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A hemivariational inequality approach to the delamination effect in theory of layered plates

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THE DELAMINATION effect in layered, laminated or sandwich plates is studied in this paper. In order to describe the debonding effect between the consecutive layers, the behaviour of the binding material is described by a generally nonmonotone and possibly multivalued law. Thus the problem can be formulated as a hemivariational inequality. This inequality is studied, taking the existence and the approximation of its solution into consideration, by using compactness and average value arguments. Finally the C^0-convergence of the introduced general discretization scheme is investigated.

Rozważa się zjawisko delaminacji w płatach warstwowych, laminowanych i sandwiczowych. Dla opisania zjawiska utraty spójności poszczególnych warstw, zachowanie się materiału wiążącego opisuje się za pomocą w ogólności niemonotonicznej i niejednoznacznej funkcji stanu. W ten sposób problem sformułować można w postaci nierówności półwariacyjnej. Istnienie jej rozwiązania oraz jego dokładność bada się rozważając jego zwartość i wartości średnie. Na koniec bada się również zbieżność C^0 wprowadzonego schematu dyskretyzacji.

Рассматривается явление деламинации в слоистых, ламинированных и сандвичообразных листах. Для описания явления потери связности отдельных слоев, поведение связывающего материала описывается при помощи в общем немонотонной и неоднозначной функции состояния. Таким образом проблему можно сформулировать в виде полувариационного неравенства. Существование ее решения и его точность исследуется, рассматривая его компактность и средние значения. Наконец, исследуется тоже сходимость C^0 введенной схемы дискретизации.

1. Introduction

IN THE PRESENT paper we formulate and study the delamination problem for layered plates in terms of hemivariational inequalities. The developed theory holds for any type of laminated and sandwich plates allowing for the debonding of the laminae. It is well known [1] that the interlaminar normal stresses may cause debonding normally to the contact area as well as interlayer slip. Both these phenomena are responsible for the strength degradation of the composite plates. Here we shall study the first effect by assuming that the binding material introduces a nonmonotone, possibly multivalued law, connecting the interlaminar stresses with the corresponding relative displacement normally to the interlayer surface. Indeed the interlaminar normal stress is considered (cf. e.g. [1, 2]) to be the main delamination cause. This law yields the variational formulation of the problem as a hemivariational inequality which permits the determination of the delamination fronts.

Due to the nonmonotonicity of the law, i.e. the lack of convexity of the corresponding "potential", the variational formulation is no longer a variational inequality (cf. e.g.

[3]) but a hemivariational inequality (cf. e.g. [4]). The theory of variational inequalities is closely connected with convex analysis: indeed we may obtain variational inequalities as expressions of the principle of virtual power—or work—if certain "generalized forces" are monotone, possibly multivalued, functions of the generalized displacements and this law can be expressed in terms of the subdifferential [5] of a convex, nonsmooth, potential called superpotential (see e.g. [6] and for relative references [4]). In the case of lack of monotonicity, i.e. if we have nonmonotone, possibly multivalued laws we may define nonconvex superpotentials [7, 8, 4] and the subdifferential ∂ has to be replaced by the generalized gradient $\bar{\partial}$, a notion recently introduced (cf. e.g. [9]) by F. H. Clarke–R. T. Rockafellar. In this case we get hemivariational inequalities, the mathematical study of which was initiated in [10, 11, 12, 4]. In contrast to the theory of variational inequalities whose study is based on monotonicity arguments the study of hemivariational inequalities is based on compactness arguments. Moreover, we obtain instead of minimum problems for the potential and complementary energy substationarity problems [8, 4, 13, 14, 15].

In the present paper the interlaminar action normal to the contact surface is simulated by a general law derived by a nonconvex superpotential. The arising hemivariational inequality is studied, taking into consideration the existence and the approximation of its solution. Then the approximation properties are studied and the C^0-convergence of a general discretization scheme presented here is proved.

2. The interlaminar superpotential law

Let us consider a layered plate consisting of m-layers. Each layer is an elastic plate and is referred to a right-handed orthogonal Cartesian coordinate system $Ox_1x_2x_3$ (Fig. 1). The plates have constant thicknesses $h_1, h_2, \ldots, h_m$, and the middle surface of each plate coincides with the respective Ox_1x_2-plane. Let Ω_j, $j = 1, 2, \ldots, m$ be open, bounded and connected subsets of R^2 and suppose that their boundaries Γ_j are Lipschitzian ($C^{0,1}$-boundary). The domains Ω_j are occupied by the plates in their undeformed state. On $\Omega'_j \subset \Omega_j \cap \Omega_{j+1}$ (Ω_j is such that $\bar{\Omega}'_j \cap \Gamma_j = \phi$ and $\bar{\Omega}'_j \cap \Gamma_{j+1} = \phi$) the plates j and $j+1$ are bonded together through an adhesive material. We denote by $\zeta_j(x)$ the deflection of the point $x = (x_1, x_2, x_3)$ and by $f_j = (0, 0, f_{3j})$, $f_{3j} = f_{3j}(x)$ (hereafter called f_j for simplicity) the distributed load of the considered plate per unit area of the middle surface (Fig. 1). In order to describe the bonding action in the Ox_3-direction by means of a phenomenological law, we split f_j into $\bar{\bar{f}}_j \in L^2(\Omega_j)$, which is the given external loading acting on the j-th plate, and $\bar{f}_j$ which denotes the interaction between the plate under consideration (plate j) and the plates $j-1$ and $j+1$, caused by the bonding material, i.e.

$$f_j = \bar{f}_j + \bar{\bar{f}}_j \quad \text{on} \quad \Omega_j. \tag{2.1}$$

$\bar{f}_j$ consists of two parts: the part $\bar{f}_j^u$ describing the influence due to the bonding with the plate $j-1$ (upper plate) and the part $\bar{f}_j^l$ describing the influence of the bonding with the plate $j+1$ (lower plate). Obviously $\bar{f}_1^u = 0$ and $\bar{f}_m^l = 0$, i.e. the upper (resp. the lower) surface of the first (resp. the last) lamina are not subjected to bonding forces.

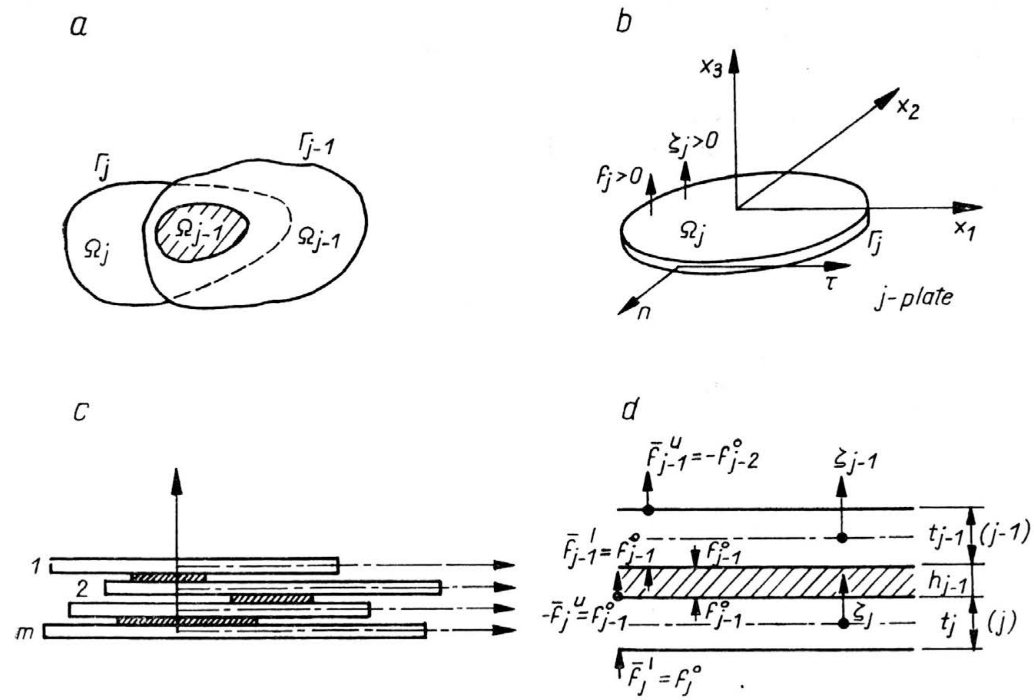

FIG. 1. Notations in the theory of layered plates.

Then we make the general assumption that the force f^0_{j-1} of the adhesive material between the $(j-1)$- and the j-plate is generally a multivalued nonmonotone function $\hat{\beta}_{j-1}$ of the relative displacement

$$[\zeta]_{j-1} = \zeta_{j-1}-\zeta_j \tag{2.2}$$

of the plates $j-1, j$.

We write that

$$-f^0_{j-1} \in \hat{\beta}_{j-1}(\zeta_{j-1}-\zeta_j) = \hat{\beta}_{j-1}([\zeta]_{j-1}) \quad \text{on} \quad \Omega'_{j-1}, \tag{2.3}$$

where

$$\Omega'_{j-1} \subset \Omega_{j-1}\cap\Omega_j, \quad \bar{\Omega}'_{j-1}\cap\Gamma_j = \phi, \quad \bar{\Omega}'_{j-1}\cap\Gamma_j = \phi$$

(cf. Fig. 1a). We note (cf. Fig. 1d) that

$$\bar{f}^u_j = -f^0_{j-1} \quad \text{and} \quad \bar{f}^l_j = f^0_j \quad \text{on} \quad \Omega'_{j-1} \tag{2.4}$$

and

$$\begin{aligned} \bar{f}^u_j &= 0 \quad \text{on} \quad \Omega_j-\Omega'_{j-1}, \\ \bar{f}^l_j &= 0 \quad \text{on} \quad \Omega_j-\Omega'_j. \end{aligned} \tag{2.6}$$

The simplest law describing the interlaminar forces and the impenetrability of the laminae is depicted in Fig. 2a. The binding material may sustain a small positive traction; then rupture occurs, which is ideally brittle (AB) or semibrittle (AC), at the point under consideration of the interface. More realistic is the diagram in Fig. 2b which describes the behaviour of an interlaminar bonding sheet with initial thickness h_j, which can be compressed up to $h'_j \leqslant h_j$. The condition of impenetrability holding for every two successive laminae is described by vertical branches OD in Fig. 2a, b. Here we can surpass the plate theory assumption of the incompressibility of plate in the Ox_3-direction by incorporating such a deformation into the $\hat{\beta}$-diagrams. Thus we allow the line OD of the inter-

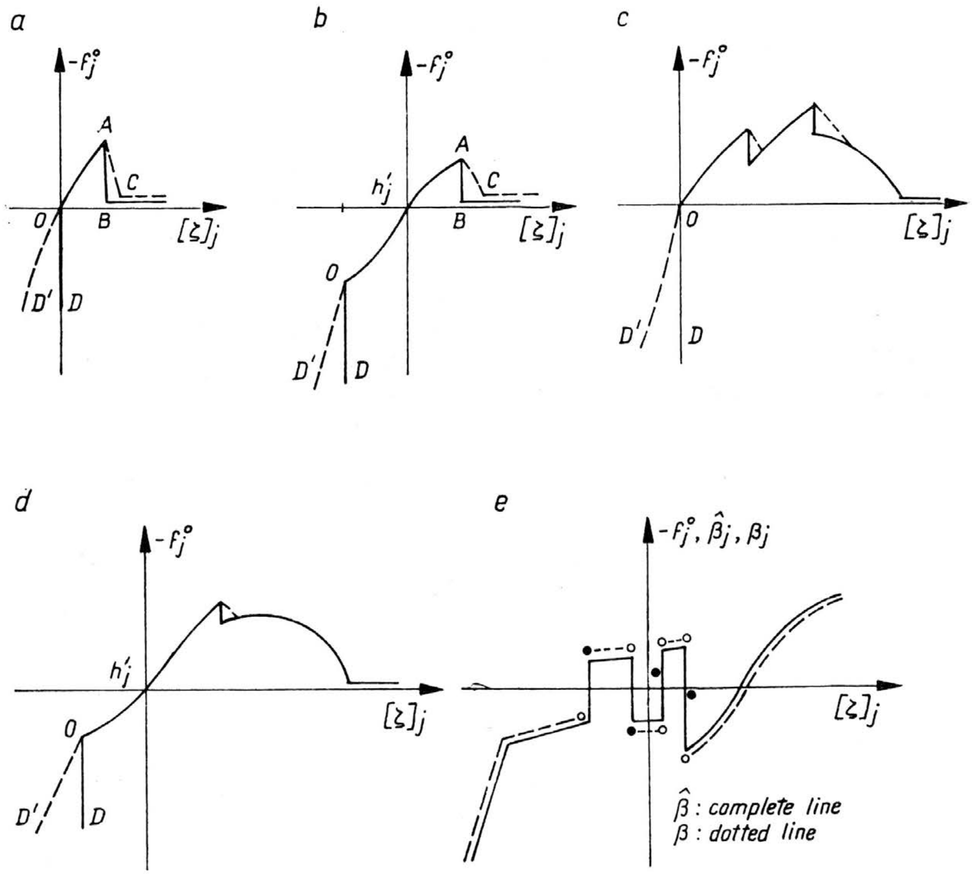

FIG. 2. Interlayer reaction-displacement diagrams.

laminar law to have a small slope (OD'). It is worth noting that the interlaminar laws can be more complicated (Fig. 2c, d) and may include local cracking and crushing effects of ideally brittle or semibrittle behaviour (cf. also [8, 4]). Note the similarity of the sawtooth diagrams of Figs. 2c, d with Scanlon's diagram of reinforced concrete in tension [27]. In the present paper we make a very general assumption, i.e. that $\hat{\beta}: \mathsf{R} \to \mathscr{P}(\mathsf{R})$ is a nonmonotone multivalued function which may include "filled in" gaps of finite length (Fig. 2e). Let β_j be a locally bounded measurable function $\beta_j: \mathsf{R} \to \mathsf{R}$, i.e. $\beta_j \in L^\infty_{\text{loc}}(\mathsf{R})$ (dotted line in Fig. 2c).

For any $\varepsilon > 0$ and $\xi \in \mathrm{R}$ we define the numbers

$$\overline{\beta}_{j\varepsilon}(\xi_1) = \operatorname*{ess\,sup}_{|\xi_1-\xi|\leqslant\varepsilon} \beta_j(\xi) \quad \text{and} \quad \underline{\beta}_{j\varepsilon}(\xi_1) = \operatorname*{ess\,inf}_{|\xi_1-\xi|\leqslant\varepsilon} \beta_j(\xi), \tag{2.7}$$

which are the increasing and decreasing function of ε, respectively. Therefore we may formulate for $\varepsilon \to 0$ the $\lim \overline{\beta}_{j\varepsilon}(\xi) = \overline{\beta}_j(\xi)$ and the $\lim \underline{\beta}_{j\varepsilon}(\xi) = \underline{\beta}_j(\xi)$ and thus we define the multivalued function $\hat{\beta}_j : \mathrm{R} \to \mathscr{P}(R)$ by setting (complete line in Fig. 2(e))

$$\hat{\beta}_j(\xi) = [\underline{b}_j(\xi), \bar{b}_j(\xi)]. \tag{2.8}$$

It is proved [16] that if $\beta_j(\xi_{\pm 0})$ exists for every $\xi \in \mathrm{R}$, then a locally Lipschitz function $\varphi_j : \mathrm{R} \to \mathrm{R}$ can be determined up to an additive constant such that

$$\hat{\beta}_j(\xi) = \bar{\partial}\varphi_j(\xi). \tag{2.9}$$

Here φ_j is defined by the relation

$$\varphi_j = \int_0^{\xi} \beta_j(t)\,dt \tag{2.10}$$

and $\bar{\partial}$ is the generalized gradient of F. H. CLARKE [9].

Then Eq. (2.3) can be written as

$$-f_{j-1}^0 \in \hat{\beta}_{j-1}(\xi) = \bar{\partial}\varphi_{j-1}(\xi), \tag{2.11}$$

which by definition, is equivalent to the hemivariational inequality

$$\varphi_{j-1}^0(\xi, z-\xi) \geqslant -f_{j-1}^0(z-\xi), \qquad \forall z \in \mathrm{R}. \tag{2.12}$$

Here $\varphi_{j-1}^0(\cdot\,,\,\cdot)$ denotes the directional derivative of F. H. Clarke which reads ([9])

$$\varphi_{j-1}^0(\xi, z) = \limsup_{\substack{\lambda\to 0_+ \\ h\to 0}} \frac{\varphi_{j-1}(\xi+h+\lambda z)-\varphi_{j-1}(\xi+h)}{\lambda} = \limsup_{\substack{\lambda\to 0_+ \\ h\to 0}} \frac{1}{\lambda} \int_{\xi+h}^{\xi+h+\lambda z} \beta_{j-1}(\xi)\,d\xi. \tag{2.13}$$

In Eq. (2.11) φ_{j-1} is the nonconvex superpotential of the multivalued law ([7, 4]).

3. Derivation of the hemivariational inequality

Further we develop a theory which holds for isotropic, orthotropic or anisotropic plates which are homogeneous or inhomogeneous. This is due to the fact that in the theory of laminated plates the laminate may exhibit different orthotropy or anisotropy in order to "tailor" a composite plate having the required properties. We write for the j-plate, considered as completely free, the principle of virtual work in the form

$$\begin{aligned}\alpha_j(\zeta_j, z_j) = \int_{\Omega_j} f_j z_j\,d\Omega_j + \int_{\Gamma_j} \overline{Q}_j(\zeta_j) z_j\,d\Gamma_j \\ - \int_{\Gamma_j} M_j(\zeta_j)\frac{\partial z_j}{\partial n_j}\,d\Gamma_j, \qquad \forall z_j \in Z_j, \qquad j = 1, 2, \ldots, m,\end{aligned} \tag{3.1}$$

assuming that $\alpha_j(\zeta_j, z_j)$ is the bilinear form of the plate's elastic energy, $\overline{Q}_j$ and M_j are, respectively, the total shearing force [17] and the bending moment at the boundary Γ_j,

$n = \{n_j\}$ is the outer unit normal vector to Γ_j and Z_j is the set of the kinematically admissible deflections ζ_j, which, since the plate is free, coincides with the classical Sobolev-space ([18, 19]) $H^2(\Omega_j)$. Equation (3.1) is written for appropriately regular functions and holds in the framework of the Kirchhoff plate theory and its generalizations for orthotropic or anisotropic, homogeneous or inhomogeneous plates; i.e. no plate stretching is considered here. However, interlayer slip may occur in the framework of the theory presented here only as a result of the Bernoulli assumption. A more complete study of the interconnection between debonding due to normal stresses and interlayer slip whould require the v. Kármán plate theory and this is not the attempt of the present paper. In the case of an isotropic, homogeneous plate we have that

$$\alpha(\zeta, z) = K \int_{\Omega} [(1-\nu)\zeta_{,\alpha\beta} z_{,\alpha\beta} + \nu \Delta\zeta \Delta z] d\Omega, \qquad \alpha, \beta = 1, 2, \tag{3.2}$$

$$M(\zeta) = -K[\nu\Delta\zeta + (1-\nu)(2n_1 n_2 \zeta_{,12} + n_1^2 \zeta_{,11} + n_2^2 \zeta_{,22})], \tag{3.3}$$

$$\bar{Q}(\zeta) = Q(\zeta) - \frac{\partial M(\zeta)}{\partial \tau} = K\left[\frac{\partial \Delta\zeta}{\partial n} + (1-\nu)\frac{\partial}{\partial \tau}[n_1 n_2(\zeta_{,22} - \zeta_{,11}) + (n_1^2 - n_2^2)\zeta_{,12}]\right]. \tag{3.4}$$

Here $K = Et^3/12(1-\nu^2)$ is the bending rigidity of the plate with E and ν the modulus of elasticity and the Poisson ratio, respectively.

From Eqs. (3.1), we obtain through addition the expression

$$\sum_{j=1}^{m} \alpha_j(\zeta_j, z_j - \zeta_j) = \sum_{j=1}^{m} \int_{\Gamma_j} \bar{Q}_j(z_j - \zeta_j) d\Gamma_j - \sum_{j=1}^{m} \int_{\Gamma_j} M_j \frac{\partial z_j - \partial \zeta_j}{\partial n_j} d\Gamma_j + \sum_{j=1}^{m} \int_{\Omega_j} \bar{\bar{f}}_j(z_j - \zeta_j) d\Omega_j + \sum_{j=1}^{m-1} \int_{\Omega'_j} f_j^0([z]_j - [\zeta]_j) d\Omega'_j, \qquad \forall z_j \in Z_j, \tag{3.5}$$

and from Eqs. (3.5) and (2.11) we get the variational expression

$$\sum_{j=1}^{m} \alpha_j(\zeta_j, z_j - \zeta_j) + \sum_{j=1}^{m-1} \int_{\Omega'_j} \varphi_j^0([\zeta]_j, [z]_j - (\zeta]_j) d\Omega'_j \geqslant \sum_{j=1}^{m} \int_{\Gamma_j} \bar{Q}_j(z_j - \zeta_j) d\Gamma_j - \sum_{j=1}^{m} \int_{\Gamma_j} M_j \frac{\partial z_j - \partial \zeta_j}{\partial n_j} d\Gamma_j + \sum_{j=1}^{m} \int_{\Omega_j} \bar{\bar{f}}_j(z_j - \zeta_j) d\Omega_j, \qquad \forall z_j \in Z_j, \tag{3.6}$$

which, due to the appearance of the terms $\varphi_j^0([\zeta], [z] - [\zeta])$, is a hemivariational inequality (see e.g. [4] ch. 4).

Until now we have not yet specified the boundary conditions of the problem. We shall assume that the boundary conditions are the classical ones of the plate theory. Note that different boundary conditions can be assigned to each plate. Thus if the j-plate is clamped on Γ_j, then we assume that (cf. also [3])

$$Z_j = \left\{ z_j | z_j \in H^2(\Omega_j), z_j = 0, \frac{\partial z_j}{\partial n} = 0 \text{ on } \Gamma_j \right\} = \overset{0}{H}{}^2(\Omega_j). \tag{3.7}$$

If the j-plate is simply supported on $\Gamma_j' \subset \Gamma_j$ ($z_j = 0$ and $M_j = 0$ on Γ_j) and free on $\Gamma_j - \Gamma_j'$ ($M_j = 0$, $Q_j = 0$ — or more generally $M_j = M_{j^0}$ and $Q_j = Q_{j^0}$, M_{j^0} and Q_{j^0} given—on $\Gamma_j - \Gamma_j'$), then

$$Z_j = \{z_j | z_j \in H^2(\Omega_j), \quad z_j = 0 \quad \text{on} \quad \Gamma_j'\}, \tag{3.8}$$

and if $\Gamma_j \equiv \Gamma_j'$ then

$$Z_j = H^2(\Omega_j) \cap \overset{0}{H}{}^1(\Omega_j). \tag{3.9}$$

We will study the hemivariational inequality (3.6) on the assumption that for each plate the boundary conditions guarantee the coerciveness of the bilinear form. We make generally the following assumption:

ASSUMPTION 1. The elastic energy function $\{\zeta_j, z_j\} \to \alpha_j(\zeta_j, z_j)$ is a continuous bilinear form on $H^2(\Omega_j) \times H^2(\Omega_j)$. Moreover the boundary conditions guarantee that $\alpha_j(\zeta_j, z_j)$ is coercive, i.e. there is a constant $c > 0$ such that

$$\alpha_j(z_j, z_j) \geqslant c||z_j||^2, \quad \forall z_j \in H^2(\Omega_j). \tag{3.10}$$

Here $||\cdot||$ denotes the classical H^2-norm [19].

This assumption is satisfied for isotropic or orthotropic homogeneous plates, if the boundary conditions do not permit a "rigid-plate" deflection, i.e. a deflection which is a polynomial of degree one in x_1 and x_2 ($q = q_0 + q_1 x_1 + q_2 q_2$). This is guaranteed, for instance, in the case of a partially clamped plate or in the case of a simply supported plate on Γ_j', on the aassumption that Γ_j' is nonrectilinear. In this context we refer the reader to [20, 3].

In the case of nonhomogeneous plates it suffices to assume that the elasticity coefficients are functions from $L^\infty(\Omega_j)$, taking values in given bounded intervals.

The boundary conditions, considered here nonhomogeneous, are incorporateed into the kinematically admissible sets Z_j which now become closed linear submanifolds of $H^2(\Omega_j)$, i.e. translations of closed linear subspaces of the space $H^2(\Omega_j)$. We note that the natural boundary conditions $M_j = M_{j^0}$ or $Q_j = Q_{j^0}$ are "complementary" to the boundary conditions $\frac{\partial \zeta_j}{\partial n} = g_j$, $\zeta_j = h_j$, where M_{j^0}, Q_{j^0} and g_j, h_j are prescribed functions on Γ_j as it is the case in classical variational methods [22]. Of course in the considered functional framework the foregoing integrals in Eqs. (3.1) shall be written as $\langle M_j(\zeta_j), \frac{\partial \zeta_j}{\partial k_j} \rangle_{1/2}$ and $\langle Q_j(\zeta_j), \zeta_j \rangle_{3/2}$, where $\langle \cdot, \cdot \rangle_{1/2}$ denotes the duality pairing on $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, and $\langle \cdot, \cdot \rangle_{3/2}$ the duality pairing on $H^{3/2}(\Gamma) \times H^{-3/2}(\Gamma)$. Note that $M_j(\zeta) \in H^{-1/2}(\Gamma)$, $\frac{\partial \zeta}{\partial n} \in H^{1/2}(\Gamma)$, $Q_j(\zeta) \in H^{-3/2}(\Gamma)$ and $\zeta_j \in H^{3/2}(\Gamma)$ by the trace theorem [21]. For example, if plate Ω_j is subjected to the boundary conditions

$$z_j = 0 \quad \text{on} \quad \Gamma_j' \subset \Gamma_j, \quad Q = Q_0 \quad \text{on} \quad \Gamma_j - \Gamma_j', \tag{3.11}$$

$$\frac{\partial z_j}{\partial n} = g_j \quad \text{on} \quad \Gamma_j'' \subset \Gamma_j, \quad M = M_0 \quad \text{on} \quad \Gamma_j - \Gamma_j'', \tag{3.12}$$

where Γ_j' is nonrectilinear, then $\alpha_j(\zeta_j, \zeta_j)$ is coercive and Eqs. (3.1) can be written, by taking now into account the boundary conditions, in the form

$$\alpha_j(\zeta_j, z_j) = \langle l_j, z_j \rangle, \quad \forall z_j \in Z_j, \tag{3.13}$$

where

$$Z_j = \left\{ z_j \in H^2(\Omega_j), \quad z_j = 0 \quad \text{on} \quad \Gamma_j', \frac{\partial z_j}{\partial n} = g_j \quad \text{on} \quad \Gamma_j'' \subset \Gamma_j \right\}$$

and l_j is a linear continuous functional on $H^2(\Omega_j)$, i.e. $l_j \in [H^2(\Omega_j)]'$, defined by

$$\langle l_j, z_j \rangle = \int_{\Omega_j} f_j z_j d\Omega + \langle Q_0, z \rangle_{3/2, \Gamma_j - \Gamma_j'} - \langle M_0, \frac{\partial z_j}{\partial n} \rangle_{1/2, \Gamma_j - \Gamma_j''}, \tag{3.14}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing on $H^2(\Omega_j) \times [H^2(\Omega_j)]'$ and $\langle \cdot, \cdot \rangle_{1/2, \Gamma_j - \Gamma_j'}$ denotes the restriction of the corresponding functional to $\Gamma_j - \Gamma_j''$ $\left(\text{i.e. if } M_0 \in L^2(\Gamma) \right.$ then the last term in Eq. (3.14) becomes $\left. \int_{\Gamma_j - \Gamma_j''} M_0 \frac{\partial z}{\partial n} d\Gamma \right)$.

We note finally that in the case of nonhomogeneous boundary conditions an appropriate translation is performed transforming the problem into a homogeneous one; thus we shall assume that Z_j is always a closed linear subspace of $H^2(\Omega_j)$. Now we can pose the general problem.

PROBLEM 1. Find $\zeta_j \in Z_j$, $j = 1, 2, \ldots, m$ such as to satisfy the hemivariational inequality

$$\sum_{j=1}^{m} \alpha_j(\zeta_j, z_j - \zeta_j) + \sum_{j=1}^{m-1} \int_{\Omega_j'} \varphi_j^0([\zeta]_j, [z]_j - [\zeta]_j) d\Omega_j' \geqslant \sum_{j=1}^{m} \langle l_j, z_j - \zeta_j \rangle, \quad \forall z_j \in Z_j \tag{3.15}$$

under assumption 1.

In the next section we shall study this hemivariational inequality.

4. The existence of the solution of the problem and its approximation

We assume, according to [23], that β_j "ultimately" increases, i.e. that for some $\xi \in \mathbf{R}$

$$\beta_j(-\infty) = \operatorname*{ess\,sup}_{(-\infty, -\xi)} \beta_j(\xi) \leqslant \operatorname*{ess\,inf}_{(+\xi, +\infty)} \beta_j(\xi) = \beta_j(+\infty), \tag{4.1}$$

which, without loss of generality and by an appropriate translation of the coordinate axes, can be written as

$$\beta_j(-\infty) = \operatorname*{ess\,sup}_{(-\infty, -\xi)} \beta_j(\xi) \leqslant 0 \leqslant \operatorname*{ess\,inf}_{(+\xi, +\infty)} \beta_j(\xi) = \beta_j(+\infty). \tag{4.2}$$

Note that it is possible in the relations (4.1) and (4.2) that $\beta_j(\pm\infty) = \pm\infty$.

In order to define the regularized problem, we consider a mollifier, that is $p \in C_c^\infty(-1, +1)$ with $p \geqslant 0$ and

$$\int_{-\infty}^{\infty} p(\xi)\,d\xi = 1. \tag{4.3}$$

Let also the following convolution product

$$\beta_{j\varepsilon} = p_\varepsilon \ast \beta_j, \qquad \varepsilon > 0 \tag{4.4}$$

be defined, where $p_\varepsilon(\xi) = \dfrac{1}{\varepsilon} p\left(\dfrac{\xi}{\varepsilon}\right)$. The regularized problem 1ε reads:

PROBLEM 1ε. Find $\zeta_{j\varepsilon} \in Z_j$, $j = 1, 2, \ldots, m$ such as to satisfy the variational equality

$$\sum_{j=1}^{m} \alpha_j(\zeta_{j\varepsilon}, z_j) + \sum_{j=1}^{m-1} \int_{\Omega'_j} \beta_{j\varepsilon}([\zeta_\varepsilon]_j)[z]_j\,d\Omega'_j = \sum_{j=1}^{m} \langle l_j, z_j\rangle, \qquad \forall z_j \in Z_j. \tag{4.5}$$

By introducing a Galerkin basis of Z_j, $j = 1, 2, \ldots, m$ and by denoting as Z_{jn} the corresponding n-dimensional subspace of Z_j, we obtain the finite-dimensional problem:

PROBLEM $1\varepsilon n$. Find $\zeta_{j\varepsilon n} \in Z_{jn}$, $j = 1, 2, \ldots, m$ such as to satisfy the variational equality

$$\sum_{j=1}^{m} \alpha_j(\zeta_{j\varepsilon n}, z_j) + \sum_{j=1}^{m-1} \int_{\Omega'_j} \beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)[z]_j\,d\Omega'_j = \sum_{j=1}^{m} \langle l_j, z_j\rangle, \qquad \forall z_j \in Z_{jn}. \tag{4.6}$$

PROPOSITION 4.1. Suppose that $\bar{\bar{f}}_j \in L^2(\Omega_j)$, that assumption 1 holds for each plate and that the relation (4.2) is satisfied. Then Problem 1 has at least one solution.

Proof. the equality (4.6) is written in the form

$$(\Lambda(\tilde{\zeta}_{\varepsilon n}), \tilde{z}) = 0, \tag{4.7}$$

for $\tilde{z} = (z_1, \ldots, z_m)$, $\forall z_j \in Z_{jn}$, $j = 1, 2, \ldots, m$.

Because of the relation (4.2) we may determine (for each $j, j = 1, 2, \ldots, m-1$), $\varrho_{j1} > 0$ and $\varrho_{j2} > 0$ such that $\beta_{j\varepsilon}(\xi) \geqslant 0$ if $\xi > \varrho_{j1}$, $\beta_{j\varepsilon}(\xi) \leqslant 0$ if $\xi < -\varrho_{j1}$ and $|\beta_{j\varepsilon}(\xi)| \leqslant \varrho_{j2}$ if $|\xi| \leqslant \varrho_{j1}$. Therefore the following holds on each interlayer:

$$\int_{\Omega'_j} \beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)[\zeta_{\varepsilon n}]_j\,d\Omega'_j = \int_{|[\zeta_{\varepsilon n}]_j| > \varrho_{j1}} \beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)[\zeta_{\varepsilon n}]_j\,d\Omega'_j + \int_{|[\zeta_{\varepsilon n}]_j| \leqslant \varrho_{j1}} \beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)[\zeta_{\varepsilon n}]_j\,d\Omega'_j \geqslant 0 - \varrho_{j1}\varrho_{j2}\,\mathrm{mes}\,\Omega'_j. \tag{4.8}$$

From the coerciveness assumption (Eq. (3.10)), the boundedness of the external loading functions and the relation (4.8), we find

$$(\Lambda(\tilde{\zeta}_{\varepsilon n}), \tilde{\zeta}_{\varepsilon n}) = \sum_{j=1}^{m} \alpha_j(\zeta_{j\varepsilon n}, \zeta_{j\varepsilon n}) + \sum_{j=1}^{m-1} \int_{\Omega'_j} \beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)[\zeta_{\varepsilon n}]_j\,d\Omega'_j - \sum_{j=1}^{m} \langle l_j, z_j\rangle \geqslant \sum_{j=1}^{m} c_j \|\zeta_{j\varepsilon n}\|^2 - \sum_{j=1}^{m-1} \varrho_{j1}\varrho_{j2}\,\mathrm{mes}\,\Omega'_j - \sum_{j=1}^{m} c'_j \|\zeta_{j\varepsilon n}\|, \tag{4.9}$$

$$c_j, c'_j \text{ const} > 0, \qquad j = 1, 2, \ldots, m.$$

Thus, according to Brouwer's fixed point theorem (cf. e.g. [24], p. 53) the relation (4.7) has a solution with $||\zeta_{j\varepsilon n}|| < r_j$, $j = 1, 2, ..., m$.

Further we shall investigate the behaviour of the solution $\tilde{\zeta}_{\varepsilon n}$ of the finite demensional problem $1_{\varepsilon n}$ as $\varepsilon \to 0$ and $n \to \infty$. Due to the fact that $\{\zeta_{j\varepsilon n}\}$ is bounded in $Z_j, j = 1, ..., m$ we may extract a subsequence again denoted by $\{\zeta_{j\varepsilon n}\}$ such as for $\varepsilon \to 0$, $n \to \infty$ to satisfy

$$\zeta_{j\varepsilon n} \to \zeta_j \quad \text{weakly in} \quad Z_j, \quad j = 1, 2, ..., m. \tag{4.10}$$

From the compact imbeddings $H^2(\Omega_j) \subset L^2(\Omega_j)$, and the relation (4.10), we get that

$$\zeta_{j\varepsilon n} \to \zeta_j \quad \text{strongly in} \quad L^2(\Omega_j), \quad j = 1, 2, ..., m \tag{4.11}$$

and

$$\zeta_{j\varepsilon n} \to \zeta_j \quad \text{a.e. in} \quad \Omega_j, \quad j = 1, 2, ..., m. \tag{4.12}$$

Further, we show that under the assumption (4.2) $\{\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)\}$ is weakly precompact in $L^1(\Omega_j')$(1). To this end, and due to the Dunford-Pettis theorem ([25], p. 239), it suffices to show that for each $\mu_j > 0$ and $\delta_j(\mu_j) > 0$ can be determined such that for $\omega_j \subset \Omega_j'$ with mes $\omega_j < \delta_j$

$$\int_{\omega_j} |\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)| \, d\Omega_j' \leqslant \mu_j. \tag{4.13}$$

From the obvious inequality (cf. [23])

$$\xi_0 |\beta_{j\varepsilon}(\xi)| \leqslant |\beta_{j\varepsilon}(\xi)\,\xi| + \xi_0 \sup_{|\xi| \leqslant \xi_0} |\beta_{j\varepsilon}(\xi)|, \tag{4.14}$$

we have

$$\int_{\omega_j} |\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)| d\Omega \leqslant \frac{1}{\xi_0} \int_{\omega_j} |\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)[\zeta_{\varepsilon n}]_j| d\Omega + \int_{\omega_j} \sup_{|[\zeta_{\varepsilon n}(x)]_j| \leqslant \xi_0} |\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)| d\Omega. \tag{4.15}$$

We obtain the boundedness of $\int_{\omega_j} |\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)\,[\zeta_{\varepsilon n}]_j| d\Omega$ as follows. We have that

$$\begin{aligned} \int_{\Omega_j'} |\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)[\zeta_{\varepsilon n}]_j| d\Omega_j' &= \int_{|[\zeta_{\varepsilon n}(x)]_j| > \varrho_{j1}} |\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)[\zeta_{\varepsilon n}]_j| d\Omega_j' \\ &+ \int_{|[\zeta_{\varepsilon n}(x)]_j| \leqslant \varrho_{j1}} |\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)[\zeta_{\varepsilon n}]_j| d\Omega_j' = \int_{|[\zeta_{\varepsilon n}(x)]_j| > \varrho_{j1}} |\ldots| d\Omega_j' - \int_{|[\zeta_{\varepsilon n}(x)]_j| \leqslant \varrho_{j1}} |\ldots| d\Omega_j' \end{aligned} \tag{4.16}$$

(1) For the present specific plate problem we have that $H^2(\Omega) \subset C^0(\overline{\Omega}) \subset L^\infty(\Omega)$ (imbeddings). Moreover β_ε is a C^∞-function and thus (4.25) results immediately without using the Dunford–Pettis theorem and the estimates (4.13)–(4.24). However, this procedure is necessary in most other types of nonmonotone nonlinearity and in any other type of functional setting. Indeed in any hemivariational inequality formulated in the H^1-space the above procedure is necessary because $H^1(\Omega) \not\subset C^0(\overline{\Omega})$. The same holds for any type of nonlinearity term $\hat{\beta}(.)$ for which the previous C^0-imbedding does not hold. For instance, let us consider in a plate problem a nonmonotone boundary condition $M \in \hat{\beta}\left(\frac{\partial \zeta}{\partial n}\right)$; then

$$\frac{\partial \zeta}{\partial u} \in H^{1/2}(\Gamma) \text{ but } H^{1/2}(\Gamma) \not\subset C^0(\Gamma).$$

In this case $Z_{jn} \subset \tilde{Z}_j = \left\{ z | z \in Z_j, \frac{\partial z}{\partial n} \in L^\infty(\Gamma) \right\} : \bigcup_r Z_{jr}$ is dense in Z_j for the H^2-norm.

$$\text{(4.16) [cont.]} \quad +2\int\limits_{|[\zeta_{\varepsilon n}(x)]_j|\leqslant\varrho_{j1}}|\dots|d\Omega'_j \leqslant \int\limits_{|[\zeta_{\varepsilon n}(x)]_j|>\varrho_{j1}}\dots d\Omega'_j+\int\limits_{|[\zeta_{\varepsilon n}(x)]_j|\leqslant\varrho_{j1}}\dots d\Omega'_j+2\int\limits_{|[\zeta_{\varepsilon n}(x)]_j|\leqslant\varrho_{j1}}|\dots|d\Omega'_j$$

$$= \int\limits_{\Omega'_j}\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)[\zeta_{\varepsilon n}]_j d\Omega'_j+2\int\limits_{|[\zeta_{\varepsilon n}(x)]_j|\leqslant\varrho_{j1}}|\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)[\zeta_{\varepsilon n}]_j|\alpha\Omega'_j.$$

In order to proceed we set in the equality (4.6) $z_1 = \tilde{z}_1 \in Z_{1n}$ such that $\tilde{z}_1 = z_2$ on Ω'_1, $z_2 = \tilde{z}_2 \in Z_{2n}$ such that $\tilde{z}_2 = z_3$ on $\Omega'_2, \dots, z_{j-1} = \tilde{z}_{j-1} \in Z_{j-1,n}$ such that $\tilde{z}_{j-1} = \zeta_{j\varepsilon n}$ on Ω'_{j-1}, $z_j = \zeta_{j\varepsilon n}$ on Ω_j, $z_{j+1} = \zeta_{j+1\varepsilon n}$ on Ω_{j+1}, $z_{j+2} = \tilde{z}_{j+2} \in Z_{j+2,n}$ such that $\tilde{z}_{j+2} = \zeta_{j+1,\varepsilon n}$ on Ω'_{j+1}, $z_m = \tilde{z}_m \in Z_{m,n}$ such that $\tilde{z}_m = z_{m-1}$ on Ω'_{m-1}.

Here we have that $\|\tilde{z}_j\| < c$, $j = 1, 2, \dots, m$. Thus we obtain from the equality (4.6) by using the continuity of the bilinear and linear forms and the fact that $\|\zeta_{j\varepsilon n}\| < c$

$$\text{(4.17)} \qquad \int\limits_{\Omega'_j}\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)[\zeta_{\varepsilon n}]_j d\Omega'_j \leqslant c.$$

From the relations (4.16), (4.17) and (4.8), we obtain that

$$\text{(4.18)} \qquad \int\limits_{\Omega'_j}\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)[\zeta_{\varepsilon n}]_j d\Omega'_j+2\int\limits_{|[\zeta_{\varepsilon n}(x)]_j|\leqslant\varrho_{j1}}|\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)[\zeta_{\varepsilon n}]_j|d\Omega'_j$$

$$\leqslant c+2\varrho_{j1}\varrho_{j2}\operatorname{mes}\Omega'_j, \qquad \forall j = 1, 2, \dots, m-1.$$

Since $\omega_j \subset \Omega'_j$,

$$\text{(4.19)} \qquad \int\limits_{\omega_j}|\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)[\zeta_{\varepsilon n}]_j|d\Omega'_j \leqslant \int\limits_{\Omega'_j}|\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)[\zeta_{\varepsilon n}]_j|d\Omega'_j \leqslant c+2\varrho_{j1}\varrho_{j2}\operatorname{mes}\Omega'_j.$$

Also from

$$\text{(4.20)} \qquad \sup_{|\xi|\leqslant\xi_0}|\beta_{j\varepsilon}(\xi)| = \sup_{|\xi|\leqslant\xi_0}|p_\varepsilon * \beta_j| = \sup_{|\xi|\leqslant\xi_0}\left|\int\limits_{-\varepsilon}^{\varepsilon}p_\varepsilon(t)\beta_j(\xi-t)\,dt\right|$$

$$\leqslant \sup_{|\xi|\leqslant\xi_0}\left|\operatorname*{ess\,sup}_{|t|\leqslant\varepsilon}\beta_j(\xi-t)\right| \leqslant \sup_{|\xi|\leqslant\xi_0}\left|\operatorname*{ess\,sup}_{|x-\xi|\leqslant1}\beta_j(x)\right| \leqslant \operatorname*{ess\,sup}_{|\xi|\leqslant\xi_0+1}\beta_j(\xi),$$

we get that

$$\text{(4.21)} \qquad \int\limits_{\omega_j}\sup_{|[\zeta_{\varepsilon n}]_j|\leqslant\xi_0}|\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)|d\Omega \leqslant \int\limits_{\omega_j}\operatorname*{ess\,sup}_{|[\zeta_{\varepsilon n}]_j|\leqslant\xi_0+1}|\beta_j([\zeta_{\varepsilon n}]_j)|d\Omega.$$

We can choose ξ_0 such that for all ε and n

$$\text{(4.22)} \qquad \frac{1}{\xi_0}\int\limits_{\omega_j}|\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)\,[\zeta_{\varepsilon n}]_j|d\Omega'_j \leqslant \frac{1}{\xi_0}(c+2\varrho_{j1}\varrho_{j2}\operatorname{mess}\Omega'_j) < \frac{\mu}{2},$$

due to the relation (4.19) and δ such that (cf. the relation (4.21)) for $\operatorname{mes}\omega < \delta$

$$\text{(4.23)} \qquad \operatorname*{ess\,sup}_{|\xi|\leqslant\xi_0+1}|\beta_j(\xi)| \leqslant \frac{\mu}{2\delta}.$$

Then

$$\text{(4.24)} \qquad \int\limits_{\omega_j}\sup_{|[\zeta_{\varepsilon n}(x)]_j|\leqslant\xi_0}|\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)|d\Omega \leqslant \operatorname*{ess\,sup}_{|[\zeta_{\varepsilon n}(x)]_j|\leqslant\xi_0+1}|\beta_j([\zeta_{\varepsilon n}]_j)|\operatorname{mes}\omega_j \leqslant \frac{\mu}{2\delta}\delta = \frac{\mu}{2}.$$

The relations (4.15), (4.22) and (4.24) yield the relation (4.13), i.e. that $\{\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)\}$ is weakly precompact in $L^1(\Omega'_j)$, for $j = 1, 2, \ldots, m-1$.

Thus, as $\varepsilon \to 0$ and $\eta \to \infty$, a subsequence again denoted by $\{\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)\}$ can be determined such that

$$(4.25) \qquad \beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j) \to \chi_j \quad \text{weakly in} \quad L^1(\Omega'_j).$$

By passing to the limit $n \to \infty$, $\varepsilon \to 0$ in the Galerkin form (4.6) and since

$$[z]_j \in Z_j \subset H^2(\Omega) \subset L^\infty(\Omega'_j) \quad \text{for} \quad \Omega'_j \subset \mathrm{R}^2 \quad j = 1, 2, \ldots, m-1,$$

we find that

$$(4.26) \qquad \sum_{j=1}^{m} \alpha_j(\zeta_j, z_j) + \sum_{j=1}^{m-1} \int_{\Omega'_j} \chi_j [z]_j d\Omega = \sum_{j=1}^{m} \langle l_j, z_j \rangle, \quad \forall z_j \in Z_j.$$

To complete the proof we shall show that

$$(4.27) \qquad \chi_j \in \bar{\beta}_j([\zeta]) \quad \text{a.e. in} \quad \Omega'_j.$$

From the relation (4.11) and by applying Egoroff's theorem (see e.g. [19]) we can find that for every $\alpha > 0$ we can determine ω_1 and ω_2 with $\operatorname{mes}\omega_1 < \alpha$ and $\operatorname{mes}\omega_2 < \alpha$ such that $\zeta_{j\varepsilon n} \to \zeta_j$ uniformly in $\Omega_j - \omega_j$, $j = 1, 2, \ldots, m-1$, where $\zeta_j \in L^\infty(\Omega_j - \omega_j)$. Actually $\zeta_{j\varepsilon n} \to \zeta_j$ strongly in $L^\infty(\Omega_j)$ due to the compact imbedding $H^2(\Omega_j) \subset C^0(\bar{\Omega}_j)$ and the imbedding $C^0(\bar{\Omega}_j) \subset L^\infty(\Omega_j)$. Due to the uniform convergence for any $\varrho > 0$, we can determine $\varepsilon_0 < \frac{\varrho}{2}$, $n_0 > \frac{2}{\varrho}$ such that for $\varepsilon < \varepsilon_0$, $n > n_0$

$$(4.28) \qquad |\zeta_{j\varepsilon n} - \zeta_j| < \frac{\varrho}{2}, \quad \forall x \in \Omega_j - \omega_j, \quad j = 1, 2, \ldots, m-1.$$

Therefore for every $\alpha > 0$ we can determine ω with $\operatorname{mes}\omega < \alpha$ such that for any $\mu > 0$ and for $\varepsilon < \varepsilon_0 < \frac{\mu}{2}$ and $n > n_0 > \frac{2}{\mu}$

$$(4.29) \qquad |[\zeta_{\varepsilon n}]_j - [\zeta]_j| < \frac{\mu}{2}, \quad \forall x \in \Omega'_j - \omega'_j.$$

But since we have used a mollifier to construct $\beta_{j\varepsilon}$, the following hold:

$$(4.30) \qquad \beta_{j\varepsilon}(\xi) = (p_\varepsilon * \beta_j)(\xi) = \int_{-\varepsilon}^{\varepsilon} \beta_j(\xi - t) p_\varepsilon(t) dt \leqslant \operatorname*{ess\,sup}_{|t| < \varepsilon} \beta_j(\xi - t),$$

and analogously

$$(4.31) \qquad \operatorname*{ess\,inf}_{|t| < \varepsilon} \beta_j(\xi - t) \leqslant \beta_{j\varepsilon}(\xi).$$

Due to the relations (4.29) (see also the numbers (2.7))

$$(4.32) \quad \beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j) \leqslant \operatorname*{ess\,sup}_{|[\zeta_{\varepsilon n}]_j - \xi| < \varepsilon} \beta_j(\zeta) \leqslant \operatorname*{ess\,sup}_{|[\zeta_{\varepsilon n}]_j - \xi| < \mu/2} \beta_j(\zeta) \leqslant \operatorname*{ess\,sup}_{|[\zeta]_j - \xi| < \mu} \beta_j(\xi) = \bar{\beta}_{j\mu}([\zeta]_j),$$

and analogously

$$(4.33) \qquad \underline{\beta}_{j\mu}([\zeta]_j) \leqslant \operatorname*{ess\,inf}_{|[\zeta]_j - \xi| < \mu} \beta_j(\zeta) \leqslant \beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j).$$

For $e \geqslant 0$ a.e. in $\Omega'_j - \omega'_j$, arbitrarily chosen with $e \in L^\infty(\Omega' - \omega'_j)$, we obtain from the relations (4.32) and (4.33):

$$\int\limits_{\Omega'_j - \omega'_j} \underline{\beta}_{j\mu}([\zeta]_j)\, e d\Omega \leqslant \int\limits_{\Omega'_j - \omega'_j} \beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)\, e d\Omega \leqslant \int\limits_{\Omega'_j - \omega'_j} \bar{\beta}_{j\mu}([\zeta]_j)\, e d\Omega, \tag{4.34}$$

and in the limits as $\varepsilon \to 0$, $n \to \infty$

$$\int\limits_{\Omega'_j - \omega'_j} \underline{\beta}_{j\mu}([\zeta]_j)\, e d\Omega \leqslant \int\limits_{\Omega'_j - \omega'_j} \chi_j e d\Omega \leqslant \int\limits_{\Omega'_j - \omega'_j} \bar{\beta}_{j\mu}([\zeta]_j)\, e d\Omega. \tag{4.35}$$

By Lebesgue's theorem, as $\mu \to 0_+$ we take

$$\int\limits_{\Omega'_j - \omega'_j} \underline{\beta}_{j}([\zeta]_j)\, e d\Omega \leqslant \int\limits_{\Omega'_j - \omega'_j} \chi_j e d\Omega \leqslant \int\limits_{\Omega'_j - \omega'_j} \bar{\beta}_{j}([\zeta]_j)\, e d\Omega. \tag{4.36}$$

Since $e \geqslant 0$ is arbitrary, the relation (4.36) implies that

$$\chi_j \in \hat{\beta}_j([\zeta]_j) \quad \text{a.e. in } \Omega'_j - \omega'_j, \quad j = 1, 2, \ldots, m-1 \tag{4.37}$$

and by taking α as small as possible we get Eq. (4.26).

5. The C^0-convergence of the approximate solution. Substationarity

In the previous section we have shown that $\zeta_{j\varepsilon n} \to \zeta_j$ weakly in Z_j. Here we shall study the strong convergence, by introducing an additional assumption.

PROPOSITION 5.1. Let all assumptions given in the previous section be valid and suppose that there exist $q_j \geqslant 1$ and constants $c_j > 0$ such that

$$|\beta_j(\xi)/ \leqslant c_j(1+|\zeta|^{q_j}), \quad \forall \xi \in \mathbb{R}, \quad j = 1, \ldots, m-1. \tag{5.1}$$

Then $\zeta_{j\varepsilon n} \to \zeta_j$ strongly as $\varepsilon \to 0$, $n \to \infty$ in Z_j.

Proof. Problem 1 (Eq. (2.28)) implies by setting $z_j = \zeta_{j\varepsilon n}$, $j = 1, \ldots, m-1$ that

$$\sum_{j=1}^{m} \alpha_j(\zeta_j, \zeta_j) \leqslant \sum_{j=1}^{m} \alpha_j(\zeta_j, \zeta_{j\varepsilon n}) + \sum_{j=1}^{m-1} \int\limits_{\Omega'_j} \varphi_j^0([\zeta]_j, [\zeta_{\varepsilon n}]_j - [\zeta]_j)\, d\Omega_j - \sum_{j=1}^{m} \langle l_j, \zeta_{j\varepsilon n} - \zeta_j \rangle. \tag{5.2}$$

From problem $1\varepsilon n$ (Eq. 3.6)) we obtain

$$\sum_{j=1}^{m} \alpha_j(\zeta_{j\varepsilon n}, \zeta_{j\varepsilon n}) = \sum_{j=1}^{m} \alpha_j(\zeta_{j\varepsilon n}, z_{jn}) + \sum_{j=1}^{m} \int\limits_{\Omega'_j} \beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)\, ([z_n]_j - [\zeta_{\varepsilon n}]_j)\, d\Omega'_j$$
$$- \sum_{j=1}^{m} \langle l_j, z_{jn} - \zeta_{j\varepsilon n} \rangle, \quad \forall z_{jn} \in Z_{jn}, \quad j = 1, 2, \ldots, m. \tag{5.3}$$

The coerciveness of the problem together with the relations (5.2) and (5.3) implies

$$\sum_{j=1}^{m} c_j \|\zeta_j - \zeta_{j\varepsilon n}\|^2 \leqslant \sum_{j=1}^{m} \alpha_j(\zeta_j - \zeta_{j\varepsilon n}, \zeta_j - \zeta_{j\varepsilon n}) = \sum_{j=1}^{m} \alpha_j(\zeta_j, \zeta_j)$$
$$+ \sum_{j=1}^{m} \alpha_j(\zeta_{j\varepsilon n}, \zeta_{j\varepsilon n}) - \sum_{j=1}^{m} \alpha_j(\zeta_j, \zeta_{j\varepsilon n}) - \sum_{j=1}^{m} \alpha_j(\zeta_{j\varepsilon n}, \zeta_j) \leqslant \sum_{j=1}^{m} \alpha_j(\zeta_j, \zeta_{j\varepsilon n}) \tag{5.4}$$

(5.4) [cont.]
$$+\sum_{j=1}^{m-1}\int_{\Omega'_j}\varphi_j^0([\zeta]_j,[\zeta_{\varepsilon n}]_j-[\zeta]_j)\,d\Omega'_j-\sum_{j=1}^{m}\langle l_j,\zeta_{j\varepsilon n}-\zeta_j\rangle+\sum_{j=1}^{m}\alpha_j(\zeta_{j\varepsilon n},z_{jn})$$
$$+\sum_{j=1}^{m-1}\int_{\Omega'_j}\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)([z_n]_j-[\zeta_{\varepsilon n}]_j)\,d\Omega'_j-\sum_{j=1}^{m}\langle l_j,z_{jn}-\zeta_{j\varepsilon n}\rangle-\sum_{j=1}^{m}\alpha_j(\zeta_{j\varepsilon n},\zeta_j)$$
$$-\sum_{i=1}^{m}\alpha_j(\zeta_j,\zeta_{j\varepsilon n}),\qquad\forall z_{jn}\in Z_{jn},\qquad j=1,2,\ldots,m.$$

We take z_{jn}, $j=1,2,\ldots,m$ such that as $n\to\infty$

(5.5) $$z_{jn}\to\zeta_j\quad\text{strongly in }\ Z_j\subset H^2(\Omega_j),\qquad j=1,2,\ldots,m.$$

From the right-hand term of the relation (5.5) it remains for us to calculate for $\varepsilon\to 0$, $n\to\infty$ the terms

(5.6) $$\sum_{j=1}^{m}\Big\{\lim\int_{\Omega'_j}\varphi_j^0([\zeta]_j,[\zeta_{\varepsilon n}]_j-[\zeta]_j)\,d\Omega+\lim\int_{\Omega'_j}\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)([z_n]_j-[\zeta_{\varepsilon n}]_j)\,d\Omega\Big\}.$$

Using the hypothesis (5.1) and the definition of φ_j^0 (Eq. (2.13)), we have for each j

(5.7) $$\int_{\Omega'_j}\varphi_j^0([\zeta]_j,[\zeta_{\varepsilon n}]_j-[\zeta]_j)\,d\Omega=\int_{\Omega'_j}\limsup_{\substack{\lambda\to 0_+\\ h\to 0}}\frac{1}{\lambda}\int_{[\zeta]_j+h}^{[\zeta]_j+h+\lambda\{[\zeta_{\varepsilon n}]_j+[\zeta]_j\}}\beta_j(t)\,dt\,d\Omega$$
$$\leqslant c_j\int_{\Omega'_j}\limsup_{\substack{\lambda\to 0_+\\ h\to 0}}\frac{1}{\lambda}\int_{[\zeta]_j+h}^{[\zeta]_j+h+\lambda\{[\zeta_{\varepsilon n}]_j+[\zeta]_j\}}(1+|t|^{q_j})\,dt\,d\Omega$$
$$=c_j\int_{\Omega'_j}(1+[\zeta]_j^{q_j})([\zeta_{\varepsilon n}]_j-[\zeta]_j)\,d\Omega.$$

Due to the compact imbedding $H^2(\Omega_j)\subset L^{q_j}(\Omega_j)$, $q_j\geqslant 1$, the fact that $\Omega'_j\subset\Omega_j$ and by means of Minkowski's inequality, we obtain

(5.8) $$[\zeta_{\varepsilon\varrho n}]_j\to[\zeta]_j\quad\text{strongly in }\ L^{q_j}(\Omega'),\qquad q_j\geqslant 1.$$

Since $[\zeta]_j\in L^\infty(\Omega'_j)$, we obtain for $\varepsilon\to 0$, $n\to\infty$

(5.9) $$\lim\int_{\Omega'_j}\varphi_j^0([\zeta]_j,[\zeta_{\varepsilon n}]_j-[\zeta]_j)\,d\Omega=0.$$

From the expression (4.25) we have

(5.10) $$\|\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)\|_{L^1(\Omega'_j)}<c_j.$$

Moreover the compact imbedding $H^2(\Omega_j)\subset C^0(\overline{\Omega}_j)$, implies that

(5.11) $$[\zeta_{\varepsilon n}]_j\to[\zeta]_j\quad\text{strongly in}\quad L^\infty(\Omega'_j).$$

Therefore, we have that

(5.12) $$\Big|\int_{\Omega'_j}|\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)([z_n]_j-[\zeta_{\varepsilon n}]_j)\,d\Omega\Big|\leqslant\|\beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)\|_{L^1(\Omega'_j)}\|[z_n]_j-[\zeta_{\varepsilon n}]_j\|_{L^\infty(\Omega'_j)}$$
$$\leqslant c_j\|[z_n]_j-[\zeta_{\varepsilon n}]_j\|_{L^\infty(\Omega'_j)}\leqslant c_j\|[z_n]_j-[\zeta]_j\|_{L^\infty(\Omega'_j)}+c_j\|[\zeta]_j-[\zeta_{\varepsilon n}]_j\|_{L^\infty(\Omega'_j)}.$$

This inequality, for $\varepsilon \to 0$, $n \to \infty$, yields

$$\lim \int_{\Omega'_j} \beta_{j\varepsilon}([\zeta_{\varepsilon n}]_j)\,([z_n]_j - [\zeta_{\varepsilon n}]_j)\,d\Omega = 0. \tag{5.13}$$

Thus the relations (5.4) with the relations (5.8) and (5.12) imply the strong convergence of $\zeta_{j\varepsilon n}$ to ζ_j in $H^2(\Omega_j)$ for $j = 1, 2, \ldots, m$, and since $H^2(\Omega_j) \subset C^0(\bar{\Omega}_j)$ is compact, we will have convergence in the $C^0(\bar{\Omega}_j)$-norm.

This last result may be interpreted in the language of the Finite Element method by choosing the finite dimensional spaces Z_{in} appropriately [26].

The study of the hemivariational inequality (3.15) leads to a substationarity problem (cf. [8, 4] p. 146) for the potential energy

$$\Pi(\zeta) = \frac{1}{2}\sum_{j=1}^{m} \alpha_j(\zeta_j, \zeta_j) + \sum_{j=1}^{m-1} \int_{\Omega'_j} \varphi_j(\zeta_j)\,d\Omega - \sum_{j=1}^{m} \langle l_j, \zeta_j \rangle, \tag{5.14}$$

of the laminated plate. The substationarity problem may be put in the form

$$\begin{gathered} 0 \in \bar{\partial}[\Pi(\zeta) + I_Z(\zeta)], \\ \zeta = \{\zeta_1, \zeta_2, \ldots, \zeta_m\} \in Z = Z_1 \times Z_2 \times \ldots \times Z_m, \end{gathered} \tag{5.15}$$

where $I_Z(\zeta) = \{0 \text{ if } \zeta \in Z,\ \infty \text{ if } \zeta \notin Z\}$. Each solution of the problem (5.15) is a solution of problem 1. The conditions under which the converse is true may be found in [4] p.150. We note here that every local minimum of the potential energy Π over Z is a substationarity point and thus a solution of the hemivariational inequality (3.15). Due to the lack of convexity, the problem has not generally a unique solution.

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References

1. R. Jones, *Mechanics of composite materials*, McGraw Hill, Scripta Book Co., New York, Washington 1975.
2. N. J. Pegano and R. Byron Pipes, *The influence of stacking sequence on laminate strength*, J. Composite Materials, p. 50–57, January 1971.
3. G. Duvaut and J. L. Lions, *Les inéquations en mécanique et en physique*, Dunod, Paris 1972.
4. P. D. Panagiotopoulos, *Inequality problems in mechanics and applications. Convex and nonconvex energy functions*, Birkhäuser Verlag, Basel-Boston-Stuttgard 1985.
5. R. T. Rockafellar, *Convex analysis*, Princeton University Press, Princeton 1970.
6. J. J. Moreau, *La notion de sur-potential et les liaisons unilatélales en élastostatique*, C. R. Acad. Sc., 954–957, 267A, Paris 1968.
7. P. D. Panagiotopoulos, *Non-convex superpotentials in the sense of F. H. Clarke and applications*, Mech. Res. Comm., **8**, 335–340, 1981.

8. P. D. PANAGIOTOPOULOS, *Nonconvex energy functions. Hemivariational inequalities and substationarity principles*, Acta Mech., **42**, 160–183, 1983.
9. F. H. CLARKE, *Nonsmooth analysis and optimization*, J. Wiley, New York 1984.
10. P. D. PANAGIOTOPOULOS, *Une généralization non-convex de la notion du sur-potential et ses applications*, C. R. Acad. Sc., **296B**, 580–584, Paris 1983.
11. P. D. PANAGIOTOPOULOS, *A systematic approach to the search for variational principles for bilateral and unilateral problems*, ZAMM, **57**, T246-T247, 1977.
12. P. D. PANAGIOTOPOULOS, *Superpotentials in the sense of Clarke and in the sense of Warga and applications*, ZAMM, **62**, Heft 415, 1982.
13. P. D. PANAGIOTOPOULOS and C. C. BANIOTOPOULOS, *A hemivariational inequality and substantionarity approach to the interface problem. Theory and prospects of applications*, Enging. Analysis, **1**, 1, 20–31, 1984.
14. P. D. PANAGIOTOPOULOS and A. V. AVDELAS, *A hemivariational inequality approach to the unilateral contact problem and substantionarity principles*, Ingenieur Archiv., **54**, 401–412, 1984.
15. C. C. BANIOTOPOULOS, *Analysis of structures with complete stress-strain laws*, Doctoral Thesis, Aristotle University, Thessaloniki 1985.
16. K. C. CHANG, *Variational methods for non-differentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl., **80**, 102–129, 1981.
17. K. GIRKMANN, *Flächentragwerke*, Springer-Verlag, Wien 1963.
18. W. I. SMIRNOV, *Lehrgang der höheren Mathematik*, Teil V., VEB Deutscher Verlag der Wissenschaften, Berlin 1962.
19. A. KUFNER, O. JOHN and S. FUČIK, *Function spaces*, Noordhoff, Leyden and Academia, Prague 1977.
20. J. NEČAS and I. HLAVÁČEK, *Mathematical theory of elastic and elastoplatic bodies. An introduction*, Elsevier, Amsterdam–Oxford–New York 1981.
21. J. L. LIONS and E. MAGENES, *Problémes aux limites non homogénes et applications*, vol. 1, 2, 3, Dunod, Paris 1968.
22. K. REKTORYS, *Variational methods in mathematics science and engineering*, D. Reidal Publishing Co., Dordrecht–Holland and Boston, U.S.A. 1977.
23. J. RAUCH, *Discontinuous semilinear differential equations and multiple valued maps*, Proc. A.M.S., **64**, 277–282, 1977.
24. J. L. LIONS, *Quelques méthodes de résolution des problémes aux limites non linéaires*, Gauthier–Villars, Dunod, Paris 1969.
25. I. EKELEND and R. TEMAM, *Convex analysis and variational problems*, North–Holland, Amsterdam and American Elsevier, New York 1976.
26. P. G. CIARLET, *The finite element method for elliptic problems*, North Holland, Amsterdam 1978.
27. H. FLOEGL and H. A. MANG, *Tension stiffening concept based on both slip*, ASCE, ST12, 108, 2681–2701, 1982.

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