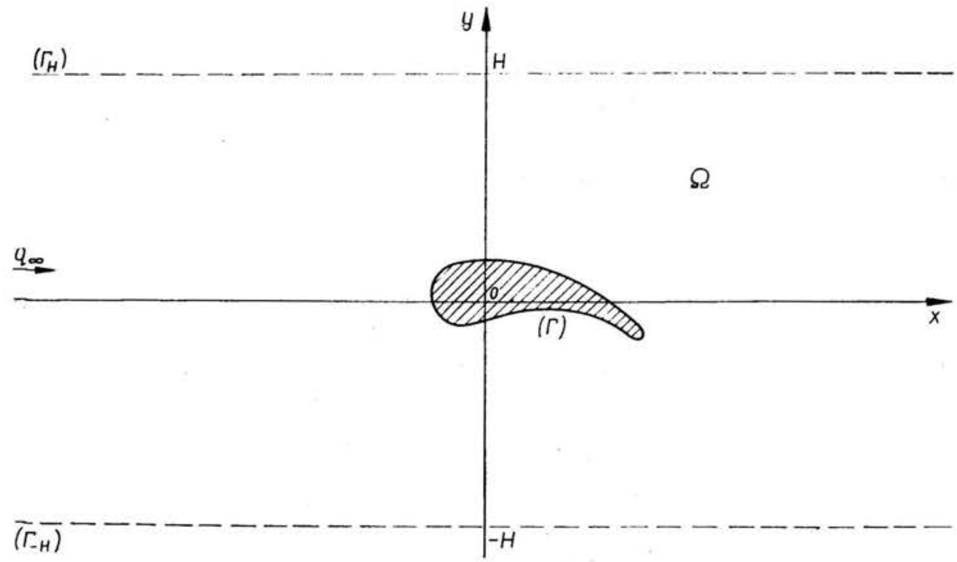

FIG. 1.

Denote the density by ϱ, the critical speed by q_c; the speed field $\mathbf{q} = (u, v)$ satisfies the relations

$$\left.\begin{aligned}&\begin{cases}\operatorname{rot}\mathbf{q} = 0\\ \operatorname{div}(\varrho\mathbf{q}) = 0\end{cases}\\ &q \leqslant q_c\text{(}^2\text{)}\end{aligned}\right\}\quad \text{throughout } \Omega, \tag{2.1, 2.2}$$

(2.1)

(2.2)

$$u - b\varrho v = L_{\pm H} \quad \text{along } \Gamma_{\pm H}, \tag{2.3}$$

$$\mathbf{q}\cdot\mathbf{n} = 0 \quad \text{along } \Gamma, \tag{2.4}$$

$$\mathbf{q} \to \mathbf{q}_\infty \quad \text{as } |x| \to \infty. \tag{2.5}$$

The irrotational character and principle of conservation of mass are expressed by the relation (2.1). The flow is subsonic as long as the inequality (2.2) holds.

The working conditions are represented by Eq. (2.3), where the right-hand sides are connected with the permeability constant P and the plenum-chamber pressures $p^c_{\pm H}$ by

$$L_{\pm H} = u_\infty + \frac{1}{\varrho_\infty u_\infty}(p_\infty - p^c_{\pm H}), \quad b = -\frac{1}{\varrho_\infty P};$$

here

$$p^c_{\pm H} \to p_\infty \quad \text{as} \quad |x| \to \infty.$$

Without loss of generality we choose henceforth $P \neq 0$; the case $P = 0$ leads to a Dirichlet problem which involves no proper difficulty.

The classical slip condition is given along Γ by the relation (2.4) where $\mathbf{n}$ is a vector normal to Γ. At last, the speed is prescribed uniform at infinity by the relation (2.5).

(2) q will denote the speed magnitude.

The limit speed is chosen for speed unit. Since the fluid is isentropic, we obtain through Bernoulli's law the speed-density relation

$$\varrho(q^2) = \varrho_0(1-q^2)^{1/(\gamma-1)}. \tag{2.6}$$

Here γ is the ratio of specific heats ($\gamma > 1$) and ϱ_0 the density when the speed vanishes.

REMARK 1. The circulation of the speed is defined as

$$\sigma = \int_\Gamma u dx + v dy. \tag{2.7}$$

We point out that σ is not a datum of the problem. However, we shall state that σ depends only on the data: u_∞ and $p^c_{\pm H}$ (see Paragraph 5.2.).

REMARK 2. In fact, the wall working-conditions proceed from the Darcy law and are defined as

$$P(p-p^c_{\pm H}) = \varrho v \quad \text{along} \quad \Gamma_{\pm H}. \tag{2.8}$$

Here p is the pressure. Actually, the walls are assumed to lie far enough from the profile so that the left-hand side in Eq. (2.8) may be linearized with respect to the conditions at infinity. The approximation order does not alter when the right-hand side is not linearized; thus we get Eq. (2.3) which will be turned into a natural boundary condition for the later transformed problem.

3. Transformation of the problem

3.1

At present our aim is to reduce the above problem into a boundary value one, involving only one partial-differential inequality throughout Ω. In view of the continuity equation, the stream function ψ and the speed $\mathbf{q}$ are related by the classical differential system

$$u = \frac{1}{\varrho}\psi_y, \quad v = -\frac{1}{\varrho}\psi_x. \tag{3.1}$$

From Eqs. (3.1) we derive (³)

$$|\nabla\psi|^2 = q^2\varrho^2(q^2). \tag{3.2}$$

The right-hand side is a strictly increasing function of q^2 under the assumption (2.2), therefore we can define in the interval

$$0 \leqslant z \leqslant t_c^2, \quad t_c = q_c\varrho(q_c^2)$$

the function $h(z)$ such that

$$h(|\nabla\psi|^2) = \frac{1}{\varrho(q^2)}. \tag{3.3}$$

h is a strictly increasing convex function over $[0, t_c^2]$ and is infinitely derivable inside $]0, t_c^2[$ (see [4]).

(³) Let $\mathbf{u} = (u_1, u_2) \in \mathbf{R}^2$, $\mathbf{v} = (v_1, v_2) \in \mathbf{R}^2$, $|\mathbf{u}| = \sqrt{u_1^2+u_2^2}$ will denote the Euclidian norm and $(\mathbf{u}, \mathbf{v}) = u_1v_1+u_2v_2$ the associated scalar product.

Using Eqs. (3.2) and (3.3), we bring the problem (2.1) ... (2.5) into the boundary-value one:

$$\left.\begin{aligned} &\operatorname{div}[2(|\nabla\psi|^2)\nabla\psi] = 0 \quad (3.4)\\ &|\nabla\psi| \leqslant t_c \quad (3.5)\end{aligned}\right\} \quad \text{throughout } \Omega,$$

$$h(|\nabla\psi|^2)\psi_y + b\psi_x = L_{\pm H} \quad \text{along } \Gamma_{\pm H}, \tag{3.6}$$

$$\Psi = 0 \quad \text{along } \Gamma, \tag{3.7}$$

$$\nabla\psi \to (0, t_\infty), \quad (t_\infty = \varrho_\infty u_\infty) \quad \text{as} \quad |x| \to \infty. \tag{3.8}$$

3.2

In order to derive an appropriate functional frame of the above problem, we look for the stream function in the form

$$\psi = \xi + w,$$

where w is a smooth raising of the condition at infinity (3.8) and does not alter the Dirichlet condition (3.7). Hence we have to determine ξ such that $\nabla\xi$ is square-summable.

More precisely, we choose an indefinitely derivable function w of support disjoined from $\mathscr{P}$ and we state that there exists a number t_1 such that ∇w obeys

$$|\nabla w(x, y)| \leqslant t_1 < t_c \quad \text{for any} \quad (x, y) \in \Omega \quad \text{(see lemma 1 in [4])}. \tag{3.9}$$

3.3

We now have to deal with the argument $h(|\nabla\psi|^2)\nabla\psi$ which is not square-summable over the unbounded domain Ω. For this purpose, we consider

$$\mathbf{Q}(\xi) = h(|\nabla(\xi+w)|^2)\nabla(\xi+w) - h(t_\infty^2)\mathbf{T} + b'\nabla\xi,$$

where

$$\mathbf{T} = (0, t_\infty), \quad {}^t\nabla\xi = (-\xi_y, \xi_x)$$

are solenoidal. The vector-valued function ${}^t\nabla\xi$ has been introduced to take into account the left-hand side of the oblique derivative condition (3.6).

LEMMA 1. If $t_\infty < t_c$, then there exists a constant $M > 0$ and a square-summable function g, depending only on the data of the problem, such that for any ξ satisfying

$$|\nabla(\xi+w)| \leqslant t_c \quad \text{a.e.} \quad (x, y) \in \Omega,$$

the estimate

$$|\mathbf{Q}(\xi)| \leqslant M|\nabla\xi| + g \quad \text{a.e.} \quad (x, y) \in \Omega$$

holds.

This is proved in the same way as for Lemma 2 in [4].

4. A functional frame

In this section we present notations and results from functional analysis which will be needed later.

Let

$$H^1(\Omega) = \{\xi \in \mathscr{D}'(\Omega);\ D^\alpha\xi \in L^2(\Omega),\ |\alpha| \leqslant 1\}$$

be the classical Sobolev space provided with the usual norm ([4]). We set $\Gamma_1 = \Gamma_H \cup \Gamma_{-H}$ and we denote by $H^{1/2}$ (Γ_1) the space of the traces over Γ_1 of the functions belonging to $H^1(\Omega)$ [5].

Define the function $x \to m(x) = \dfrac{1}{\sqrt{1+x^2}}$, and introduce the weighed space

$$V = \{\xi \in \mathscr{D}'(\Omega);\ m\xi \in L^2(\Omega),\ D^\alpha\xi \in L^2(\Omega),\ |\alpha| = 1,\ \xi_{|\Gamma} = 0\}$$

provided with the norm:

$$\xi \to [|m\xi|^2_{L^2(\Omega)} + \|\nabla\xi\|^2_{L^2(\Omega)}]^{1/2}.$$

Let V denote the subspace of V spanned by the infinitely differentiable functions which are of bounded support dispoined from $\mathscr{P}$.

LEMMA 2. The Hilbert space V is the closure of V and the norm defined above is equivalent to the following one

$$\xi \mapsto \|\xi\| = \|\nabla\xi\|_{L^2(\Omega)}.$$

First we refer to classical density properties in Sobolev spaces [5]. Hence we deduce it suffices to show that V is the completion of bounded-support functions. Then, using a truncation process, we establish that any function in V can be approximated by a sequence of functions which are of bounded support [6]. Using integration by parts, the second statement is proved in the same way as for an analogous norm equivalence in [7].

We denote by V' and $H^{-1/2}(\Gamma_1)$ the dual spaces ([5]) of V and $H^{1/2}(\Gamma_1)$. Given $L \in V'$ and $\xi \in V$ (resp. $l \in H^{-1/2}(\Gamma_1)$ and $\chi \in H^{1/2}(\Gamma_1)$), the value $L(\xi)$ (resp. $l(\chi)$) is written as $\langle L, \xi\rangle_{V',V}$ (resp. $\langle l, \chi\rangle_{H^{-1/2}(\Gamma_1),\, H^{1/2}(\Gamma_1)}$), where the brackets indicate the pairing between V' and V (resp. $H^{-1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$).

5. A variational formulation

5.1

To take into account the condition (3.5), we consider

$$K = \{\xi \in V;\ |\nabla(\xi + w)| \leqslant t_c, \quad \text{a.e. } (x, y) \in \Omega\}.$$

From the inequality (3.9) we observe that $0 \in K$ and it is classical to state that K is a closed convex subset of V. We introduce the semi-linear functional

$$a(\xi, \chi) = \int_\Omega (\mathbf{Q}(\xi), \nabla\chi)\, dx\, dy \quad \text{for} \quad (\xi, \chi) \in K \times V.$$

([4]) $L^2(\Omega)$ is the space of square-summable functions over Ω provided with the norm $|u|_{L^2(\Omega)} = \left(\int_\Omega |u(x)|^2 dx\right)^{1/2}$. $\mathscr{D}'(\Omega)$ is the space of generalized functions.

([5]) The dual space of a Hilbert space is the space consisting of all continuous linear forms defined over the Hilbert space.

LEMMA 3. For any $(\xi, \chi) \in K \times V$, the estimates

$$|a(\xi, \chi)| \leqslant [M\|\xi\| + |g|_{L^2(\Omega)}] \cdot \|\chi\|,$$

$$a(\xi, \xi-\chi) - a(\chi, \xi-\chi) \geqslant \frac{1}{\varrho_0} \|\xi-\chi\|^2,$$

$$a(\xi, \xi) \geqslant \frac{1}{\varrho_0} \|\xi\|^2 - |g|_{L^2(\Omega)} \cdot \|\xi\|$$

hold.

Moreover, for fixed $(\xi, \chi, \zeta) \in K \times K \times V$, the function

$$t \mapsto a(\xi + t(\chi - \xi), \zeta)$$

is continuous over R.

Proof. As an application of Lemma 1, we derive the first estimate; the second and third are consequences of the following [4]:

$$(\mathbf{Q}(\xi) - \mathbf{Q}(\chi), \quad \nabla(\xi-\chi)) \geqslant \frac{1}{\varrho_0} |\nabla(\xi-\chi)|^2,$$

$$(\mathbf{Q}(\xi), \nabla\xi) \geqslant \frac{1}{\varrho_0} |\nabla\xi|^2 - g. |\nabla\xi|.$$

At last, applying the Lebesgue theorem in the integrand of a, we complete the proof of the lemma.

The right-hand side of the oblique derivative condition (3.6) is taken into account by introducing the distribution L defined along Γ_1 as

$$L = L_{\pm H} - [bw_x + h(t_\infty^2) t_\infty].$$

It is easily seen that if $\xi \in V$, then $m\xi \in H^1(\Omega)$, so that its trace lies in $H^{1/2}(\Gamma_1)$. Furthermore, if we assume that $\frac{L}{m} \in H^{-1/2}(\Gamma_1)$, we get the continuity estimate [6]

$$\left| \left\langle \frac{L}{m}, m\xi \right\rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)} \right| \leqslant C \left| \frac{L}{m} \right|_{H^{-1/2}(\Gamma_1)} \|\xi\|, \quad \xi \in V. \tag{5.1}$$

It follows that L induces a continuous linear form over V defined by

$$\langle L, \xi \rangle_{V',V} = \left\langle \frac{L}{m}, m\xi \right\rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)}, \quad \xi \in V. \tag{5.2}$$

THEOREM 1. *If $t_\infty < t_c$ and if $\frac{L}{m} \in H^{-1/2}(\Gamma_1)$, then there exists one function $\xi \in K$ and one at the most such that*

$$a(\xi, \chi-\xi) \geqslant \langle L, \chi-\xi \rangle_{V',V}, \quad \forall \chi \in K. \tag{5.3}$$

Moreower, if the data are such that the function $\psi = \xi + w$ satisfies

$$\begin{gathered} |\nabla\psi| \leqslant t_c - \varepsilon \quad \text{a.e. } (x, y) \in \Omega, \quad \varepsilon > 0, \\ D^\alpha\psi \in L^2(\Omega), \quad |\alpha| = 2, \end{gathered} \tag{5.4}$$

then $\psi = \xi + w$ is the unique solution of Eqs. (3.4)–(3.7) *in $w + V$.*

(6) C means any constant which depends only on the data of the problem and not on the other parameters.

P r o o f. Since Lemma 3 holds, we deduce that a is continuous over $K \times V$ so that it defines the operator A acting from K to V' by

$$\langle A\xi, \chi \rangle_{V',V} = a(\xi, \chi), \qquad (\xi, \chi) \in K \times V.$$

Then we observe that A is strictly monotone, hemi-continuous and coercitive. Therefore, we state the relations (5.3) to be a consequence of Eqs. (5.1) and (5.2) and the monotonicity method [8]. The proof of the second statement follows the same lines as in the proof of Theorem 1 in [4].

5.2

We now show that if we assume the relations (5.4), then the circulation given by Eq. (2.7) depends only on the data of the problem.

THEOREM 2. *Assuming the relations* (5.4), *suppose the difference* $(p^c_H - p^c_{-H})$ *is summable over* R. *Then the formula*

$$\sigma = \frac{1}{\varrho_\infty u_\infty} \int_{-\infty}^{\infty} [p^c_{-H}(x) - p^c_H(x)]\, dx$$

holds.

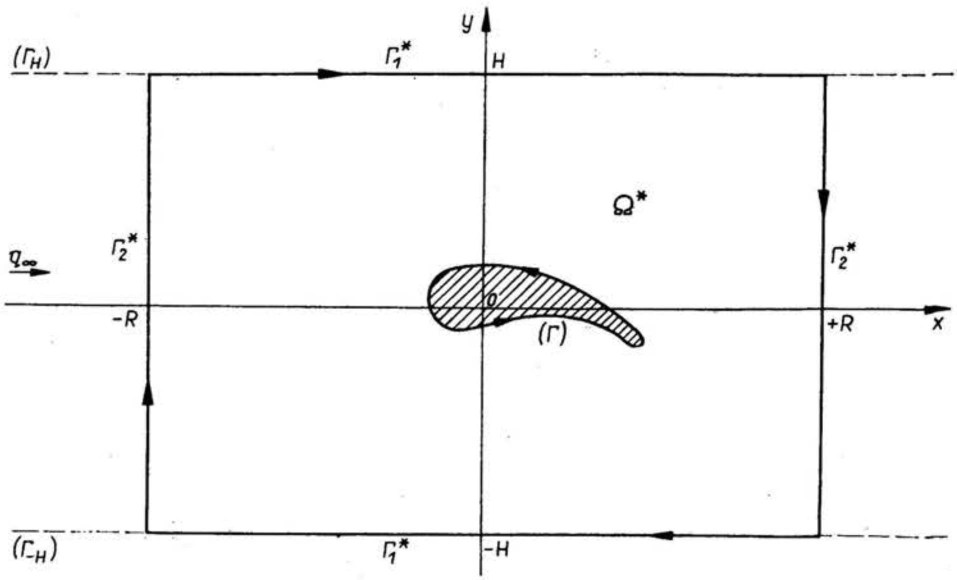

FIG. 2.

P r o o f. Given $R > 0$, Ω^* denotes the region exterior to $\mathscr{P}$ and is enclosed in the rectangle $[-R, R] \times [-H, H]$.

Let Γ_1^* (resp. Γ_2^*) be the horizontal (resp. vertical) sides of the rectangle (see Fig. 2). We set $\Gamma^* = \Gamma_1^* \cup \Gamma_2^*$.

Using Eqs. (2.7), (3.1), (3.2) and (3.3), we have by Green's formula

$$-\sigma = \int_{\Gamma^*} h(|\nabla\psi|^2)(\psi_y dx - \psi_x dy) = I_1 - I_2, \tag{5.5}$$

where we set

$$I_1 = \int_{\Gamma_1^*} h(|\nabla\psi|^2)\psi_y dx, \quad I_2 = \int_{\Gamma_2^*} h(|\nabla\psi|^2)\psi_x dy.$$

When written out, the integral I_2 takes the form

$$I_2 = \int_{-H}^{H} h(|\nabla\psi|^2)\psi_x(-R, y)dy + \int_{H}^{-H} h(|\nabla\psi|^2)\psi_x(R, y)dy,$$

and we have, by Schwarz's inequality,

$$(I_2)^2 \leqslant C \int_{-H}^{H} [\psi_x^2(-R, y) + \psi_x^2(R, y)]dy.$$

From the relations (5.4) we observe that ψ_x belongs to $H^1(\Omega)$; then, using the Babitch extension [5], we show the estimate

$$(I_2)^2 \leqslant C \int_{|x| \geqslant R} [\psi_x^2 + |\nabla\psi_x|^2]dxdy.$$

Since C does not depend on R, we derive by the Lebesgue theorem

$$I_2 \to 0 \quad \text{as} \quad R \to \infty. \tag{5.6}$$

If we take into account the oblique-derivative condition (3.6) then we obtain

$$I_1 = \int_{-R}^{R} [L_H(x) - L_{-H}(x)]dx - bI,$$

where we set

$$I = \int_{-R}^{R} \int_{-H}^{H} \psi_{xy} dx dy;$$

Following the same way as above for I_2, we show that

$$I \to 0 \quad \text{as} \quad R \to \infty,$$

hence we have

$$I_1 \to 0 \quad \text{as} \quad R \to \infty. \tag{5.7}$$

Gathering together the relations (5.5), (5.6) and (5.7), we conclude that

$$\sigma = \int_{-\infty}^{\infty} [L_H(x) - L_{-H}(x)]dx.$$

Theorem 2 is proved.

6. Approximation involving bounded open sets

In order to justify the numerical approach involving bounded domains, we actually solve the problem (3.4)–(3.8) in the bounded sets

$$\Omega^* = \Omega \cap \{(x, y); |x| < R\},$$

where parameter $R > 0$ is expected to increase to infinity.

6.1

We now look for a function ψ^* which satisfies the boundary value problem

$$\left.\begin{array}{l} \operatorname{div}[h(|\nabla\Psi^*|^2)\nabla\Psi^*] = 0 \\ |\nabla\Psi^*| \leqslant t_c \end{array}\right\} \quad \text{thoughout } \Omega^*,$$

$$(6.1) \qquad h(|\nabla\Psi^*|^2)\Psi^*_y + b\Psi^*_x = L_{\pm H} \quad \text{along } \Gamma^*_1,$$

$$\Psi^* = 0 \quad \text{along } \Gamma,$$

$$\Psi^* = t_\infty y \quad \text{along } \Gamma^*_2.$$

As in the previous section we introduce the space

$$V^* = \{\xi \in H^1(\Omega^*); \xi_{|\Gamma \cup \Gamma^*_2} = 0\},$$

provided with the norm

$$\xi \to ||\xi||^* = ||\nabla\xi||_{L^2(\Omega^*)}.$$

We point out that the weight does not appear any more in V^* since Ω^* is bounded. Let us consider the non-empty closed convex subset of V^*

$$K^* = \{\xi \in V^*, |\nabla(\xi+w)| \leqslant t_c, \text{ a.e. } (x, y) \in \Omega^*\}$$

and the functional defined over $K^* \times V^*$ as

$$a^*(\xi, \chi) = \int_{\Omega^*} (\mathbf{Q}(\xi), \nabla\chi)\, dx\, dy.$$

We denote by L^* a suitable approximation of L that we will state precisely in the sequel. Proceeding as in the proof of Theorem 1, we obtain the following:

THEOREM 3. *If $t_\infty < t_c$ and if $L^* \in V^*$, then there exists a unique function $\xi^* \in K^*$ such that*

$$(6.2) \qquad a^*(\xi^*, \chi-\xi^*) \geqslant \langle L^*, \chi-\xi^*\rangle_{V^{*\prime}, V^*}, \quad \forall \chi \in K^*.$$

Moreover, if the function $\Psi^* = \xi^*+w$ satisfies

$$(6.3) \qquad \begin{cases} |\nabla\Psi^*| \leqslant t_c - \varepsilon \quad \text{a.e.} \quad (x, y) \in \Omega^*, \quad \varepsilon > 0 \\ D^\alpha\Psi^* \in L^2(\Omega^*), \quad |\alpha| = 2, \end{cases}$$

then $\Psi^* = \xi^*+w$ is the unique solution of Eqs. (6.1) *in $w+V^*$.*

6.2

At present we investigate the connection between Ψ and Ψ^* as R increases to infinity. For all $\chi \in V^*$, we denote by $\tilde{\chi}$ its zero-extension over Ω, thus $\chi \in V$.

L is approximated by L^* in such a way that

$$\langle L, \tilde{\chi}\rangle_{V', V} = \langle L^*, \chi\rangle_{V^{*\prime}, V^*}; \quad \forall \chi \in V^*.$$

We introduce an infinitely derivable function α which obeys

$$\alpha(x) = 0 \quad \text{for} \quad |x| \leqslant \operatorname{sh}(\tilde{R}/2),$$
$$\alpha(x) = 1 \quad \text{for} \quad |x| \geqslant R,$$
$$0 \leqslant \alpha(x) \leqslant 1, \quad |\alpha'(x)| \leqslant (C_1/\tilde{R}) \cdot m(x) \quad \text{for any } x.$$

Here C_1 does not depend on $\tilde{R} = \operatorname{Arg\,sh} R$ and the function Arg sh acts as a primitive of the weight function m.

Using interpolation results [5], we show that the relation

$$\left\langle \frac{\alpha L}{m}, \zeta \right\rangle = \left\langle \frac{L}{m}, \alpha\zeta \right\rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)}, \quad \forall \zeta \in H^{1/2}(\Gamma_1)$$

defines a continuous linear functional $\dfrac{\alpha L}{m} \in H^{-1/2}(\Gamma_1)$.

Following the same lines as in the proof of Theorem 3 in [4], we state such a result:

THEOREM 4. *If we assume that there exists* $\varepsilon > 0$ *such that*

$$|\nabla \Psi| \leqslant t_c - \varepsilon, \quad |\nabla \Psi^*| \leqslant t_c - \varepsilon, \quad \text{a.e. } (x, y) \in \Omega$$

then the functions $\tilde{\Psi}^* = \tilde{\xi}^* + w$ *converge in* $w + V$ *towards* $\Psi = \xi + w$ *as* R *increases to infinity. Moreover, we have the asymptotic estimate*

$$\|\Psi - \tilde{\Psi}^*\| \leqslant C \left\{ \frac{1}{\tilde{R}} \left[\left| \frac{L}{m} \right|_{H^{-1/2}(\Gamma_1)} + |g|_{L^2(\Omega)} \right] + \left| \frac{\alpha L}{m} \right|_{H^{-1/2}(\Gamma_1)} \right\}.$$

REMARK 3. Note that $\left| \dfrac{\alpha L}{m} \right|_{H^{-1/2}(\Gamma_1)}$ vanishes as R goes to infinity. Furthermore, for suitable choices of L we get an estimate in $0(1/\tilde{R})$. For instance, it is easily seen that if we take $p^c_{\pm H} = p_\infty$, then L vanishes for any R large enough.

7. An algorithm to compute speed distribution

Henceforth we shall work in the bounded open set Ω^* and for the sake of simplicity we agree to omit the stars from now on.

On the other hand we assume the flow to be totally subsonic so that the function Ψ actually represents the stream function. We extend the function $\lambda \to \varrho(\lambda)$ by a function g continuously differentiable and strictly decreasing on R such that

$$g(\lambda) = \varrho(\lambda), \qquad \forall \lambda \leqslant q_c^2,$$
$$g(\lambda) \geqslant a > 0, \quad \forall \lambda \in \mathrm{R}, \quad a \in \mathrm{R},$$
$$\lambda \to z(\lambda) = \lambda g^2(\lambda) \quad \text{is inversible all over R},$$

and we denote h as the analogous function to Eq. (3.3) which satisfies

$$h(z) = 1/g(\lambda), \quad \forall \lambda \in \mathrm{R}.$$

By setting

$$\mathbf{e} = \nabla \Psi$$
$$\mathbf{m} = h(|\mathbf{e}|^2)\mathbf{e},$$

the equation which defines the stream function as long as $q \leqslant q_c$,

(7.1) $$(h(|\nabla\Psi|^2)\nabla\Psi + b^t\nabla\Psi, \nabla\chi) = \langle L_{\pm H}, \chi\rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)}, \quad \forall\chi \in V \text{ (}^7\text{)}$$

is turned into

$$\operatorname{rot} \mathbf{e} = 0,$$

$$(\mathbf{m} + b^t\mathbf{e}, \nabla\chi) = \langle L_{\pm H}, \chi\rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)}, \quad \forall\chi \in V.$$

According to an analogous device as in [4], we construct a sequençe $\mathbf{m}^n$, $\mathbf{e}^n \in [L^2(\Omega)]^2$ with the following algorithm: compute $\mathbf{m}^0$, $\mathbf{e}^0$ in $[L^2(\Omega)]^2$ such that

(7.2) $$(\mathbf{m}^0 + b^t\mathbf{e}^0, \nabla\chi) = \langle L_{\pm H}, \chi\rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)}, \quad \forall\chi \in V.$$

Get step $n+1$ from step n through

(7.3) $$\mathbf{e}^{n+1} = g(|\mathbf{m}^n|^2)\mathbf{m}^n,$$

compute $\Psi^{n+1} = \xi^{n+1} + w$, $\xi^{n+1} \in V$, solution of the variational problem

(7.4) $$(\nabla\Psi^{n+1}, \nabla\chi) = (\mathbf{e}^{n+1}, \nabla\chi), \quad \forall\chi \in V,$$

(7.5) $$\mathbf{m}^{n+1} + b^t\mathbf{e}^{n+1} = \mathbf{m}^n + b^t\mathbf{e}^n + \nu(\nabla\Psi^{n+1} - \mathbf{e}^{n+1}), \quad \nu \in \mathrm{R}, \quad \nu > 0.$$

THEOREM 5. *Assume that*

(7.6) $$|b| < 1/\varrho_0, \quad 0 < \nu < 2(1/\varrho_0 - |b|);$$

thus the algorithm (7.2)–(7.5) *is convergent and, more precisely,*

$$\xi^n \to \xi \quad \text{in} \quad V, \mathbf{e}^n \to \nabla\Psi \quad \text{in} \quad [L^2(\Omega)]^2, \quad \mathbf{m}^n \to \mathbf{m} = {}^t\mathbf{q} \quad \text{in} \quad [L^2(\Omega)]^2,$$

where

$$\Psi = \xi + w, \quad \mathbf{q} = (h(|\nabla\Psi|^2)\Psi_y, \quad -h(|\nabla\Psi|^2)\Psi_x)$$

are the functions connected to the solution of Eq. (7.1).

THEOREM 5 is proved just in the same way as for the proof of Theorem 4 in [4].

From Eqs. (7.2) and (7.5) we deduce that the equality

(7.7) $$(\mathbf{m}^n + b^t\mathbf{e}^n, \nabla\chi) = \langle L_{\pm H}, \chi\rangle_{H^{-1/2}H^{1/2}(\Gamma_1)}, \quad \forall\chi \in V$$

holds at each step. Since the algorithm is convergent, we get at limit

$$\nabla\Psi = e,$$

hence

$$\operatorname{rot} \mathbf{e} = 0 \quad \text{in } \Omega \Leftrightarrow \operatorname{div}(g(q^2)\mathbf{q}) = 0 \quad \text{in} \quad \Omega$$

and, integrating Eq (7.7) by parts,

$$\operatorname{div} \mathbf{m} = 0 \quad \text{in } \Omega \Leftrightarrow \operatorname{rot} \mathbf{q} = 0 \quad \text{in } \Omega,$$

$$\mathbf{m}_2 + b\mathbf{e}_1 = L_{\pm H} \quad \text{on } \Gamma_1 \Leftrightarrow u - b\varrho v = u + \frac{1}{\varrho_\infty u_\infty}(p_\infty - p^c_{\pm H}),$$

that is to say we have computed the speed distribution in Ω whenever the flow is subsonic.

(7) The scalar product defined over $[L^2(\Omega)]^2$ is also denoted as the scalar product over R^2.

8. First numerical experiments and comments

We first note that at each step we have to compute Eqs. (7.4). This is very easy because we deal with the harmonic operator associated to the Dirichlet-Neumann boundary conditions, hence the matrix is symmetric, definite and positive.

To find the starting point $\mathbf{m}^0$, $\mathbf{e}^0$, we compute once and for all the stream function Ψ^0 corresponding to the associated incompressible problem (see [2]). Then we set

$$\mathbf{e}^0 = \nabla\Psi^0, \quad \mathbf{m}^0 = \frac{1}{\varrho_\infty}\mathbf{e}^0,$$

hence the condition (7.2) is automatically realized.

Naturally, we are anxious to find appropriate wind-tunnel corrections to simulate the flight through an infinite atmosphere. Therefore we have first studied the speed distribution as the profile is set in an unbounded domain.

Note that the algorithm (7.2)–(7.5) still works. Just set

$$b = 0, \quad p^c_{\pm H} = p_\infty,$$

the conditions corresponding to completely open walls and choose in addition

$$V = H^1_0(\Omega)$$

to take into account the fact that

$$\Psi \to t_\infty y, \quad \text{as} \quad |x|^2+|y|^2 \to \infty.$$

With the finite element method of order 1, we have computed the flow past an ellipse the thickness ratio of which was 0.1. We have tried experiments with a large range of Mach numbers at infinity and angles of attack from 0 to 7 degrees. In practice, the condition (7.3) is written with the expression of p whatever the velocity modulus may be. It occurs that the algorithm still works for supercritical flows with no shock. Figure 3, 4 and 5 give an exam-

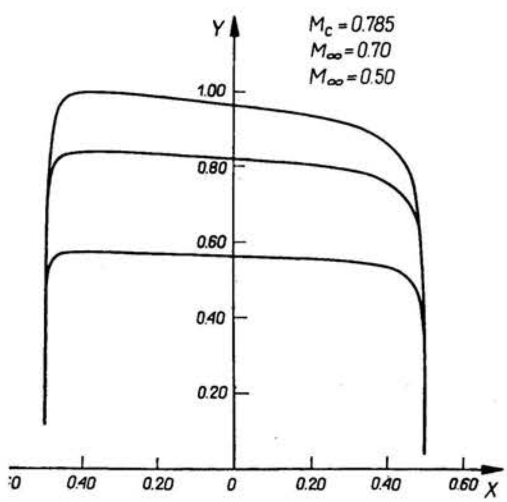

FIG. 3. Subcritical speed distributions, expressed in Mach number, along the ten per cent ellipse with an angle of attack of 1 degree.

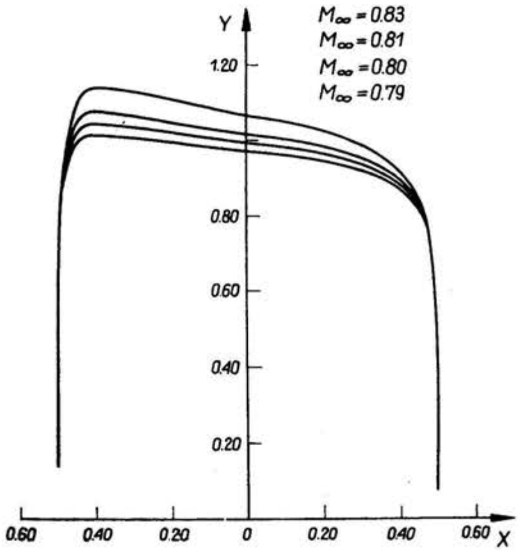

FIG. 4. Transonic speed distributions along the ten per cent ellipse in the same conditions.

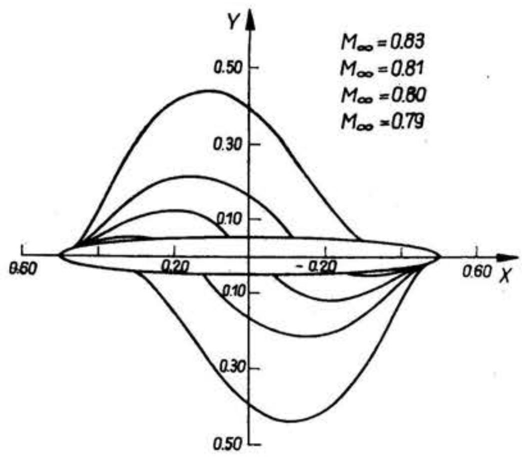

FIG. 5. Supercritical pockets past the ten per cent ellipse with an angle of attack of 1 degree.

ple of the speed-distribution computed along the profile. More complete results have been presented in [9].

At last, we would like to discuss the constraint appearing in the inequalities (7.6)

$$|b| < 1/\varrho_0.$$

In terms of permeability it is equivalent to

$$P > \left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{1/\gamma-1}, \tag{8.1}$$

so that the largest restriction corresponds to $M_\infty = M_c$. In practice, the inequality (8.1) appears to be quite reasonable. For instance, for the 10% ellipse without incidence (that is to say the angle of attack which corresponds to the largest critical Mach number) we have found $M_c = 0.8$; that implies $P > 1.35$.

To conclude, we shall say that the weighted Sobolev space allows the statement of the first theorem, according to our knowledge, to prove the existence and uniqueness of the stream function of the flow between permeable walls. The convergence theorem justifies the usual way to set boundary conditions at a finite distance. On the other hand, the algorithm provides an easy calculus of the speed distribution; furthermore, the first numerical experiments in an infinite atmosphere show that the method is fast and efficient.

Finally, since we can fit the circulation, we actually expect to deal with the profiles which present a sharp trailing edge. Indeed, we have already obtained some results on the way to take into account the Kutta-Joukowski condition as the profile lies in an infinite atmosphere (see [10]).

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