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Settling velocity of Newtonian suspensions

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THE MAIN point of the theory of sedimentation is the relation between the settling velocity and the spatial distribution of suspended particles. We consider spherical particles settling in infinite fluid, at low Reynolds number conditions. The nearest neighbour distribution function of randomly distributed spheres is exploited, to modify a cell model approach. Inside a cell of a radius equal to the characteristic length of suspension, the Stokes equation is solved. The striking result of our calculations is the presence of the overshoot of the relative settling velocity. This overshoot was also experimentally found [1]. For large volume concentrations of suspended particles ($\phi < 0.4$), the proposed approach satisfactorily agrees with available experimental data.

Praca dotyczy sedymentacji zawiesiny, w której cieczą nośną jest ciecz lepka, a fazą rozproszoną — kuliste krople cieczy o dowolnej lepkości. Głównym problemem jest wpływ przestrzennego rozkładu rozproszonych kropli na średnią prędkość sedymentacji. Prędkość tę obliczono zakładając losowe rozmieszczenie kropli cieczy, opisane przez funkcję rozkładu odległości do najbliższego sąsiada. Oddziaływania hydrodynamiczne między kroplami opisano zmodyfikowanym modelem komórkowym. Rezultatem pracy, na który chcielibyśmy zwrócić uwagę, jest wystąpienie, przy koncentracji objętościowej kropli poniżej 1%, średniej prędkości sedymentacji większej niż prędkość opadania pojedynczej kropli. Efekt ten obserwowano w kilku pracach eksperymentalnych. Obliczono również średnią prędkość sedymentacji jako funkcję koncentracji objętościowej kropli dla gęstych zawiesin (dla koncentracji objętościowej poniżej 0,4).

В теории седиментации основную роль играет соотношение между скоростью осаждания взвешенных частиц и их пространственным распределением. В работе рассматривается осаждание сферических частиц в неограниченной жидкой среде при условии, что число Рейнольдса мало. Предлагается модифицированная клеточная схема, в которой употребляется функция распределения до ближайшего соседа. В окрестности каждой сферической частицы решается задачу Стокса. Неожиданным результатом наших вычислений является превышение относительной скорости осаждания для концентрации около 1 процента. Это превышение было также найдено экспериментально [1]. Для больших объемных концентраций предложенный метод удовлетворительно согласуется с экспериментальными данными.

1. Introduction

THE SEDIMENTATION of equi-sized spherical particles, dispersed in the Newtonian and incompressible fluid, is not fully yet understood. In the simplest case, to be considered here, the fluid is unbounded and at rest far from the particles. It is assumed that sedimentation takes place at low Reynolds number conditions. The settling velocity is established under the combined influence of hydrodynamic interactions between dispersed particles and the gravitation force. It will be regarded as a function of the viscosity ratio $\eta = \bar{\mu}/\mu$ ($\bar{\mu}$ — the coefficient of viscosity of the suspended particles, μ — the viscosity of the ambient fluid), and the volume concentration ϕ of the suspended particles $\left(\phi = \frac{4}{3}\pi a^3 n,\ a\right.$ — the radius of the suspended particles, n — the number density of particles$\Big)$. The qualitative description of the sedimentation is usually the following one: up to a certain concentration the particles settle individually and they obey the Stokes' formula:

$$U_0 = -\frac{2ga^2(\bar{\varrho}-\varrho)(1+\eta)}{3\mu(2+3\eta)}, \tag{1.1}$$

where the usual notation is used:

g — the acceleration of gravity,

$\bar{\varrho}$ — density of particles,

ϱ — density of the ambient fluid.

At higher concentration the rate of sedimentation diminishes with increasing concentration.

However, some experimental evidence exists showing that dilute suspensions settle faster than the individual Stokes' particle. Interesting experimental results have recently been obtained by BARFOD [1]. Figure 1 presents a typical behaviour of dilute suspensions

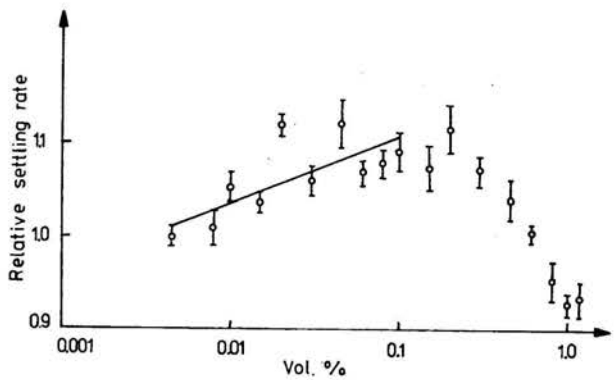

FIG. 1. Experimental data for the relative settling rate of dilute suspensions obtained by BARFOD [1].

showing that the relative rate of sedimentation (i.e. the ratio of the average settling velocity U to the Stokes' settling velocity U_0) exhibits an overshoot at these concentrations. The exact position of the maximum relative rate and its height is difficult to assess. According to Barfod's interpretation, the observed overshoot is due to the particulate spatial distribution of settling particles, namely, due to appearance of clouds of particles which have a greater settling rate than single particles.

For the simplest case of just two particles, this effect has been considered theoretically by GOLDMAN, COX, BRENNER [2]; they have found the linear and angular velocities of two settling spheres to be a function of their relative positions (of the separation and of the orientation of their line-of-centers relative to the direction of gravity).

The discussion of the settling rate dependence on a spatial distribution of particles has started with BURGER's paper [3]. In the first part of this paper Burgers studied random dilute suspensions of rigid spheres, assuming that the centres of spheres in the suspension take with an equal probability all positions, under the only condition that no two particles overlap. He obtained a correction to the Stokes settling velocity, proportional to the first power of particle concentration:

$$U/U_0 = 1-6.88\,\phi. \tag{1.2}$$

For ordered suspensions the correction in terms of the one-third power of volume concentration is reported:

$$U/U_0 = 1-\text{const}\cdot\phi^{1/3}, \tag{1.3}$$

where the numerical value of the proportionality coefficient depends on the assumed lattice structure.

Also so-called cell models [9] lead to the power law $\phi^{1/3}$. The controversial point of the power-law dependence was critically examined by SAFFMAN [11].

An extensive discussion of the sensitivity of the settling velocity to the arrangement of particles has been given in BATCHELOR's paper [4]. Batchelor developed a theory which enables us to calculate the relative settling velocity to order ϕ:

$$U/U_0 = 1-6.55\phi. \tag{1.4}$$

In the present paper the average settling velocity is examined using the method already applied for calculating the effective viscosity of suspensions [12]. This method enables us to calculate the average settling rate for the relatively broad range of volume concentrations, up to $\phi \approx 0.4$.

It is shown that the assumed form of spatial distribution of suspended particles leads to the overshoot of the average settling rate for small volume concentrations.

The proposed approach enables us also to calculate the dispersion coefficient which describes the fluctuations of the relative settling velocity.

2. A binomial spatial distribution of particles

We assume, like some of the previously mentioned papers, a random distribution of suspended particles. The analytical form of description of that particle arrangement is a separate problem considered, f.e., in statistical geometry [5]. We have chosen here a way of description which takes explicitly into account the finite size of particles. In view of that finite size the parameter ϕ_m giving the maximum volume concentration is introduced. It is defined by the following relation, describing the maximum number of M particles, which can be embedded in a volume V:

$$M = V\phi_m \Big/ \left(\frac{4}{3}\pi a^3\right). \tag{2.1}$$

The quantity of our interest is the probability $B(N, V)$ that the volume V contains exactly N particles, $N \leqslant M$. This probability is described by the relation of binomial form:

$$B(N, V) = \binom{M}{N} P^N(1-P)^{M-N}, \tag{2.2}$$

where $P = \phi/\phi_m$, $P \in [0, 1]$.

The conditions under which that relation can be used were discussed in [8].

A similar distribution of particles was used by BUYEVICH [6] who regarded the fluctuations of the number of particles in dense dispersed systems, and by SMITH [7] who studied experimentally the spatial distribution of spheres falling in a viscous liquid.

The present analysis involves the nearest neighbour distribution function related to the random distribution of spheres. It was shown in [8] that it is of the form

$$f(a, R) = 0 \quad \text{for} \quad R < a,$$

(2.3)
$$f(a, R) = -24R^2 \frac{\phi_m}{a^3} \ln\left(1-\frac{\phi}{\phi_m}\right)\left(1-\frac{\phi}{\phi_m}\right)^{8\phi_m\left(\frac{R^3}{a^3}-1\right)} \quad \text{for} \quad R \geqslant a.$$

Here R denotes half of a distance to the nearest neighbour sphere.

3. Dissipation of energy due to sedimentation

To model hydrodynamic interactions between suspended particles a modified cell model of suspension is employed. Let us imagine that each particle is surrounded by a spherical fluid envelope of the radius b, equal to the characteristic length of the suspension:

$$b^3 = \frac{3\phi_m}{4\pi n}. \tag{3.1}$$

The particle moves downward with a velocity U.

The mathematical problem posed involves the solution to the creeping motion equations inside the envelope:

(3.2)
$$\begin{gathered} \bar{\mu}\Delta\mathbf{u} = \nabla\bar{p}, \quad r \leqslant a, \\ \mu\Delta\mathbf{v} = \nabla p, \quad a \leqslant r \leqslant b, \\ \nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{v} = 0, \end{gathered}$$

where $\bar{p}$, $\mathbf{u}$ — the pressure, and velocity field inside the suspended particle, p, $\mathbf{v}$ — the pressure and velocity field in the ambient fluid in the region $a \leqslant r \leqslant b$, with the boundary conditons imposed at the surface of the particle ($r = a$), and at the outer spherical envelope ($r = b$). At $r = a$, the velocity and the tangential components of normal stress are continuous, and the radial component of velocity is equal to Un_1 (n_i — the directional cosine):

(3.3)
$$\begin{gathered} v_i = u_i, \quad i = 1, 2, 3 \\ \varepsilon_{kli} n_l n_j (\bar{\sigma}_{ij} - \sigma_{ij}) = 0, \\ n_i v_i = n_i u_i = U n_1, \end{gathered}$$

$\bar{\sigma}_{ij}$, σ_{ij} — the stress tensor components, respectively, inside the suspended particle and in the ambient fluid.

At $r = b$, the radial component of velocity and the tangential stress vanish:

(3.4)
$$\begin{gathered} n_i v_i = 0, \\ \sigma_{r\theta} = 0. \end{gathered}$$

That kind of boundary conditions was used first by HAPPEL [9]. Having imposed these boundary conditions, we arrive at cells which do not exchange energy with their surroundings. On the other hand, the boundary conditions proposed earlier (see [9] for references) do not lead to the energetically independent cells.

The pressure and velocity field inside the envelope were calculated with the aid of the Lamb's general solution to the creeping motion equations. For the suspended particles the results are

$$u_i = U\left\{\bar{A}\delta_{i1}+\bar{B}\left(\frac{1}{5}\frac{r^2}{a^2}\,\delta_{i1}-\frac{1}{10}\frac{r^2}{a^2}\,n_1 n_i\right)\right\},$$

$$\bar{p} = U\bar{B}\mu r n_1/a^2,$$

where

$$\bar{A} = \frac{1}{\Delta}\left[-2\gamma^6(\eta-1)+\gamma^5(3\eta-3)-\gamma(3\eta+2)+2\eta+3\right],$$

$$\bar{B} = \frac{10}{\Delta}(\gamma^5-1), \qquad \gamma = a/b,$$

$$\Delta = -2\gamma^6(\eta-1)+\gamma^5(3\eta-2)-\gamma(3\eta+2)+2\eta+2,$$

and, respectively, for the ambient fluid

$$v_i = U\left\{A\delta_{i1}+B\left(\frac{1}{5}\frac{r^2}{a^2}\,\delta_{i1}-\frac{1}{10}\frac{r^2}{a^2}\,n_1 n_i\right) + C\left(\frac{a^3}{r^3}\,\delta_{i1}-3\frac{a^3}{r^3}\,n_1 n_i\right)+D\left(\frac{1}{2}\frac{a}{r}\,\delta_{i1}+\frac{1}{2}\frac{a}{r}\,n_1 n_i\right)\right\}, \tag{3.5}$$

$$p = U\{B\mu r n_1/a^2+D\mu n_1 a/r^2\},$$

where

$$A = -\frac{\gamma}{\Delta}\left[2\gamma^5(\eta-1)-3\eta-2\right], \quad B = \frac{10}{\Delta}\gamma^5\eta, \quad C = \frac{1}{2\Delta}\eta,$$

$$D = \frac{1}{\Delta}\left[2\gamma^5(\eta-1)-3\eta-2\right].$$

As it was discussed by HAPPEL [9], the resulting motion is of a circulatory type. This way the return, upward flow of fluid, observed between settling particles is taken into account.

In the present work a cell model approach is modified by introducing variable radii R of spherical envelopes surrounding suspended particles. To describe the variation of these radii, we adopt the nearest neighbour distribution function $f(a, R)$. Owing to the features of that function, the whole space occupied by the suspension is divided into non-overlapping cells (envelopes) which can be considered independently of one another.

Outside these envelopes the fluid is assumed to be at rest.

For an evaluation of the settling velocity we will need the average rate of energy dissipation in the suspension.

Inside an envelope of the radius R, the energy dissipation is presented in the form

$$E(a, R) = \begin{cases} 2\pi\mu a U^2 F(a/R), & a \leqslant R \leqslant b, \\ 2\pi\mu a U^2 F(a/b), & b \leqslant R \leqslant \infty, \end{cases} \tag{3.6}$$

where

$$F(a/R) = \frac{1}{25}\eta\bar{B}^2+24C^2\left(1-\frac{a^5}{R^5}\right)+\frac{1}{25}B^2\left(\frac{R^5}{a^5}-1\right)+2D^2\left(1-\frac{a}{R}\right)$$
$$-8CD\left(1-\frac{a^3}{R^3}\right)-\frac{2}{5}BD\left(\frac{R^2}{a^2}-1\right).$$

As envelopes surrounding different particles are not overlapping, the average dissipation per particle is

$$(3.7)\qquad <E> = 2\pi\mu aU^2\int\limits_a^\infty F(a/R)f(a, R)dR = 2\pi\mu aU^2\langle F\rangle.$$

The dissipation takes place only in envelopes since outside these envelopes the velocity field, by the assumption, vanishes.

4. The average settling velocity

To determine the average settling velocity one can calculate the time rate of change of potential energy P_t of fluid and particles in a volume V:

$$(4.1)\qquad P_t = \frac{4}{3}\pi a^3(\bar{\varrho}-\varrho)g\sum_{k=1}^{N}U^{(k)},$$

where $U^{(k)}$ denotes a settling velocity of particle k.

As usual, the average settling velocity U is determined through

$$(4.2)\qquad U \underset{\mathrm{df.}}{=} \lim_{V\to\infty}\frac{P_t}{\frac{4}{3}\pi a^3(\bar{\varrho}-\varrho)g\cdot N},$$

i.e. U yields the same rate of change of potential energy as is described by P_t.

To calculate U we take advantage of the fact that the change of potential energy is equal to minus the rate of energy dissipation:

$$(4.3)\qquad P_t = -\sum_{k=1}^{N}E^{(k)}(a, R) = -2\pi\mu aU^2\sum_{k=1}^{N}F^{(k)}(a/R).$$

In view of Eq. (3.7) we simply have

$$(4.4)\qquad P_t = -n2\pi\mu aU^2\langle F\rangle.$$

From Eqs. (4.2), and (4.4) it follows that

$$(4.5)\qquad \langle U\rangle = -\frac{2ga^2}{3\mu\langle F\rangle}.$$

Further, the relative settling velocity will be inquired:

$$(4.6)\qquad \left\langle\frac{U}{U_0}\right\rangle = \frac{2+3\eta}{(1+\eta)}\frac{1}{\langle F\rangle}.$$

The use of an explicit form of the distribution function enables also to give an expression for the dispersion coefficient σ:

$$\sigma^2 = \left\langle \left(\frac{U}{U_0} \right)^2 \right\rangle - \left\langle \frac{U}{U_0} \right\rangle^2, \tag{4.7}$$

or

$$\sigma^2 = \left[\frac{2+3\eta}{(1+\eta)\langle F \rangle} \right]^2 \left[\frac{\langle F^2 \rangle}{\langle F \rangle^2} - 1 \right],$$

which is the measure of fluctuations of settling velocities.

5. Results

For dilute suspensions we have arrived at a correction to the Stokes' settling velocity, proportional to the one-third power of the concentration ϕ. That result is a reminiscence

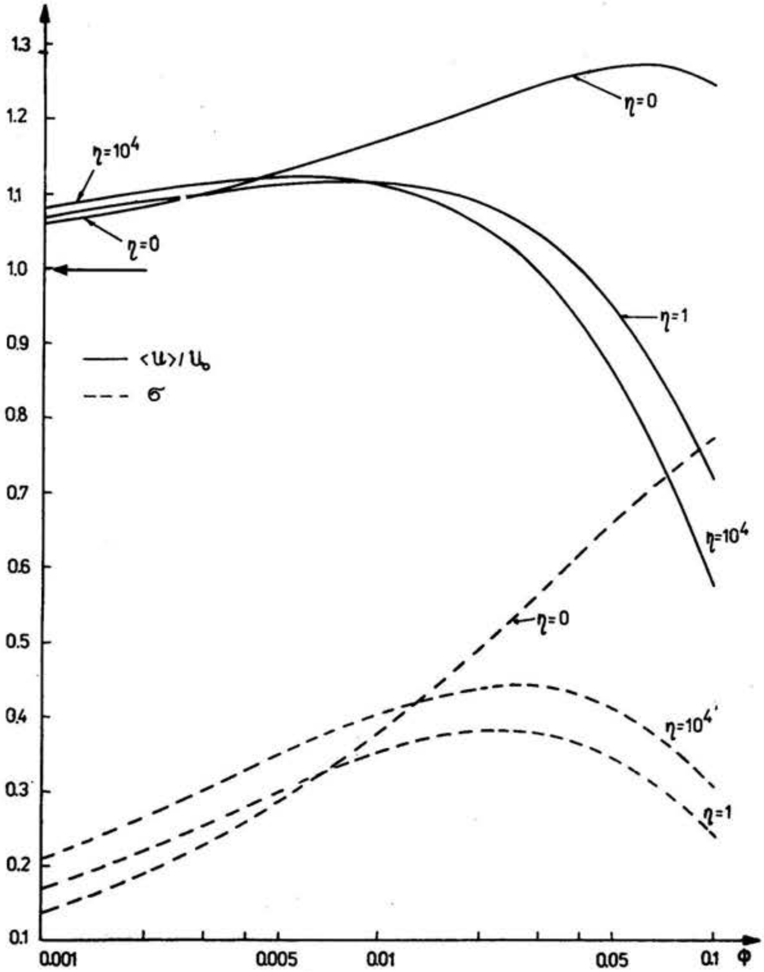

FIG. 2. The relative settling rate (solid lines), and the dispersion coefficient (dotted lines), for viscosity ratios equal to $\eta = 10^4$, $\eta = 1$, and $\eta = 0$, at small volume concentrations ϕ.

of the cell model employed. Such a dependence appears in all approaches, exploiting that model [9]:

$$\left\langle \frac{U}{U_0} \right\rangle \xrightarrow[\phi \to 0]{} 1 - Q\left(\frac{\phi}{\phi_m}\right)^{1/3} \dots \tag{5.1}$$

However, due to the assumed particle arrangement, the coefficient Q is negative; this feature leads to the overshoot of the relative settling velocity

$$\begin{gathered} Q = 3\{1 - 230.4\phi_m^3 \chi - e^{-8\phi_m}(0.5 + 28.8\phi_m^2 + 6\phi_m)\}, \\ \chi = \sum_{n=0}^{\infty} \frac{(-1)^n (8\phi_m)^n}{\left(\frac{8}{3} + n\right) n!}, \quad \chi \approx 0.01266, \\ \phi_m = 0.74, \quad Q = -0.71. \end{gathered} \tag{5.2}$$

We are aware of the fact that the presence of the overshoot is not generally accepted. Here only one of the possible explanations of the observed behaviour is discussed.

Figure 2 shows the relative settling velocity and the dispersion coefficient for suspensions up to the concentration ϕ equal to 0.1. The non-monotonic dependence should be noted. The position of the overshoot depends on the viscosity ratio η; also the dispersion curve exhibits the maximum in the same region with a slight shift towards greater concentrations.

For rigid particles the predicted relative settling velocity has the maximum equal to 1.12 at about $\phi = 0.006$. The numbers given here and the curves plotted in Figs. 2, 3 and 4 are results of numerical calculations, according to the formulae (4.6) and (4.7).

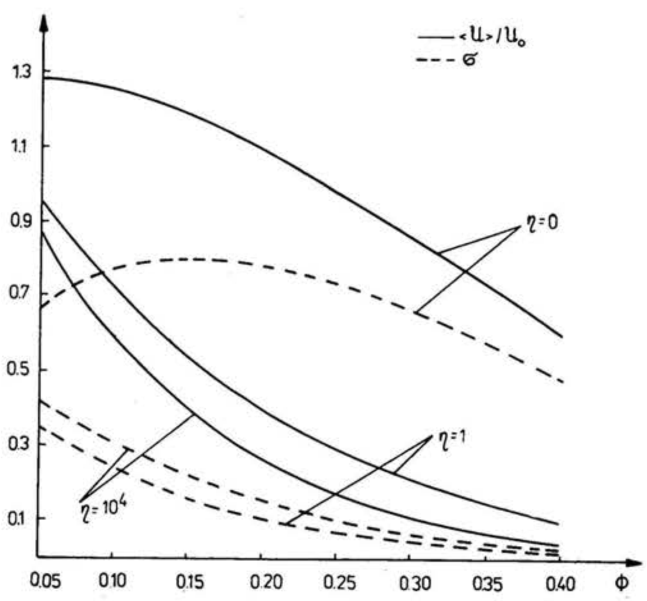

FIG. 3. The relative settling rate (solid lines), and the dispersion coefficient (dotted lines), for viscosity ratios equal to $\eta = 10^4$, $\eta = 1$, and $\eta = 0$, at intermediate volume concentrations ϕ.

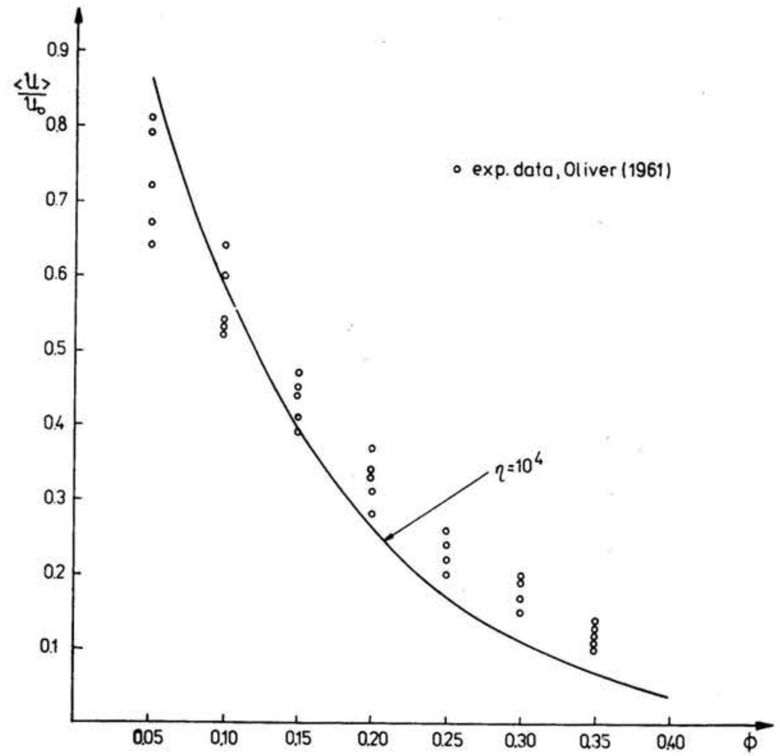

FIG. 4. Comparison of the calculated relative settling rate with the experimental data selected by OLIVER [10].

A tentative comparison with experimental findings is not discouraging, as Barfod reported the maximum in the range $0.001 < \phi < 0.002$, and the maximum relative settling velocity changing between $1.12 < U/U_0 < 1.4$.

Figure 3 presents the behaviour of the settling velocity and the dispersion for denser suspensions. Here the relative settling velocity decreases with the concentration ϕ, as it is expected. The high values of the dispersion are of interest. This feature is attributed to the slow asymptotic decrease, as r^{-1}, of the velocity disturbance due to the individual falling particle (Stokes' particle).

The last figure, Fig. 4, gives the comparison with the experimental data selected by OLIVER [10]. Certainly, the denser the suspension, the more questionable is the present analysis based on a purely geometrical distribution function. In view of this fact the comparison is even more promising than it was expected.

Finally, it should be mentioned that any cell model provides only approximate results and it always serves as a tool because of the lack of more rigorous methods.

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