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On static calculation of fishing nets

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THE SYSTEM of equations governing the statics of a simple model of fishing nets is derived. The existence and uniqueness of solutions are discussed. For a single line the problem is reduced to a system of three equations in three unknowns. The properties of this system allow for a globally convergent solution method. Testing examples as well as conclusions for the problem of rectangular nets complete the paper.

Wyprowadzono układ równań statyki prostego modelu sieci rybackiej. Przedyskutowano problemy istnienia i jednoznaczności rozwiązań. W przypadku pojedynczej liny zagadnienie sprowadza się do układu trzech równań z trzema niewiadomymi. Własności takiego układu pozwalają na zastosowanie globalnie zbieżnych metod rozwiązywania. Pracę kończą przykłady obliczeń i wnioski.

Выведена система уравнений статики простой модели рыболовной сети. Обсуждены проблемы существования и единственности решений. В случае единичного каната задача сведена к системе трех уравнений с тремя неизвестными. Свойства такой системы позволяют применить глобально сходящиеся методы решения. Работу окончивают примеры расчетов и следствия.

1. Introduction

FOR THE LAY-OUT of fishing nets it is desirable to have approximate but effective methods to calculate the acting line-forces and the positions of the knots. To this end various physical models were introduced [1, 2, 3] and the resulting systems of equations were solved by several numerical methods [2]. In some cases one prefers to model segments of the real lines as stiff or elastic rods [2]. Despite the simplicity of such models, the related systems of equations are essentially nonlinear and hence their solution is expensive. Unfortunately, in general, there is no known natural configuration as, for example, in the case of prestressed line constructions, so that the introduction of linearized models is not useful.

Generally, one is interested in keeping the number of unknowns as small as possible. Consequently, one prefers to introduce the nodal positions as unknowns rather than the line-forces. For this choice the dimension of the equations system is smaller by a factor 1/2, provided the net under consideration is rectangular [2]. However, if the model consists of stiff rods, a substitution of the line-forces in terms of the positions is impossible. So we have a paradoxical situation: the number of unknown variables increases, while the number of degrees of freedom decreases.

For the case of rectangular nets, we found a trick to obtain again a system with only three unknowns per node. We treat such nets as a connection of two families of lines, say rows and columns. Now, our variables are the interactions between rows and columns in the nodes. The system of equations comes from the condition that the dislocation of the nodes has to vanish, i.e. the nodal positions have to be the same if calculated for rows as for columns. The key to this approach is a reliable algorithm for the solution of single line problems. The aim of the present paper is to discuss the difficulties connected even with this apparently simple case and to describe an algorithm for the solution of the occurring type of nonlinear equations. The idea of this algorithm is to calculate the position of the last node from the position and the force acting on the first node. Then that force and latter on all other unknowns are determined from the equality between the calculated position and prescribed boundary condition. A shooting algorithm of this kind was already tested in [2], but its convergence was not satisfactory. However, in [2] a discretized version of Newton's method was used. We studied the exact Jacobian and obtained under appropriate assumptions a globally covergent method.

2. The mathematical model

We introduce a set of nodes $Y = \{y_i\}$, $i \in I$, $y_i \in R^3$. Two nodes y_i and y_j, $i \neq j$, are said to be connected, shortly

$$y_i \operatorname{con} y_j$$

if there is a line between them. The neighbourhood of the node y_i is defined as

$$U_{yi} := \{y_j \in Y : y_i \operatorname{con} y_j\}.$$

If the cardinality of all U_{yi} is not greater than two, we have a single line. Usually, for fishing nets the relation

$$|U_{yi}| \leqslant 4 \;\forall\, i \in I$$

is valid (Fig. 1).

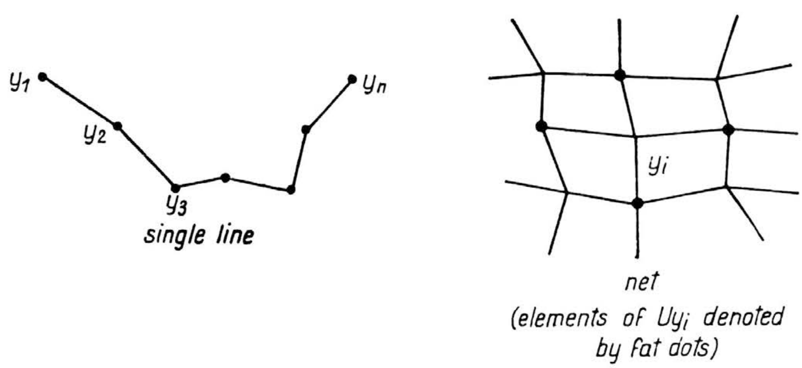

FIG. 1.

A solution of a static problem has to satisfy the following conditions:

$$(1)_1 \qquad \sum_{j: y_j \in U_{yi}} F_{ij} + E_i = 0, \quad i \in I \setminus I_0 \quad \text{(balance of forces)},$$

$$(1)_2 \qquad \|y_i - y_j\| = l_{ij}, \quad i < j, \quad y_i \text{ con } y_j \text{ (length of lines)},$$

$$(1)_3 \qquad F_{ij} = f_{ij}(y_i - y_j) = -F_{ji}$$

$$\text{(constitutive relations)},$$

$$(1)_4 \qquad f_{ij} \geqslant 0$$

$$(1)_5 \qquad y_i = g_i, \quad i \in I_0 \quad \text{(boundary conditions)}.$$

Here I_0 is the subset of I corresponding to boundary points, E_i the external force acting on the i-th node, l_{ij} the length of the line between the nodes y_i and y_j (if there is a line) and F_{ij} describes the interaction between these nodes.

It is worthwhile to note that the unknown tensions f_{ij} can be interpreted as Lagrange multipliers corresponding to the constraint $(1)_2$. Usually, for bilateral constraints there is no inequality posed on the multipliers, while non-negative f_{ij} corresponds to the unilateral constraint $\|y_i - y_j\| \leqslant l_{ij}$. For that reason the condition $(1)_4$ is frequently dropped during the calculation process and used after it as a criterion for the choice of proper solutions.

If there are no vanishing line-forces, Eqs. $(1)_1$–$(1)_5$ can be reduced to the form

$$(2) \qquad A(F)F = r(E, g).$$

Here the vector F denotes the totality of unknown forces, E the given tractions and g the prescribed boundary positions. Having solved the system (2), the remaining unknowns are easily calculated from Eqs. $(1)_2$, $(1)_3$ and $(1)_5$.

The matrix-valued function $A(F)$ occurring in the system (2) splits into a constant part for the equilibrium conditions $(1)_1$ and a variable part for the geometrical compatability $(1)_2$, $(1)_3$ and $(1)_5$, i.e.

$$(3) \qquad A(F) = \begin{pmatrix} A_c \\ \hline A_v(F) \end{pmatrix}.$$

As an example let us write down the matrix $A(F)$ for a single line clamped at both ends:

$$(3') \qquad A(F) = \left(\begin{array}{cccccc} I & -I & & & & \\ & I & -I & & 0 & \\ & 0 & I & -I & & \\ & & & \dots & I & -I \\ \hline \frac{l_1}{\|F_1\|} I & & \dots & & & \frac{l_{n-1}}{\|F_{n-1}\|} I \end{array}\right).$$

Here we used the notation

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Equation (2) reads for this case

$$(2') \qquad A(F) \quad \begin{pmatrix} E \\ \hline g_n - g_1 \end{pmatrix}.$$

Before constructing algorithms for the solution of this system of equations, let us consider what we may expect from those solutions.

3. Existence and uniqueness of solutions

If we drop the condition (1.4) and assume E to be a given constant vector, then an existence proof can be easily carried out via the construction of a potential and the argument that a continuous function on a compact set takes its bounds. Excluding special cases the solution is obviously not unique. The situation becomes much more complicated if E depends on the actual configuration, i.e.

$$E = \varepsilon(Y)$$

with a given function ε. Such a dependence occurs very often in mechanics of fishing equipment, some typical examples of functions ε are listed in [2]. Let us call a configuration $Y = (y_1, \ldots, y_n)^t$ admissible if it obeys the boundary conditions (1.5) and the line-length conditions (1.2). The set of all admissible configurations M is a smooth manifold in R^{3n-3}. Now, we can characterize the equilibrium configurations as such points of M in which the vector of forces $\varepsilon(Y)$ is perpendicular to the tangent space T_y. We denote the tangent component of the field $\varepsilon(Y)$ by $\varepsilon_t(Y)$ and assume it to be continuous. By a well-known theorem [4] the set of zeros of the field ε_t is non-empty, if the Euler characteristic of M is different from zero. For a single line modelled by $n-1$ rods and fixed at one of the ends, M is isomorphic to the product

$$\underbrace{M = S^2 \times S^2 \ldots \times S^2}_{n-1 \text{ times}}.$$

Hence its Euler characteristic equals $2^{n-1} \neq 0$. This implies the existence of solutions to Eqs. (1.1), (1.2), (1.3) and (1.5) even for the case of solution depending forces.

Unfortunately, this proof is non-constructive, and it cannot be generalized to more complicated nets or boundary conditions. Indeed, even for two rods clamped at $(0,0,0)^t$ and $(1,0,0)^t$ with $l_{12} = l_{23} \geqslant \frac{1}{2}$ and $E_2 = (0, -y_{23}, y_{22})^t$ we always have $\varepsilon_t(Y) \neq 0$, hence no equilibrium can be found (Fig. 2).

Analogously, for larger numbers of rods, such whirling forces can be introduced.

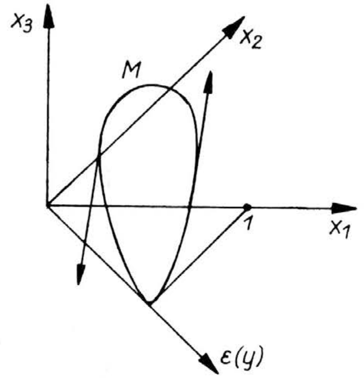

FIG. 2. Line composed of two rods, fixed at the ends.

Now the question arises whether for a non-empty set of equilibrium solutions there is always a solution satisfying the relation (1.4). The answer is negative again, as we can see by another example.

We take three rods of length 2 fixed at the same points as above. If we assume $E_2 = E_3 = mg(0,0-1)^t$, then either the middle rod or the two others are pressed in equilibrium (Fig. 3).

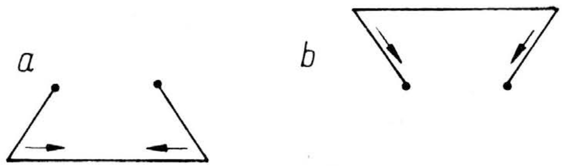

FIG. 3. Line composed of three rods.

Of course, this example has nothing to do with a real line. By refinement, taking, e.g., six rods of unit length, we obtain solutions satisfying the relation (1.4).

Finally, let us show that the condition (1.4) does not suffice to remove the non-uniqueness of solutions of Eqs. (1.1), (1.2), (1.3) and (1.5). Take two rods fixed as in Fig. 2 above. For the force $E_2 = (0, y_{22}, y_{23})^t$ each configuration is in equilibrium.

It follows from the above considerations that we need further assumptions in order to construct a globally convergent algorithm for the solution of Eq. (2′). This will become more evident in the course of the next section.

4. Single line under constant forces

If the matrix $A(F)$ is non-singular for all forces, then Eq. (2) may be transformed in fixed point form

(4) $$F = A^{-1}(F)r =: \varphi(F)$$

and hence a simple iteration can be applied. This method was exploited by several authors [1, 3], but its convergence is unsatisfactory. Other methods were tested in [2], but most of them fail to work for stiff rods or are too expensive. For the case of a single line a shooting algorithm was tested in [2]. However, this algorithm required good initial guesses which usually are not available. The aim of this section is to provide suitable modifications and appropriate assumptions in order to make the convergence of a shooting algorithm for Eq. (2′) global.

First, let us reduce Eq. (2′) to a three-dimensional system of equations. We denote $F = F_1$ and obtain from the constant block of (2)

(5) $$F_i = F - \sum_{j \leqslant i} E_j.$$

Now the variable block of Eq. (2a) takes the form

(6) $$h(F) = g$$

with

$$(7) \qquad h(F) = \sum_i \frac{l_i}{\|F - \sum_{j\leqslant i} E_j\|}\left(F - \sum_{j\leqslant i} E_j\right)$$

and $g = g_n - g_1$.

Let $S_i = \sum_{j\leqslant i} E_j$, $i = 2, 3, \dots n-1$, then the domain Dom (h) is just the open set $D = R^3 \setminus \{S_2, S_3, \dots S_{n-1}\}$. Further we see that the function h is smooth in D, the Jacobian being

$$(8) \qquad H(F) = \sum \frac{l_i}{\|F_i\|}\left(I - \frac{F_i}{\|F_i\|} \otimes \frac{F_i}{\|F_i\|}\right)$$

with F_i from Eq. (5). The application of Newton's method to Eq. (6) is thus connected with two main difficulties:

a) the iteration may lead out of the domain D,

b) the matrix $H(F)$ may be singular (at least numerically).

If one of the above situations occurs, then we want to modify Newton's method so that a new iterated in D with a smaller defect $\|h(F)-g\|^2$ is found. As a preparation we prove now the following:

LEMMA 1. The matrix $H(F)$ is positive semi-definite for all $F \in D$. It is singular iff F is parallel to all E_i, $i = 2, 3, \dots n-1$.

P r o o f. Let us consider the sum (8). In a suitable system of coordinates each of the terms has the form

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{with} \quad \alpha = \frac{l_i}{\|F_i\|} > 0.$$

Hence $H(F)$ is a sum of positive semi-definite matrixes of rank 2, and it is singular if the kernels of all these matrices are equal. On the other hand, the kernel of the i-th term in Eq. (8) is spanned by F_i. Hence $H(F)$ is singular if all forces F_i, $i = 1, 2, \dots n-1$, are parallel, and that case occurs due to Eq. (5) if $F \| E_2 \| E_3 \dots \| E_{n-1}$. As a consequence we see that $H(F)$ is always regular if not all of the external forces are parallel. Further, if the defect $\|g-h(F)\|^2$ takes a local minimum, then either F is a solution of Eq. (6) or

$$(9) \qquad g - h(F) \| F \| E_2 \| E_3 \dots \| E_{n-1}.$$

This relation indicates ill-posed situations, i.e. solution is either geometrically impossible or trivial. The first case occurs if the lines are too short, i.e. $\Sigma\, l_i < \|g\|$ or if the discre-

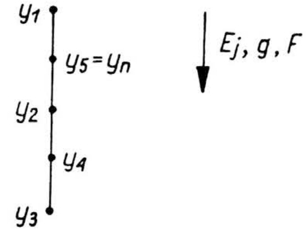

FIG. 4. Trivial solution; $n = 5$, $l_1 + l_2 = l_3 + l_4 + \|g\|$.

tization is too coarse so that there is no equilibrium without pressed rods. Trivial solutions exist if the expression (9) is valid and there is a number k, $1 \leqslant k \leqslant n-1$, such that $\sum\limits_{i\leqslant k} l_i = \sum\limits_{i>k} l_i + ||g||$ (Fig. 4).

Note that a small disturbance of the parameters l_i of this problem leads to non-existence of the solution.

The above motivates the following assumption.

A. If all external forces are parallel, then g is linearly independent of them.

Now we can formulate

LEMMA 2. If the assumption A holds, then each local minimum point of the defect $||g-h(F)||^2$ is a solution of Eq. (6). For the proof it suffices to observe that A violates the condition (9). The remark preceding the expression (9) yields the thesis.

In order to avoid the difficulty a) mentioned above, let us now extend h to the whole R^3. Unfortunately, there is no continuous extension. Thus we define

$$\tilde{h}(F, b) = \sum_{i=1}^{n-1} l_i v_i$$

with

$$v_i = \begin{cases} \dfrac{F_i}{||F_i||} & \text{for} \quad ||F_i|| \neq 0, \\ b_i & \text{else} \end{cases}$$

for $F \in R^3$, $\quad b \in B = \underbrace{S^2 \times S^2 \times \ldots \times S^2}_{n-1 \text{ times}}.$

Now we choose $\bar{h}(F)$ such that

$$||\bar{h}(F)-g|| = \inf_B ||\tilde{h}(F, b)-g||.$$

This is always possible since B is compact and $\tilde{h}(F, \cdot)$ continuous.

As a matter of course we note that

$$\bar{h}|_D = h \tag{10}$$

and that $||g-\bar{h}(F)||^2$ is lower semi-continuous.

Let us consider now the behaviour of h for increasing modulus of the force F. Since $R^3 \setminus D$ is bounded, we may restrict ourselves to h. It is geometrically obvious that

$$\lim_{M\to\infty} \inf_{||F||=M} ||h(F)-g||^2 = \Big(\sum_{i=1}^{n-1} l_i - ||g||\Big)^2 =: c^2. \tag{11}$$

Without proof we note that the following lemma 3 holds:

LEMMA 3. If $\sum\limits_{i=1}^{n-1} l_i > ||g||$ the set

$$\{F: ||g-\bar{h}(F)||^2 < c^2\}$$

is not empty.

As a straightforward consequence of the lemmas we can formulate

THEOREM 1. *If* $\Sigma\, l_i > ||g||$, *then the function* $F \mapsto ||g-\bar{h}(F)||^2$ *takes its minimum at some point* $F^* \in R^3$. *If further A is valid and* F^* *belongs to D, then* F^* *is a solution of Eq.* (6).

On the other hand it follows from our considerations that for $F^* \in D$ there are two possibilities. Either $||g-h(F^*)||^2 = 0$ and hence F^* determines a solution of the conditions $(1)_1$–$(1)_5$ which is not a solution of Eq. (2), or the conditions $(1)_1$–$(1)_5$ have no solution.

Consequently, the following assumption is useful

B. If $||g-\bar{h}(F^*)|| = \min||g-h(F)||$, then $F^* \in D$.

Now we are ready for the formulation of our modification of Newton's method. We put

$$\Delta(F,\lambda) = \begin{cases} \lambda H(F)^{-1}(g-h(F)) & \text{for} \quad |H(F)| > 0, \quad F \in D, \\ \dfrac{\lambda}{\sum\limits_{F_i \neq 0} \dfrac{l_i}{F_i}}(g-h(F)) & \text{else} \end{cases} \tag{12}$$

with F_i from Eq. (5).

If we start near a solution $F^* \in D$ with the iteration

$$F^{k+1} = F^k + \Delta(F^k, 1) \tag{13}$$

then the definition (12) leads to Newton's method. Now our aim is to study the iteration

$$F^{k+1} = F^k + \Delta(F^k, \lambda_k) \tag{14}$$

with a suitable choice of positive stepsizes λ_k. To this end we assume $\Sigma l_i > g$, A, B and, additionally,

C. $\sum\limits_{F_i=0} l_i < \left\| \sum\limits_{F_i=0} \dfrac{l_i}{||F_i||} F_i - g \right\|$.

The most crucial point is to show that $\Delta(F,\lambda)$ is a direction of descent for the defect $||g-h(F)||^2$.

THEOREM 2. *Let* $\Sigma l_i > ||g||$ *and* A, B *as well as* C *be valid. Then, for each* $F \in R^3$ *either F is a solution of Eq.* (6) *or there is a positive number* ε *such that* $0 < \lambda \leqslant \varepsilon$ *implies*

$$||\bar{h}(F+\Delta(F,\lambda))-g|| < ||\bar{h}(F)-g||.$$

P r o o f. a) Let $F \in D$. Then it is sufficient to verify

$$\Delta(F,\lambda)\nabla||g-h(F)||^2 < 0 \quad \text{for} \quad g \neq h(F).$$

But

$$\Delta(F,\lambda)\nabla||g-h(F)||^2 = \begin{cases} -2\lambda||g-h(F)||^2 & \text{if} \quad |H(F)| > 0, \\ -2\lambda \sum \dfrac{l_i}{||F_i||}(g-h(F))^t H(F)(g-h(F)) & \text{else}. \end{cases}$$

Due to assumption A the above product is indeed negative unless $g = h(F)$.

b) Let $F \notin D$. We put $I' = \{i \in I, F_i \neq 0\}$, $I'' = \{i \in I\colon F_i = 0\}$ and denote summation over I' by Σ', summation over I'' by Σ''.

Let further $h_0(F) = \Sigma' \frac{l_i}{||F||} F_i$. Then, with respect to C,

$$\bar{h}(F) = h_0(F)+\Sigma'' l_i \frac{g-h_0(F)}{||g-h_0(F)||}$$

and

$$\bar{h}(F+\delta) = h_0(F+\delta)+\Sigma'' l_i \frac{g-h_0(F)}{||g-h_0(F)||}$$

for each δ of the form $\delta = (g-h_0(F))\alpha^2$.

Denote

$$h_1 = \Sigma'' l_i \frac{g-h_0(F)}{||g-h_0(F)||}.$$

Since $\Delta(F, \lambda)$ has the above form (cf. Eq. (12)), we are through if

$$\Delta(F, \lambda)\nabla||h_0(F)+h_1-g||^2 < 0 \quad \text{for} \quad h_0(F)+h_1 \neq g.$$

But this term equals

$$2\lambda \sum{}' \frac{l_i}{||F_i||} (g-\bar{h}(F))^t H_0(g-\bar{h}(F))$$

with

$$H_0 = \sum{}' \frac{l_i}{||F_i||} \left(I - \frac{F_i}{||F_i||} \otimes \frac{F_i}{||F_i||}\right).$$

Since H_0 has the same form as H, the proof is finished.

REMARK. The choice of the factor $\frac{1}{\sum\limits_{F_i \neq 0} \frac{l_i}{F_i}} =: \beta$ in Eq. (12) is not essential for the proof and seems to be unmotivated. But it turns out that βI is a generalized inverse o H if $F \in D$ and $|H| = 0$. Indeed, in this case we have

$$\beta IH = H\beta I = I_{\mathrm{Im}H}.$$

5. Numerical results

Now we want to give some numerical examples. The calculations were carried out on a HP 9845 B. We found it to be ineffective to check the assumptions of theorems 1 and 2 before starting the algorithm. Thus we proceed with the descent algorithm for $\frac{1}{2}||\bar{h}(F)-g||^2$ until numerical convergence is indicated. Then we examine whether a solution is found or which of the assumptions is violated.

For an illustration we choose three representative configurations. In the first one all assumptions are fulfilled, in the second one we have $||g|| = \Sigma l_i$ and in the last one there is no solution because of the too coarse discretization. We expect square convergence in

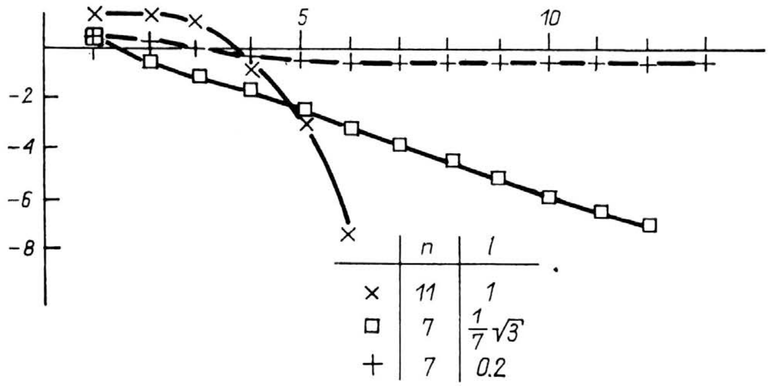

FIG. 5. For all three examples are $g = (1, 1, 1)^t$, $E_i = E = (0, -1, 0)^t$ and $l_i = l$ $(i = 1, 2, \ldots, n)$.

the first case and linear convergence else. In Fig. 5 we plotted $\log \frac{1}{2} \|\bar{h}(F)-g\|^2$ via the number of steps. Those curves affirm our expectation. The initial guess was in all examples $F^0 = (0, 0, 0)^t$. This point doesn't belong to the convergence region of Newton's method, it doesn't belong even to Dom (h). The different behaviour of the first curve outside that region and inside of it is evident.

6. Outlook

Our present interest is focussed on applications of the algorithm for a single line as a tool for the solution of more complicated net problems. As an example let us briefly discuss the case of a rectangular net. We assume

$$I = \{(p, q)\ p = 1, \ldots n_r, q = 1, \ldots n_c\}$$

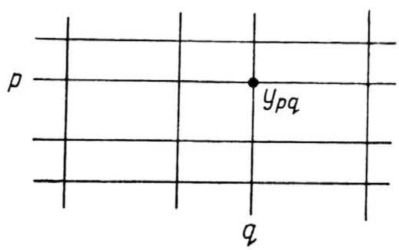

FIG. 6.

with n_r the number of rows and n_c the number of columns (Fig. 6). Let further

$$Uy_{p,q} = \{y_{\tilde{p},\tilde{q}}: |\tilde{p}-p|+|\tilde{q}-q| \leqslant 1\}.$$

Boundary conditions are given on

$$I_0 := \{(p, q)\ p = 1 \vee p = n_r, q = 1 \vee q = n_c\}.$$

For the above net, Eqs. $(1)_1$–$(1)_5$ reduce to n_r+n_c single line problems coupled by unknown nodal interactions. Let us denote by $N_{p,q}$ the nodal force in the intersection of the p-th row and the q-th column, acting on the column. We can calculate the position $y^c_{p,q}$ solving the problem (6) for the q-th column with the external forces $\frac{1}{2} E_{\tilde{p},q}+N_{\tilde{p},q}$ $(\tilde{p} = 2, 3, \ldots n_r-1)$. Analogously, we can calculate $y^r_{p,q}$ solving Eq. (6) for the p-th row with the external forces $\frac{1}{2} E_{p,\tilde{q}}-N_{p,\tilde{q}}$ $(\tilde{q} = 2, 3, \ldots n_c-1)$.

Finally, our system of equations takes the form

$$\Delta y(N) = 0 \tag{15}$$

with

$$\Delta y_{pq} := y^c_{p,q}-y^r_{p,q}.$$

Its dimension is indeed just 3 times the number of free nodes, while it usually equals 3 times the number of rods. Moreover, the assignment $N \mapsto \Delta y$ turns out to possess similar properties as $F \mapsto h$, stated in the lemmas of Sect 4. So its Jacobian is symmetric and positive semi-definite with respect to an appropriate inner product. We hope to give a discussion of an effective procedure for the solution of Eq. (15) in a forthcoming paper.

Last but not least, it should be mentioned that the considerations of this paper concerning stiff rods may be applied to elastic ones as well. If there is at least one non-stiff rod, then the matrix H will be regular in each point of D, hence our results may be strengthened in the elastic case.

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