Archives of Mechanics • Archiwum Mechaniki Stosowanej • 30, 4–5, pp. 645–656, Warszawa 1978

Pressure wave pattern in a liquid filling an elastic pipe

A. SZUMOWSKI (WARSZAWA)

A shock wave generated in a liquid produces a tensile wave in the wall of a pipe. Due to the tensile wave, which usually propagates faster than the shock, there emerges in the liquid a pressure wave preceding the shock. Moreover, in certain circumstances, cavitation zones appear in front of the shock. As a result, the complex pressure wave composed of the preceding wave, cavitation zones and the shock wave is formed. The shock front is gradually attenuated along the pipe. This complex pressure wave in an inviscid liquid filling a semi-infinite pipe of circular cross-section is considered theoretically in the paper. One-dimensional motion of the liquid is assumed. The influence of this assumption and of some simplifications performed in the equations of pipe motion on the calculation results are shown.

Fala uderzeniowa wytworzona w cieczy powoduje oscylacje ścianki rury. Oscylacje te rozprzestrzeniają się wzdłuż rury zwykle z prędkością większą niż wynosi prędkość fali w cieczy. Na skutek tego przepływ przed czołem fali uderzeniowej zostaje wstępnie zaburzony. W pewnych przypadkach, w przepływie tym mogą wystąpić strefy kawitacji. W wyniku działania wymienionych czynników fala uderzeniowa w cieczy ulega stopniowemu tłumieniu. W pracy, na drodze teoretycznej, badano opisane zjawiska w przypadku cieczy nielepkiej wypełniającej półnieskończoną rurę o przekroju kołowym, zakładając przy tym jednowymiarowy ruch cieczy. Zbadano wpływ tego założenia oraz wpływ pewnych uproszczeń w równaniach ruchu rury na wyniki obliczeń.

Ударная волна, генерированная в жидкости, вызывает колебания стенки трубы. Эти колебания распространяются по трубе со скоростю, которая преимущественно большая чем скорость волны в жидкости. В следстве того течение перед ударной волной сразу возмущено. В некоторых случаях, в этом течении могут возникнуть кавитационные зоны. В результате действительности вышеприведенных факторов ударная волна постепенно затухается. В статии теоретическом путём исследовано эти явления в невяской жидкости выполняющей пол-бесконечную трубу круглого сечения. При этом предположено одномерное движение жидкости. Исследовано влияние этого предположения и некоторых упрощений принятых в уравнениях движения трубы на результаты вычислений.

Nomenclature

- A bore area of pipe,
- a velocity of sound in liquid, $a = (K/\varrho_L)^{\frac{1}{2}}$,
- c velocity of sound in pipe wall, $c = [E/(1-\nu^2)\varrho_w]^{\frac{1}{2}}$,
- D $Eh^3/12(1-\nu^2)$,
- E Youngs modulus of pipe wall material,
- E_p $Eh/(1-\nu^2)$,
- G $E/2(1+\nu)$,
- h thickness of pipe wall,
- I $h^2/12$,
- I_0, I_1 Bessel's functions,
- K bulk modulus of liquid,

http://rcin.org.pl

M bending moment,
N_x, N_θ longitudinal and circumferential forces,
p pressure,
p_s $\varrho_L a \vartheta_0$,
p_v pressure of saturated vapour,
Q shear force,
r radial coordinate,
R mean radius of pipe,
t time,
$\bar{t}$ $t\,c/R$,
u longitudinal displacement of pipe wall element,
w radial diplacement of pipe wall element,
w_s $p_s R^2/Eh$,
x coordinate along the pipe,
φ angular deformation of pipe wall element,
ν Poisson's ratio,
ϱ density,
ϑ liquid velocity,
ϑ_0 initial liquid velocity.

Suffixes

L liquid,
W wall material,
r radial direction,
x longitudinal direction.

1. Introduction

THE FIRST significant constribution to the solution of water hammer problems include those of KORTEWEG [1], JOUKOWSKI [2], LAMB [3] and ALLIEVI [4].

The most common and practically applied theory is Joukovski's. This theory predicts that pressure waves travel without change of shape and velocity

$$a_L = \frac{a}{\left(1+\frac{2R}{h}\frac{K}{E}\right)^{\frac{1}{2}}}. \tag{1.1}$$

According to this theory, the pressure rise due to the rapid flow stopping is

$$\Delta p = \varrho_L a \vartheta_0. \tag{1.2}$$

In Joukowski's theory it is assumed that the flow properties are uniform across any section of the pipe and that the deflection of the pipe is proportional to the instantaneous pressure in the liquid.

In fact, the wave phenomenon resulting from the rapid stopping of the flow is much more complicated. Let us take into consideration an extreme case, when as an effect of this stopping a shock wave emerges (Fig. 1a). The pressure jump in this shock causes, among other things, radial oscillations of the pipe wall. Because of these oscillations at

http://rcin.org.pl

the surface of the fluid, waves are generated and then are propagated inside the pipe (Fig. 1b). Finally, the flow across any section becomes non-uniform. Moreover, the oscillations of the pipe propagate along the pipe wall with the velocity of sound specific for a material of the wall. In most cases this velocity is greater than the velocity of sound for the liquid.

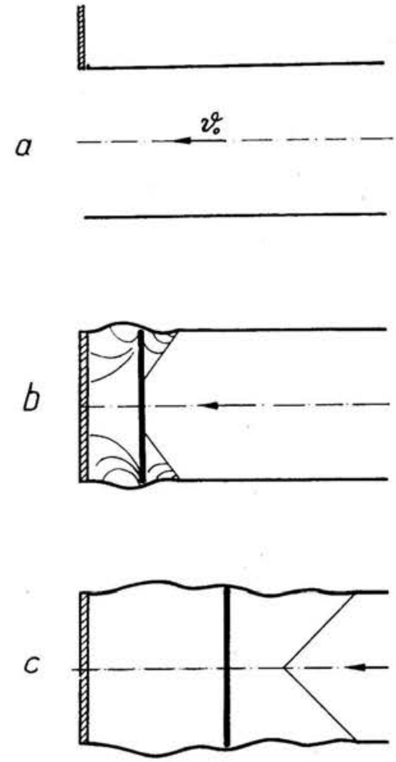

FIG. 1. Physical model of the investigated waves motion.

Thus the wave in the pipe wall moves ahead of the shock front and disturbes the initial flow properties. Eventually, the shock wave is preceded by a weak cone wave (Fig. 1c).

One can expect that due to the wall oscillations, the pressure distribution along the pipe becomes non-uniform and varies in time.

When the initial pressure is sufficiently low, the pressure ahead of the shock front can locally decrease to the saturated vapour pressure.

Of particular significance to this problem is a theoretical study by SKALAK [5], whose model of a transient water-hammer phenomenon includes axi-symmetric motion of the liquid as well as the inertia of the pipe wall.

By means of Fourier and Laplace transforms of the equations of motion for the pipe wall and for the liquid, Skalak obtained an asymptotic solution, which is valid after a sufficiently long time from the moment of the flow stopping.

An example of Skalak's results is shown in Fig. 2 where pressure, longitudinal stress and radial displacement distributions along a pipe are given.

It is specific in Skalak's solution that these distributions are uniform between the preceding and the main waves as well as behind the main wave.

http://rcin.org.pl

Further investigations of this problem made by KING and FRIEDERIC [6] concerned the preliminary phase of the wave formation when the shock front travels a distance up to 3 radii.

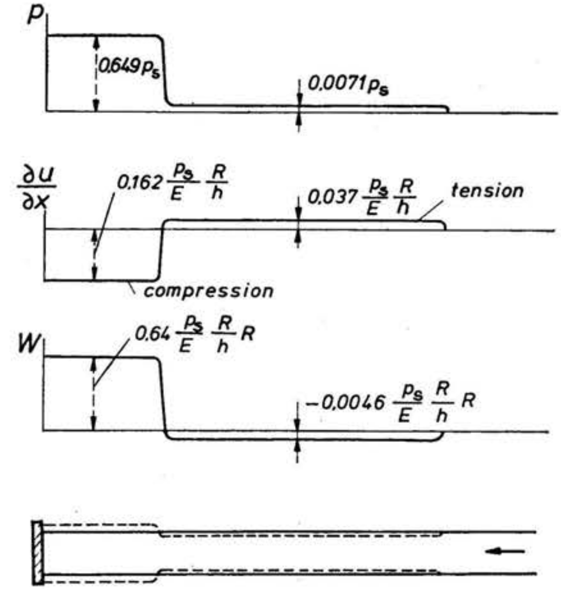

FIG. 2. Example of Skalak's results [5].

The authors gave pressure distributions in the case where longitudinal and shear forces as well as momentum in the wall were neglected.

Naturally, in this case, the existence of the precursor wave and the attenuation of the shock wave were not predicted.

In all of the previous investigations continuous flow of the liquid was assumed; that means cavitation was neglected. The aim of this work is to investigate the formation of the wave pattern generated by sudden termination of a uniform flow at the end of a semi-infinite tube including all internal forces and inertia of the pipe wall as well as the possible discontinuities of the liquid.

2. Governing equations

In the case of axi-symmetric loading of the pipe, one can distinguish four internal forces shown in Fig. 3. A pipe element is displaced and deformed as a results of the action of these forces (Fig. 4).

Using the notation shown in Figs. 3 and 4 and designating a comma as the x-derivative and a point as the t-derivative, one can write the equations of pipe motion as follows:

Equation of motion in the axial direction

$$N_x' = \varrho_W h\left(\ddot{u} + \frac{h^2}{12R}\ddot{\varphi}\right), \tag{2.1}$$

http://rcin.org.pl

in the radial direction

$$Q' - \frac{N_\theta}{R} + p\left(1 - \frac{h}{2R}\right) = \varrho_W h \ddot{w}, \tag{2.2}$$

and the equation of rotary motion

$$M' - Q = \frac{\varrho_W h^3}{12}\left(\frac{\ddot{u}}{R} + \ddot{\varphi}\right). \tag{2.3}$$

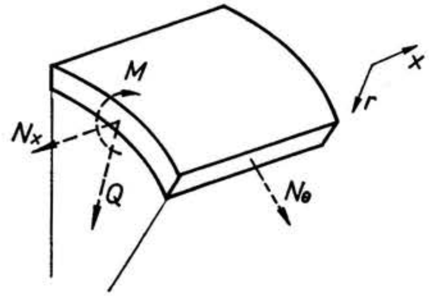

FIG. 3. Internal forces in the pipe element.

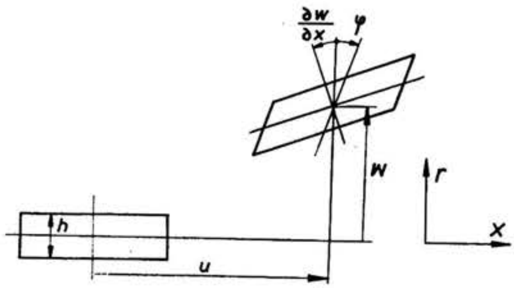

FIG. 4. Displacements and deformation of the pipe element.

Moreover, we have the following four equations which describe the relations between forces and deformations for the case of an elastic pipe:

$$\begin{aligned} N_x &= E_p\left(u' + \nu\frac{w}{R} + \frac{h^2}{12R}\varphi'\right), \\ N_\theta &= E_p\left[\nu u' + \frac{w}{R}\left(1 + \frac{h^2}{12R^2}\right)\right], \\ M &= D\left(\frac{u'}{R} + \varphi'\right), \\ Q &= k^2 Gh(\varphi + w'), \end{aligned} \tag{2.4}$$

where $k^2 = 0.91$ for $\nu = 0.3$ [7].

Eliminating forces from the equations of motion by means of Eqs. (2.4), one obtain the following [7]:

$$\left[E_p\frac{\partial^2}{\partial x^2} - \varrho_W h\frac{\partial^2}{\partial t^2}\right]u + \left[E_p\frac{\nu}{R}\frac{\partial}{\partial x}\right]w + \left[\frac{D}{R}\frac{\partial^2}{\partial x^2} - \varrho_W\frac{J}{R}\frac{\partial}{\partial t^2}\right]\varphi = 0, \tag{2.5}$$

http://rcin.org.pl

$$\left[E_p\frac{\nu}{R}\frac{\partial}{\partial x}\right]u+\left[\frac{E_p}{R^2}+\frac{D}{R^4}-k^2Gh\frac{\partial^2}{\partial x^2}+\varrho_W h\frac{\partial^2}{\partial t^2}\right]w-\left[k^2Gh\frac{\partial}{\partial x}\right]\varphi = p\left(1-\frac{h}{2R}\right), \tag{2.6}$$

$$\left[\frac{D}{R}\frac{\partial}{\partial x^2}-\varrho_W\frac{J}{R}\frac{\partial^2}{\partial t^2}\right]u-\left[k^2Gh\frac{\partial}{\partial x}\right]w+\left[D\frac{\partial^2}{\partial x^2}-k^2Gh-\varrho_W J\frac{\partial^2}{\partial t^2}\right]\varphi = 0. \tag{2.7}$$

When one includes only circumferential forces and radial inertia which seem to be the most important factors of the pipe motion, Eqs. (2.5), (2.6) and (2.7) reduce to the following:

$$h\varrho_W\frac{d^2w}{dt^2}+E\frac{w}{R}\frac{h}{R} = p\left(1-\frac{h}{2R}\right). \tag{2.8}$$

The axi-symmetric motion of inviscid fluid is governed by the following equations.

Equation of continuity

$$\frac{\partial(\varrho_L\vartheta_r)}{\partial r}+\frac{\varrho_L\vartheta_r}{r}+\frac{\partial(\varrho_L\vartheta_x)}{\partial x}+\frac{\partial\varrho_L}{\partial t} = 0, \tag{2.9}$$

and two momentum equations

$$\frac{\partial\vartheta_x}{\partial t}+\vartheta_x\frac{\partial\vartheta_x}{\partial x}+\vartheta_r\frac{\partial\vartheta_x}{\partial r}+\frac{1}{\varrho_L}\frac{\partial p}{\partial x} = 0, \tag{2.10}$$

$$\frac{\partial\vartheta_r}{\partial t}+\vartheta_x\frac{\partial\vartheta_r}{\partial x}+\vartheta_r\frac{\partial\vartheta_r}{\partial r}+\frac{1}{\varrho_L}\frac{\partial p}{\partial r} = 0. \tag{2.11}$$

Linearizing and neglecting terms with ϑ_r, ϑ_x one can reduce Eqs. (2.9), (2.10) and (2.11) to the axi-symmetric wave equation with respect to the velocity potential ϕ

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2\phi}{\partial t^2}. \tag{2.12}$$

On the other hand, when one-dimensional flow is assumed one obtains the following:

Equation of continuity

$$\frac{\partial(\varrho_L\vartheta A)}{\partial x}+\frac{\partial(\varrho_L A)}{\partial t} = 0, \tag{2.13}$$

and momentum equation

$$\frac{\partial\vartheta}{\partial t}+\vartheta\frac{\partial\vartheta}{\partial x}+\frac{1}{\varrho_L}\frac{\partial p}{\partial x} = 0. \tag{2.14}$$

In these equations the radial deformation of the pipe is taken into account.

The equations governing the motion of the pipe wall are of the hyperbolic type and have the following characteristics:

$$\frac{dx}{dt}\mp c_l = 0, \tag{2.15}$$

$$\frac{dx}{dt}\mp ck\sqrt{\frac{G}{E}(1-\nu^2)} = 0, \tag{2.16}$$

$$\frac{dx}{dt} = 0. \tag{2.17}$$

http://rcin.org.pl

Physically, the characteristics given by Eq. (2.15) correspond to the propagation of axial forces and momenta, the characteristics given by Eq. (2.16) to the shear forces and the characteristic given by Eq. (2.17) corresponds to the path line.

Neglecting the convection derivative in the momentum equation (Eq. 2.14), we have the following characteristics for equations of one-dimensional flow of the liquid:

$$\frac{dx}{dt} \mp a = 0. \tag{2.18}$$

Along these characteristics we have

$$dp \pm \varrho_L a d\vartheta = -2\frac{\varrho_L a^2}{R}\frac{\delta w}{\delta t}dt, \tag{2.19}$$

where $\delta w/\delta t$ is derivative along path line.

In the present investigation the case is studied when $c > a$; the velocity of sound in a steel pipe is about 4 times greater than in water. The characteristics corresponding to this case for the positive x-direction are drawn in Fig. 5.

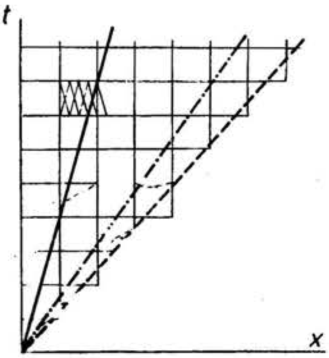

FIG. 5. Model of numerical calculations.

3. Numerical calculations and results

The disturbed region in the x, t plane due to the rapid stopping of the flow is determined by the angle between the t-axis and the characteristic $dx/dt = c$. Numerical calculations of the flow properties and wall displacements in this region are carried out by the method of characteristics for the equation of motion of the liquid Eqs. (2.13), (2.14) and by the method of finite defferences for the equations of motion of the pipe wall Eqs. (2.5), (2.6) and (2.7). For this purpose the rectangular net is built in which the ratio of the t-step to the x-step is limited by the characteristic $dx/dt = c$. In these calculations it is taken that $\Delta t = \Delta x/a \cdot n$, where $n =$ entier $(c/a)+1$. This formula fulfills the limitation mentioned above and simultaneously secures the sharp shock wave in the calculations. In order to achieve this purpose each x-step (Δx) is devided into n equal segments. At the ends of these segments flow properties are calculated.

In the case when the pressure of the liquid at any section of the pipe decreases below the pressure of the saturated vapour (p_v), it is assumed at this section that $p = p_v$.

http://rcin.org.pl

The following initial conditions for fluid motion are assumed:

$$\vartheta(0,0) = 0, \quad \vartheta(x > 0, 0) = -\vartheta_0,$$
$$p(0,0) = p_s, \quad p(x > 0, 0) = 0,$$

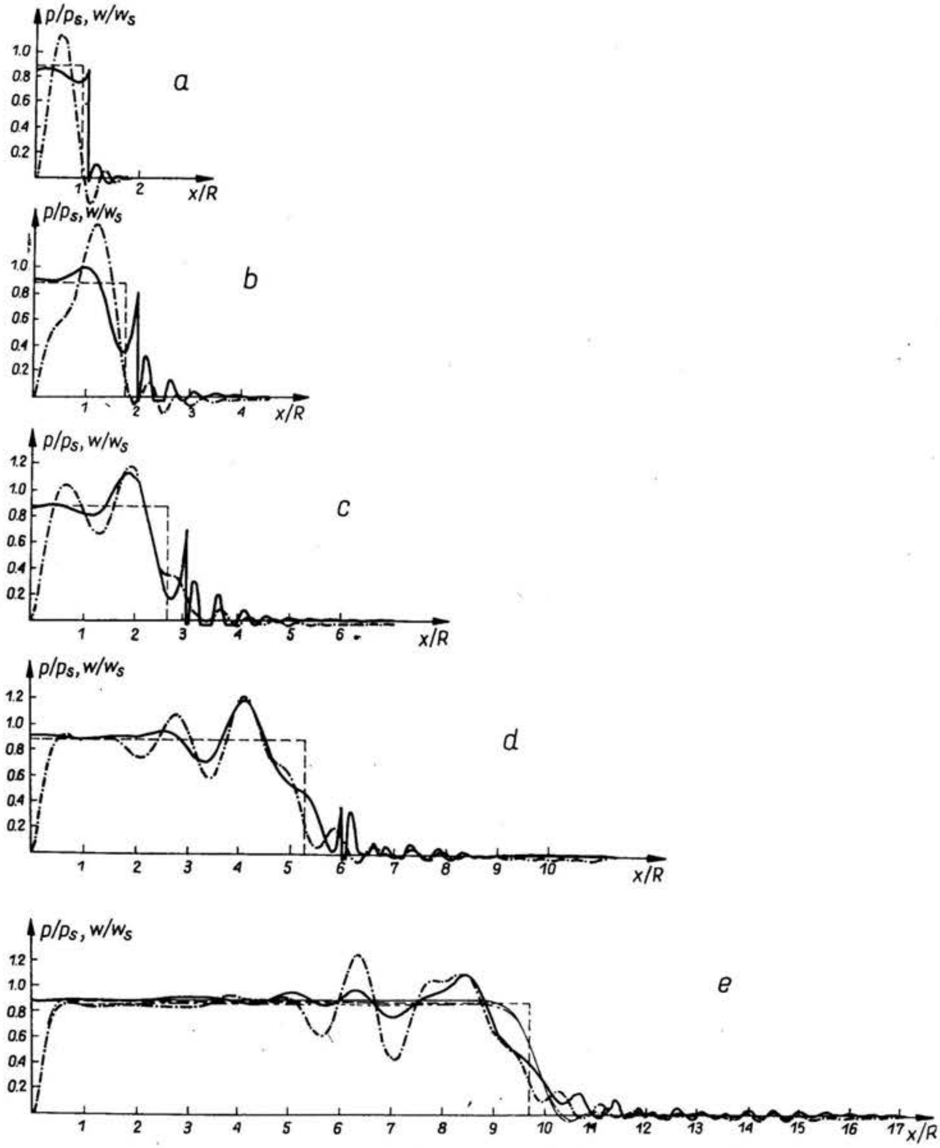

FIG. 6. Pressure and radial displacement distributions along the pipe

pressure: unsteady motion of the pipe, quasi-steady motion of the pipe (e), Joukowski's theory;

radial displacement: unsteady motion, quasi-steady motion,

$a/\bar{t} = 3.63$, $b/\bar{t} = 7.26$, $c/\bar{t} = 10.9$, $d/\bar{t} = 21.79$, $e/\bar{t} = 39.96$.

http://rcin.org.pl

which corresponds to rapid stopping of the flow. Moreover, it is assumed that wall displacements and their first derivatives are initially equal to 0.

A semi-infinite pipe whose end is rigid and motionless is considered. At this end we have the following boundary conditions:

$$w(0,t) = u(0,t) = \varphi(0,t) = 0,$$
$$\frac{\partial w}{\partial x}(0,t) = \frac{\partial u}{\partial x}(0,t) = \frac{\partial \varphi}{\partial x}(0,t) = 0.$$

The results of calculations carried out with the data $h/R = 0.06598$, $a/c = 0.2753$, $\varrho_L a^2/(\varrho_W c^2) = 0.009596$ are shown in Figs. 6-9. Figure 6 indicates the successive stages of the wave motion. One can observe that the pressure distribution before the shock front as well as behind it is non-uniform.

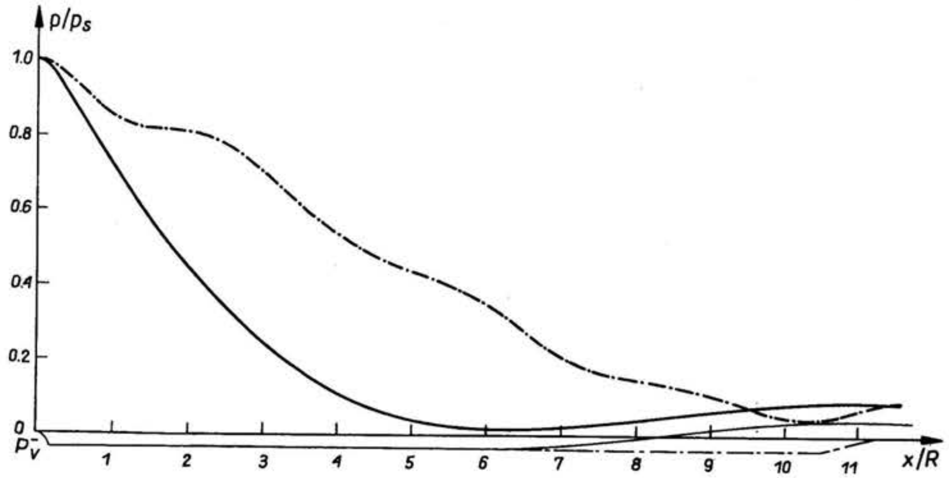

FIG. 7. Shock wave attenuation along the pipe (———) unsteady and (—·—·—) quasi-steady motion of the pipe.

Immediately behind the shock, pressure decreases because the wall in this region has the highest outward radial velocity. After the shock wave is passed, the wall returns and the pressure increases again.

It can also be seen that the strength of a shock wave is gradually diminished. At the stage when the shock front reaches a position equal to 11 radii (Fig. 6e), the shock is practically no longer visible.

When the shock wave is sufficiently strong, regions of cavitation emerge ahead of the shock front.

In Fig. 6e the pressure and the radial displacement of the wall in the case of quasi-steady deformation of the pipe are also shown. One can observe that both the pressures and the radial displacements corresponding to the quasi-steady and the unsteady motion of the pipe are very close to one another at a sufficient distance behind the main wave.

A good approximation of the pressure distribution in the main wave could be obtained from Joukowski's theory.

http://rcin.org.pl

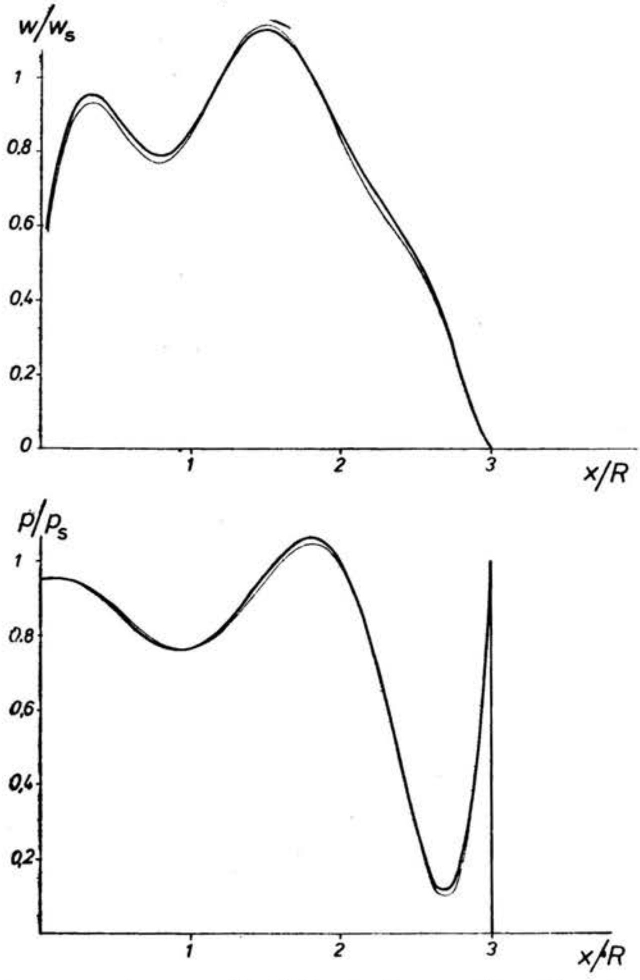

FIG. 8. Radial displacement and pressure distributions in the case of one-dimensional (thick lines) and axi-symmetric (thin lines) flow.

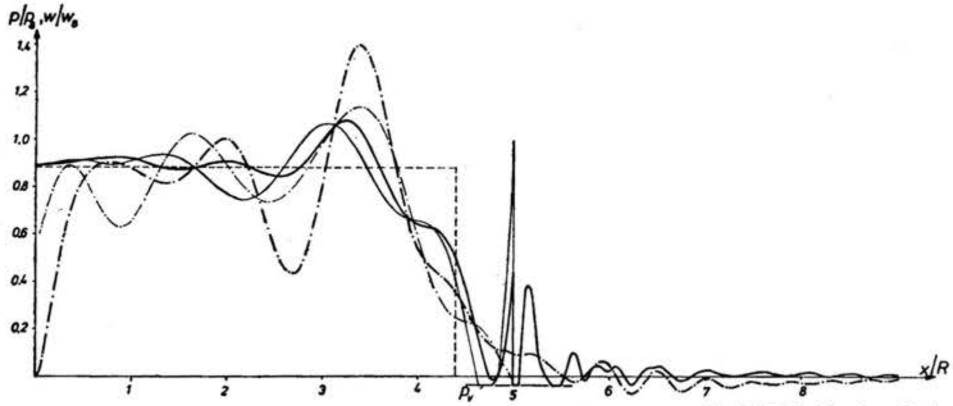

FIG. 9. Radial displacement and pressure distributions obtained by means of full (thick lines) and simplified (thin lines) equations of pipe motion (see Fig. 6).

http://rcin.org.pl

Shock wave attenuation determined in the case of the quasi-steady and the unsteady deformation of the pipe is shown in Fig. 7 where the pressures behind as well as ahead of the shock front are given. One can see that the shock wave has been almost fully attenuated at a distance of about 10 radii. Immediately ahead of the shock front, when it is sufficiently strong, the cavitation zone appears.

In all of these calculations a one-dimensional model of the motion of the liquid was assumed. In fact, as mentioned above, the flow is axi-symmetric. The influence of this assumption on the result of calculations was examined in the following manner, which is similar to that of KING and FRIEDERIC [6].

One can convert the axi-symmetric wave equation Eq. (2.12) by means of the Hankel transform

$$\varkappa(\pi_i, x, t) = \frac{2}{R_z^2}\int\limits_0^{R_z} rJ_0(\pi_i r)\phi(x, r, t)dt,$$

where $R_z = R(1-h/2R)$, π_i are non-negative roots of $J_1(\pi_i R_z) = 0$, to this form

$$\frac{\partial^2\varkappa_i}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2\varkappa_i}{\partial t^2}-\pi_i^2\varkappa+R_zJ_0(\pi_iR_z)\left(\frac{\partial\phi}{\partial r}\right)_{r=R_z} = 0. \tag{3.1}$$

The derivative $(\partial\phi/\partial r)_{r=R_z}$ is determined by the boundary condition between the liquid and the wall, namely

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=R_z} = \frac{\partial w}{\partial t}.$$

Introducing new variables

$$\vartheta_i = \frac{\partial\varkappa_i}{\partial x}, \quad p_i = \varrho_L\frac{\partial\varkappa_i}{\partial t},$$

one obtains the following:

$$\frac{\partial\vartheta_i}{\partial t}+\frac{1}{\varrho_L}\frac{\partial p_i}{\partial x} = 0, \tag{3.2}$$

$$\frac{\partial\vartheta_i}{\partial x}+\frac{1}{\varrho_La^2}\frac{\partial p_i}{\partial t} = \pi_i^2\varkappa_i+\frac{2}{R_z}\frac{\partial w}{\partial t}, \tag{3.3}$$

which can be recognized as the momentum and continuity equations with successive pairs of variables p_i and ϑ_i for i from 0 to infinity. The compatibility equations along the characteristics of Eqs. (3.2) and (3.3) are then

$$\frac{1}{\varrho_La}dp_i\pm d\vartheta_i = \left(\pi_i^2\varkappa_i-\frac{2}{R}\frac{\partial w}{\partial t}\right)adt. \tag{3.4}$$

The wall pressure is equal to the following sum:

$$p_{r=R_z} = \sum_{i=0}^{\infty} p_i. \tag{3.5}$$

The results of calculations using this method for i from 0 to 4 are compared with the corresponding ones for the one-dimensional model (Fig. 8). For simplicity Eq. (2.8) is used.

http://rcin.org.pl

The lines corresponding to the one-dimensional and axi-symmetric model are very close to each other. Thus one can conclude that in this case the one-dimensional model of the flow is accurate.

In Fig. 9 pressure and radial displacement distributions along a pipe, obtained by means of full and simplified equations of pipe wall motion Eqs. (2.5), (2.6), (2.7) and (2.8), respectively, are compared. One can see that the corresponding distributions are far away one from the other. Thus not only circumferential forces and radial inertia Eq. (2.8) but also longitudinal, shear forces and bending moment as well as longitudinal and rotary inertias have essential influence on the investigated wave motion.

References

1. D. I. KORTEWEG, *Ueber die Fortpflanzungsgeschwindigkeit des Schalles in elastisches Rohren*, Annalen der Physik and Chemie, **9**, 5, 1878.
2. N. JOUKOWSKI, *Ueber den hydraulischen Stoss in Wasserleitungsrohren*, Memoirs Acad. Imper. Sci., **9**, 5, Petersburg 1900.
3. H. LAMB, *On the velocity of sound in a tube, as affected by the elasticity of the walls*, Manchester Literary and Philosophical Society, Memoirs and Proc., **42**, 9, 1898.
4. L. ALLIEVI, *Allgemeine Theorie ueber die veraenderliche Bewegung des Wassers in Leitungen*, J. Springer, Berlin 1909.
5. R. SKALAK, *An extension of the theory of water hammer*, Water Power, Dec. 1955.
6. W. KING, D. FRIEDERIK, *Transient elastic waves in a fluid-filled cylinder*, J. Engng. Mech. Div., Oct. 1968.
7. G. HERMANN, I. MIRSKY, *Three-dimensional and shell theory analysis of axially symmetric motion of cylinders*, J. Appl. Mech., **23**, Trans ASME, **78**, Series E, 1956.

TECHNICAL UNIVERSITY OF WARSAW.

Received November 7, 1977.

http://rcin.org.pl