Arch. Mech., 38, 1–2, pp. 13–24, 1986

Some existence results in stationary problems of solid mechanics with unilateral constraints for displacements and stresses

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THE PAPER deals with some class of stationary problems with unilateral constraints imposed both on displacements and stresses. The existence of solutions to such problems is discussed. The method proposed is based on the suitably constructed family of approximation sets.

W pracy rozpatruje się pewną klasę zagadnień stacjonarnych z więzami jednostronnymi nałożonymi zarówno na przemieszczenia jak i naprężenia. Zbadano problem istnienia rozwiązań tych problemów. Zaproponowano metodę opartą na odpowiednio skonstruowanej rodzinie zbiorów przybliżających.

В работе рассматривается некоторый класс стационарных задач с односторонними связями, наложенными так на перемещения, как и на напряжения. Исследована проблема существования решений этих задач. Предложен метод опирающийся на соответственно построенном семействе приближающих множеств.

1. Introduction

IN THE PAPER we are to discuss the existence of solutions to some class of stationary problems of solid mechanics with unilateral constraints imposed both on displacements and stresses. The analysis will be carried out under assumptions of the infinitesimal theory of elasticity.

Let Ω denotes a regular bounded region in the Euclidean space R^m and let $R^{(m\times m)}$ be the space of all real-valued symmetric matrixes $m\times m$. Throughout this study by $\Delta \subset R^{(m\times m)}$ a proper convex closed subset of $R^{(m\times m)}$ with $0 \in \Delta$ and int $\Delta \neq \emptyset$ will be denoted.

We are interested in problems in which the stress field T is subject to the following condition:

$$T(x) \in \Delta, \qquad x \in \Omega. \tag{1.1}$$

Independently of the stress constraints defined above there are also some restrictions imposed on the displacement field u, namely

$$u \in \Xi, \tag{1.2}$$

where Ξ is a certain convex set of all admissible displacement fields, known in every problem under consideration.

The aim of the paper is to present some existence results for problems with the constraints (1.1) and (1.2). Such problems have been encountered in the theory of slender and textile-type materials [5, 15]. The case in which only stress constraints (1.1) are taken into account (for example, the Hencky plasticity case) has been analysed by many authors, cf. [1, 6, 7, 10, 11]. For problems with displacement constraints (1.2) only, the reader is referred to [3].

We start our investigation with the assumption that

$$H_0^1(\Omega)^m = \{v \in H^1(\Omega)^m : v|_{\partial_1 \Omega} = 0\},$$

$\partial_1 \Omega$ being a part of the boundary $\partial\Omega$ with a positive $m-1$ dimensional measure, and

$$L^2(\Omega)^{(m\times m)} = \{S = (S_{ij}) : S_{ij} = S_{ji},\ S_{ij} \in L^2(\Omega), \quad i, j = 1, 2, \ldots, m\}$$

are taken as the displacement space and the stress space, respectively. Let us denote by L the operator which assigns to any displacement field v the symmetric part of its gradient ∇v, i.e.,

$$Lv = \frac{1}{2}(\nabla v + \nabla v^T).$$

Let the material properties of the body be determined by the operator K: $L^2(\Omega)^{(m\times m)} \to$ $\to L^2(\Omega)^{(m\times m)}$ which is assumed to be demicontinuous and strong monotone, i.e. it is continuous from $L^2(\Omega)^{(m\times m)}$ into $L^2(\Omega)_w^{(m\times m)}$ (1), and

$$(1.3) \qquad (KS-KT, S-T)_{L_2} \geqslant \beta\|S-T\|_{L^2}^2, \quad S, T \in L^2(\Omega)^{(m\times m)}, \quad \beta > 0,$$

$(\cdot, \cdot)_{L^2}$ and $\|\cdot\|_{L^2}$ being the inner product and the norm in the Hilbert space $L^2(\Omega)^{(m\times m)}$, respectively.

The body force b and surface traction p are assumed to lie in $L^2(\Omega)^m$ and $L^2(\Gamma)^m$, $\Gamma \equiv \partial\Omega - \partial_1 \Omega$, respectively. According to the condition (1.1) the set of all admissible stress fields will be defined by

$$(1.4) \qquad \Sigma = \{S \in L^2(\Omega)^{(m\times m)} : S(x) \in \Delta, \quad \text{a.e. } x \in \Omega\}.$$

In the sequel the set Ξ of all admissible displacement fields is assumed to be nonempty convex and closed in $H_0^1(\Omega)^m$.

Summing up, we are to deal with such problems in which the displacement field u and stress field T are subject to

$$(1.5) \qquad u \in \Xi$$

and

$$(1.6) \qquad T \in \Sigma,$$

respectively. In this case the governing relations take the form of the following system of two variational inequalities [8, 14]:

$$(1.7) \qquad \begin{aligned} &\int_\Omega \operatorname{tr}[T(Lv-Lu)]\,d\omega - \int_\Omega b\cdot(v-u)\,d\omega - \int_\Gamma p\cdot(v-u)\,ds \geqslant 0, \quad \forall v \in \Xi, \\ &\int_\Omega \operatorname{tr}[(KT-Lu)(S-T)]\,d\omega \geqslant 0, \quad \forall S \in \Sigma, \\ &\qquad u \in \Xi, \quad T \in \Sigma, \end{aligned}$$

(1) $L^2(\Omega)_w^{(m\times m)}$ denotes the space $L^2(\Omega)^{(m\times m)}$ endowed with the weak topology.

or, equivalently,

$$\int_{\Omega} \mathrm{tr}[TLv]\,d\omega + \int_{\Omega} \mathrm{tr}[KT(S-T)]\,d\omega - \int_{\Omega} \mathrm{tr}[SLu]\,d\omega$$

(1.8)
$$- \int_{\Omega} b\cdot(v-u)\,d\omega - \int_{\Gamma} p\cdot(v-u)\,ds \geqslant 0, \quad \forall v\in \Xi, \quad \forall S\in\Sigma,$$

$$u\in\Xi, \qquad T\in\Sigma.$$

Let $L^*: L^2(\Omega)^{(m\times m)} \to (H_0^1(\Omega)^m)^*$, $(H_0^1(\Omega)^m)^*$ being the dual of $H_1^0(\Omega)^m$, be the adjoint of L. It assigns to any $S\in L^2(\Omega)^{(m\times m)}$ element $L^*S\in(H_0^1(\Omega)^m)^*$ such that

$$\langle L^*S,\ v\rangle_{H^1} \overset{\mathrm{df}}{=} (S, Lv)_{L^2}, \qquad v\in H_0^1(\Omega)^m,$$

$\langle\cdot,\ \cdot\rangle_{H^1}$ being the pairing over $(H_0^1(\Omega)^m)^*\times H_0^1(\Omega)^m$. Putting

(1.9)
$$\langle f, v\rangle_{H^1} \overset{\mathrm{df}}{=} \int_{\Omega} b\cdot v\,d\omega + \int_{\Gamma} p\cdot v\,ds, \qquad v\in H_0^1(\Omega)^m,$$

the system (1.6) can be rewritten as

(1.10)
$$\begin{cases} L^*T - f\in\partial\,\mathrm{ind}_{\Xi}(u),\\ Lu-KT\in\partial\,\mathrm{ind}_{\Sigma}(T),\end{cases}$$

where $\partial\,\mathrm{ind}_{\Xi}$ and $\partial\,\mathrm{ind}_{\Sigma}$ are the subdifferentials of the indicator functions of Ξ and Σ, respectively.

The following existence result for systems of the type (1.10) has been obtained.

THEOREM 1. [8]. *Let Ξ' and Σ' be closed convex nonempty subsets of $H_0^1(\Omega)^m$ and $L^2(\Omega)^{(m\times m)}$, respectively, and let f' be an element of $(H_0^1(\Omega)^m)^*$. Define $\alpha_{(f',\Xi')}: L^2(\Omega)^{(m\times m)}\to$ $\to(-\infty,\infty]$ according to the formula*

$$\alpha_{(f',\Xi')}(S) \overset{\mathrm{df}}{=} \mathrm{ind}^*_{\Xi'}(-L^*S+f'), \qquad S\in L^2(\Omega)^{(m\times m)},$$

$\mathrm{ind}^*_{\Xi'}$ *being the conjugate of* $\mathrm{ind}_{\Xi'}$. *Suppose that $K': L^2(\Omega)^{(m\times m)}\to L^2(\Omega)^{(m\times m)}$ is demicontinuous and strong monotone and that the mapping*

$$\partial\,\mathrm{ind}_{\Sigma'}+\partial\alpha_{(f',\Xi')}$$

is maximal monotone. Then there exists a pair $(u', T')\in\Xi'\times\Sigma'$ such that

$$\begin{cases} L^*T'-f'\in-\partial\,\mathrm{ind}_{\Xi'}(u')\\ Lu'-K'T'\in\partial\,\mathrm{ind}_{\Xi'}(T')\end{cases}$$

holds.

From Theorem 1 it follows that the existence solution problem related to the system (1.10) can be reduced to the investigation of the maximal monotonicity condition for mapping

(1.11)
$$\partial\,\mathrm{ind}_{\Sigma}+\partial\alpha_{(f,\Xi)},$$

where $\alpha_{(f,\Xi)}(S)\overset{\mathrm{df}}{=}\mathrm{ind}^*_{\Xi}(-L^*S+f)$, $S\in L^2(\Omega)^{(m\times m)}$. Unfortunately, for the set Σ defined by the relation (1.4) there are many physical situations with sets Ξ and loadings f for which the mapping (1.11) is not maximal monotone (unconstrained displacements, for instance). Hence it can be easily deduced that, in general, the system (1.10) has no solutions.

in spaces $H_0^1(\Omega)^m$ and $L^2(\Omega)^{(m\times m)}$. However, as it will be shown in Sect. 2 and Sect. 3, the family $\{\Sigma_\varepsilon\}_{\varepsilon>0}$ of approximation sets to Σ may be constructed in $L^2(\Omega)^{(m\times m)}$ with the properties

1) $\Sigma_\varepsilon \supset \Sigma, \quad \varepsilon > 0,$

2) $\bigcap_{\varepsilon>0} \Sigma_\varepsilon = \Sigma,$

3) $\partial \mathrm{ind}_{\Sigma_\varepsilon} + \partial \alpha_{(f,\Xi)}$ is maximal monotone (provided that f satisfies the modified safe load condition).

So, Theorem 1 implies the existence of solutions to the system (1.10) with Σ replaced by Σ_ε, i.e.

$$
(1.12) \qquad \begin{cases} L^*T - f \in -\partial \mathrm{ind}_\Xi(u), \\ Tu - KT \in \partial \mathrm{ind}_{\Sigma_\varepsilon}(T). \end{cases}
$$

In Sect. 4, using suitable a priori estimations for the relation (1.12) and assuming among other things that Range $(\partial \mathrm{ind}_\Xi)$ is closed in $(H_0^1(\Omega)^m)^*$, the existence of solutions to the following problem

$$
(1.13) \qquad \int_\Omega \mathrm{tr}[TLv]\,d\omega + \int_\Omega \mathrm{tr}[KT(S-T)]\,d\omega - \int_\Omega \mathrm{tr}[STu] - \int_\Omega b\cdot(v-u)\,d\omega
$$

$$
-\int_\Gamma p\cdot(v\,ds - du) \geqslant 0, \quad \forall v \in \Xi, \quad \forall S \in \Sigma \cap C_0(\Omega)^{(m\times m)},
$$

$$
u \in \Xi_{BD}, \quad T \in \Sigma, \quad du \in M(\Gamma)^m,
$$

will be deduced. Above Ξ_{BD} stands for the weak (star) closedness of Ξ in $BD(\Omega)$, $BD(\Omega)$ being the space of bounded deformation, [7, 12], $C_0(\Omega)$ is the space of all continuous functions with compact support in Ω, $M(\Gamma)$ being the space of bounded measures on Γ.

This result allows us to conclude that if $BD(\Omega)$ and $L^2(\Omega)^{(m\times m)}$ are taken as the displacement space and the stress space, respectively, then it is possible to realize the constraints determined by sets of the form Ξ_{BD} and Σ in the sense of satisfying the problem (1.13).

2. Approximation sets

The aim of this Section is to construct in $L^2(\Omega)^{(m\times m)}$ the family $\{\Sigma_\varepsilon\}_{\varepsilon>0}$ of such sets that

1) $\Sigma_\varepsilon \supset \Sigma, \quad \varepsilon > 0,$

2) $\bigcap_{\varepsilon>0} \Sigma_\varepsilon = \Sigma,$

3) $\mathrm{int}\Sigma_\varepsilon \neq \emptyset, \quad \varepsilon > 0,$

4) if $T_\varepsilon \in \Sigma_\varepsilon$, $\varepsilon > 0$, and $T_\varepsilon \to T$ weakly in $L^2(\Omega)^{(m\times m)}$ as $\varepsilon \to 0$, then $T \in \Sigma$.

To this end, let us denote by ω_ε: $R^m \to R$, $\varepsilon > 0$, a function having the following properties:

$$
\omega_\varepsilon(x) \geqslant 0, \quad \omega_\varepsilon(x) = \omega_\varepsilon(-x), \quad x \in R^m,
$$

ω_ε is continuous in R^m,

$$\operatorname{supp}\omega_\varepsilon = \{x \in R^m : ||x|| \leqslant \varepsilon\},$$

$$\int_{R^m} \omega_\varepsilon(x)dx = 1, \quad \varepsilon > 0.$$

To any $S \in L^2(\Omega)^{(m\times m)}$ will be assigned $\omega_\varepsilon \ast S \in C(\bar{\Omega})^{(m\times m)}$ according to the formula

$$\omega_\varepsilon \ast S(x) \stackrel{\text{df}}{=} \int_{\Omega} S(y)\omega_\varepsilon(x-y)dy, \quad x \in \Omega.$$

It allows us to define

$$(2.1) \qquad \Sigma_\varepsilon = \{S \in L^2(\Omega)^{(m\times m)} : \omega_\varepsilon \ast S \in \Sigma\}, \quad \varepsilon > 0,$$

as a convex closed subset of $L^2(\Omega)^{(m\times m)}$.

Setting

$$\gamma_\varepsilon(x) \stackrel{\text{df}}{=} \int_{\Omega} \omega_\varepsilon(x-y)dy, \quad x \in \Omega, \quad \varepsilon > 0,$$

and

$$\gamma_\varepsilon = \inf_{x\in\Omega} \gamma_\varepsilon(x), \quad \varepsilon > 0,$$

we obtain immediately (by the regularity and the boundness of Ω):

$$(2.2) \qquad 0 < \gamma_\varepsilon < 1, \quad \varepsilon > 0.$$

PROPOSITION 1. Let π be a closed convex nonempty subset of $R^{(m\times m)}$. Define $\pi_\varepsilon = \operatorname{conv}(\gamma_\varepsilon \pi \cup \pi)$. Suppose that $S \in L^2(\Omega)^{(m\times m)}$ is such that $S(x) \in \pi$ for a.e. $x \in \Omega$. Then $\omega_\varepsilon \ast S(x) \in \pi_\varepsilon$ for every $x \in \Omega$.

Proof. We argue by the contradiction. Suppose that there exists $x_0 \in \Omega$ such that $\omega_\varepsilon \ast S(x_0) \notin \pi_\varepsilon$. Using the standard separation argument for convex sets we can find $w \in R^{(m\times m)}$ with

$$(2.3) \qquad (z, w)_{R^m} \geqslant (\omega_\varepsilon \ast S(x_0), w)_{R^m} + \nu, \quad \forall z \in \pi_\varepsilon,$$

where ν is a positive constant, $(\cdot, \cdot)_{R^m}$ denotes the inner product in $R^{(m\times m)}$. By the assumption we have $S(x) \in \pi$ for a.e. $x \in \Omega$. Thus, using the definition of π_ε and the inequalities (2.2) we get

$$\gamma_\varepsilon(x_0) S(x) \in \pi_\varepsilon, \quad \text{a.e. } x \in \Omega.$$

Hence, according to the inequalities (2.2) we easily establish that

$$\gamma_\varepsilon(x_0)\big(S(x), w\big)_{R^m} \geqslant \big(\omega_\varepsilon \ast S(x_0), w\big)_{R^m} + \nu, \quad \text{a.e. } x \in \Omega.$$

Multiplying both sides of the above inequality by $\omega_\varepsilon(x-x_0)$ and integrating over Ω we find

$$\gamma_\varepsilon(x_0)\big(\omega_\varepsilon \ast S(x_0), w\big)_{R^m} \geqslant \gamma_\varepsilon(x_0)\Big(\big(\omega_\varepsilon \ast S(x_0), w\big)_{R^m} + \nu\Big),$$

which, due to $\gamma_\varepsilon(x_0) \geqslant \gamma_\varepsilon > 0$, leads to the contradiction.

REMARK 1. If $0 \in \pi$, then $\gamma_\varepsilon \pi \subset \pi$ and, consequently, $\pi_\varepsilon = \pi$ for any $\varepsilon > 0$.

COROLLARY 1. From Proposition 1 and Remark 1 it follows immediately that

$$\Sigma_\varepsilon \supset \Sigma, \quad \varepsilon > 0.$$

Let $\tilde{\Delta} \subset \Delta$ be a closed convex nonempty subset of Δ with $\operatorname{dist}(\tilde{\Delta}, \partial\Delta) \geqslant \delta > 0$ for some positive constant δ, $\partial\Delta$ being the boundary of Δ.

PROPOSITION 2. Let $T_0 \in L^2(\Omega)^{(m\times m)}$ be such that

$$T_0(x) \in \tilde{\Delta}, \quad \text{a.e. } x \in \Omega.$$

Then

$$T_0 \in \operatorname{int}\Sigma_\varepsilon, \quad \varepsilon > 0.$$

P r o o f. Denoting by $\tilde{\Delta}_\varepsilon = \overline{\operatorname{conv}}(\gamma_\varepsilon \tilde{\Delta}_\varepsilon \cup \tilde{\Delta}_\varepsilon)$ and taking into account that $0 \in \Delta$, it can be easily verified that

$$\operatorname{dist}(\tilde{\Delta}_\varepsilon, \partial\Delta) \geqslant \gamma_\varepsilon \delta. \tag{2.4}$$

Now, for $S \in L^2(\Omega)^{(m\times m)}$ we have

$$\|\omega_\varepsilon \ast S(x) - \omega_\varepsilon \ast T_0(x)\|^2_{R^m} \leqslant M_\varepsilon^2 \|S - T_0\|^2_{L^2}, \quad x \in \Omega,$$

where

$$M_\varepsilon = \sup_{x\in\Omega} \left\{ \left(\int_\Omega \omega_\varepsilon^2(y-x)\,dy \right)^{\frac{1}{2}} \right\},$$

and $\|\cdot\|_{R^m}$ is the norm in $R^{(m\times m)}$. If $\|S - T_0\|_{L^2} \leqslant \gamma_\varepsilon \delta / M_\varepsilon$, then we get the following estimation:

$$\|\omega_\varepsilon \ast S(x) - \omega_\varepsilon \ast T_0(x)\|_{R^m} \leqslant \delta\gamma_\varepsilon, \quad x \in \Omega. \tag{2.5}$$

From Proposition 1 it follows that $\omega_\varepsilon \ast T_0(x) \in \tilde{\Delta}_\varepsilon$; hence owing to the inequalities (2.4) and (2.5) we arrive at

$$\omega_\varepsilon \ast S(x) \in \Delta, \quad x \in \Omega.$$

It leads directly to

$$S \in \Sigma_\varepsilon, \quad \varepsilon > 0,$$

which proves the assertion.

PROPOSITION 3. Let T_ε be a sequence with $T_\varepsilon \in \Sigma_\varepsilon$, $\varepsilon > 0$. Suppose that T_ε converges weakly to T as $\varepsilon \to 0$. Then $T \in \Sigma$.

P r o o f. In the first step it will be shown that $\omega_\varepsilon \ast T_\varepsilon$ converges weakly to T as $\varepsilon \to 0$. From the boundness of T_ε the boundness of $\omega_\varepsilon \ast T_\varepsilon$ can be easily deduced. As it is known in such a case it suffices to establish that

$$(\omega_\varepsilon \ast T_\varepsilon, \varphi)_{L^2} \to (T, \varphi)_{L^2}, \quad \text{as} \quad \varepsilon \to 0,$$

holds for any $\varphi \in C_0(\Omega^{(m\times m)}$, $C_0(\Omega)$ being the space of continuous functions with compact support in Ω. For $\varphi \in C_0(\Omega)^{(m\times m)}$ we have

$$\begin{aligned}(\omega_\varepsilon \ast T_\varepsilon, \varphi)_{L^2} &= \int_\Omega \operatorname{tr}\Big[\varphi(x) \int_\Omega T_\varepsilon(y)\omega_\varepsilon(x-y)\,dy\Big]dx \\ &= \int_\Omega \operatorname{tr}\Big[T_\varepsilon(y) \int_\Omega \varphi(x)\omega_\varepsilon(y-x)\,dx\Big]dy = \int_\Omega \operatorname{tr}[T_\varepsilon(y)\omega_\varepsilon \ast \varphi(y)]\,dy \\ &= \int_\Omega \operatorname{tr}[T_\varepsilon(y)\,(\omega_\varepsilon \ast \varphi(y) - \varphi(y))]\,dy + \int_\Omega \operatorname{tr}[T_\varepsilon(y)\varphi(y)]\,dy.\end{aligned}$$

Using the estimation

$$\left|\int_{\Omega} \operatorname{tr}\left[T_\varepsilon(y)\left(\omega_\varepsilon \ast \varphi(y)-\varphi(y)\right)\right] dy\right| \leqslant \|T_\varepsilon\|_{L^2}\|\omega_\varepsilon \ast \varphi-\varphi\|_{L^2}$$

and taking into account

$$\|\omega_\varepsilon \ast \varphi-\varphi\|_{L^2} \to 0, \quad \text{as} \quad \varepsilon \to 0,$$

we conclude that

$$(\omega_\varepsilon \ast T_\varepsilon, \varphi)_{L^2} \to (T, \varphi)_{L^2}, \quad \text{as} \quad \varepsilon \to 0.$$

Further, from $T_\varepsilon \in \Sigma_\varepsilon$ we get $\omega_\varepsilon \ast T_\varepsilon \in \Sigma$. Since Σ is closed and convex it is weakly closed. Thus T as a weak limit of $\omega_\varepsilon \ast T_\varepsilon \in \Sigma$ has to belong to Σ. The proof is complete.

COROLLARY 2. From Corollary 1 and Proposition 3 we obtain immediately

$$\bigcap_{\varepsilon>0} \Sigma_\varepsilon = \Sigma.$$

Due to (1–4) elements of the family $\{\Sigma_\varepsilon\}_{\varepsilon>0}$ may be regarded as approximation sets to Σ. Their further important properties will be given in the next Section.

3. Approximation problems and a priori estimates

Let us consider the problem consisting in finding a pair (u, T) such that the relation (1.10) holds with Σ replaced by Σ_ε, $\varepsilon > 0$, i.e.,

$$\left\{\begin{array}{l} L^*T-f \in -\partial \operatorname{ind}_{\Xi}(u), \\ Lu-KT \in \partial \operatorname{ind}_{\Sigma_\varepsilon}(T). \end{array}\right. \tag{3.1}$$

The analysis will be carried out under the following hypothesis concerning f given by the relation (1.9)

(H.1) **Safe load condition.** There exist $u_0 \in \Xi$, a closed convex $\tilde{\Delta} \subset \Delta$ with $\operatorname{dist}(\tilde{\Delta}, \partial\Delta) \geqslant \delta > 0$ and $T_0 \in L^2(\Omega)^{(m\times m)}$ such that

(i) $L^*T_0-f \in -\partial \operatorname{ind}_{\Xi}(u_0)$,

or, equivalently,

$$\int_{\Omega} \operatorname{tr}[T_0(Lv-Lu_0)]d\omega - \int_{\Omega} b(v-u_0)d\omega - \int_{\Gamma} p(v-u)ds \geqslant 0, \quad \forall v \in \Xi,$$

(ii) $T_0(x) \in \tilde{\Delta}$ for a.e., $x \in \Omega$.

PROPOSITION 4. Under Hypothesis (H.1) mapping

$$\partial \operatorname{ind}_{\Sigma_\varepsilon} + \partial\alpha_{(f,\Xi)}, \quad \varepsilon > 0, \tag{3.2}$$

is maximal monotone.

P r o o f. From (i) of (H.1) it follows that

$$-L^*T_0+f \in \partial \operatorname{ind}_{\Xi}(u_0) \Leftrightarrow u_0 \in \partial \operatorname{ind}^*_{\Xi}(-L^*T_0+f) \Rightarrow -Lu \in \partial\alpha_{(f,\Xi)}(T_0). \tag{3.3}$$

On the other hand, (ii) of (H.1) implies by Proposition 2 that

$$T_0 \in \operatorname{int}\Sigma_\varepsilon, \quad \varepsilon > 0,$$

and, consequently,

(3.4) $$T_0 \in \operatorname{int}[\operatorname{dom}(\partial \operatorname{ind}_{\Sigma_\varepsilon})], \quad \varepsilon > 0,$$

where "dom" is used to denote an effective domain. Combining the relations (3.3) and (3.4), we obtain

$$\operatorname{dom}(\partial \alpha_{(f,\Xi)}) \cap \operatorname{int}[\operatorname{dom}(\partial \operatorname{ind}_{\Sigma_\varepsilon})] \neq \emptyset, \quad \varepsilon > 0,$$

which implies the maximal monotonicity of (3.2), [2].

Theorem 1 and Proposition 4 lead to the following existence result.

THEOREM 2. *Suppose that K is demicontinuous and strong monotone from $L^2(\Omega)^{(m\times m)}$ into $L^2(\Omega)^{(m\times m)}$, Ξ is a closed convex nonempty subset of $H_0^1(\Omega)^m$. Moreover, let Σ_ε, $\varepsilon > 0$, be given by the relation* (2.1) *and let Hypothesis* (H.1) *hold. Then there exists a pair $(u, T) \in \Xi \times \Sigma_\varepsilon$ satisfying the relation* (3.1).

Let us denote by $(u_\varepsilon, T_\varepsilon) \in \Xi \times \Sigma_\varepsilon$, $\varepsilon > 0$, a solution of (3.1). It means that the following relations have to hold:

(3.5) $$\begin{aligned} &(T_\varepsilon, Lv - Lu_\varepsilon)_{L^2} - \langle f, v - u_\varepsilon \rangle_{H^1} \geqslant 0, \quad \forall v \in \Xi, \\ &(KT_\varepsilon, S - T_\varepsilon)_{L^2} - (Lu_\varepsilon, S - T_\varepsilon)_{L^2} \geqslant 0, \quad \forall S \in \Sigma_\varepsilon, \end{aligned}$$

or, equivalently,

(3.6) $$\int_\Omega \operatorname{tr}[T_\varepsilon Lv]\, d\omega + \int_\Omega \operatorname{tr}[KT_\varepsilon(S - T_\varepsilon)]\, d\omega - \int_\Omega \operatorname{tr}[S\, Lu_\varepsilon]\, d\omega - \int_\Omega b \cdot (v - u_\varepsilon)\, d\omega - \int_\Gamma p \cdot (v - u_\varepsilon)\, ds \geqslant 0, \quad \forall v \in \Xi, \quad \forall S \in \Sigma_\varepsilon.$$

Now, let us pass to a priori estimations for the relation (3.1).

PROPOSITION 5. There exists a positive constant C, not depending on ε, such that

(3.7) $$\|T_\varepsilon\|_{L^2} \leqslant C, \quad \varepsilon > 0.$$

P r o o f. Putting $S = T_0$ and $v = u_0$ in the relations (3.5) (u_0 and T_0 being the same as in (H.1)), we get

$$(KT_\varepsilon, T_0 - T_\varepsilon)_{L^2} - (T_0 - T_\varepsilon, Lu_0)_{L^2} - \langle -L^* T_0 + f - (-L^* T_\varepsilon + f), u_0 - u_\varepsilon \rangle_{H^1} \geqslant 0,$$

which, by the monotonicity of $\partial \operatorname{ind}_\Xi$, yields

$$(KT_\varepsilon, T_0 - T_\varepsilon)_{L^2} - (T_0 - T_\varepsilon, Lu_0)_{L^2} \geqslant 0.$$

This inequality together with the relations (1.3) amount to

$$(KT_0, T_0 - T_\varepsilon)_{L^2} - (T_0 - T_\varepsilon, Lu_0)_{L^2} \geqslant \beta \|T_0 - T_\varepsilon\|_{L^2}^2.$$

Hence we easily obtain the boundness of T_ε.

Further investigations will be continued under the following stronger assumption related to K:

(H.2) K is a demicontinuous strong monotone operator from $L^2(\Omega)^{(m\times m)}$ into $L^2(\Omega)^{(m\times m)}$ mapping bounded sets into bounded sets.

PROPOSITION 6. Let us suppose that (H.1) and (H.2) hold. Then there exists a positive constant C not depending on ε, such that

$$||Lu_\varepsilon||_{L^1} \leqslant C, \quad \varepsilon > 0, \tag{3.8}$$

where $||\cdot||_{L^1}$ is the norm in the space $L^1(\Omega)^{(m\times m)}$, given by

$$||E||_{L^1} = \int\limits_\Omega \mathrm{tr}[E\,\mathrm{sgn}\,E]d\omega, \quad E \in L^1(\Omega)^{(m\times m)}.$$

P r o o f. Since $T_0(x) \in \tilde{\Delta}$ for a.e. $x \in \Omega$ and $\mathrm{dist}(\tilde{\Delta}, \partial\Delta) \geqslant \delta > 0$, so seeing that

$$||\mathrm{sgn}\,Lu_\varepsilon(x)||_{R^m} \leqslant m, \quad \text{a.e. } x \in \Omega,$$

we get

$$T_0(x) + \frac{\delta}{m}\,\mathrm{sgn}\,Lu_\varepsilon(x) \in \Delta, \quad \text{a.e. } x \in \Omega.$$

It implies

$$S_\varepsilon = T_0 + \frac{\delta}{m}\,\mathrm{sgn}\,Tu_\varepsilon \in \Sigma \subset \Sigma_\varepsilon, \quad \varepsilon < 0.$$

Putting $S = S_\varepsilon$ in the inequality $(3.5)_2$ the following estimation can be easily obtained:

$$||Lu_\varepsilon||_{L^1} \leqslant \frac{m}{\delta}||KT_\varepsilon||_{L^2}||T_0 - T_\varepsilon||_{L^2} + \frac{m}{\delta}||Tu_0||_{L^2}||T_0 - T_\varepsilon||_{L^2} + m||KT_\varepsilon||_{L^2}\,\mathrm{mes}(\Omega)^{\frac{1}{2}}.$$

By Proposition 5 and Hypothesis (H.2) this implies the desired estimation. The proof is complete.

4. General problem

Our purpose in this Section is to answer the following two questions. What kind of displacement constraints can be realized in $BD(\Omega)$ if the stress field is subject to Σ given by the relation (1.4)? In what sense is a solution of the problem under consideration understood? The second question arises from the fact that

$$\int\limits_\Omega \mathrm{tr}[TLu]$$

may be not well defined for $u \in BD(\Omega)$ and $T \in L^2(\Omega)^{(m\times m)}$.

We restrict ourselves to such sets Ξ that the following hypothesis holds:

(H.3) Ξ is a closed convex nonempty subset of $H_0^1(\Omega)^3$ with

$$\mathrm{Range}(\partial\,\mathrm{ind}_\Xi)$$

closed in $(H_0^1(\Omega)^3)^*$.

For instance, an arbitrary closed convex cone satisfies (H.3).

PROPOSITION 7. Let the inequalities (3.5) be satisfied. Then under Hypotheses (H.2) and (H.3) there exists $T \in \Sigma$ such that

$$\begin{cases} T_{\varepsilon_n} \to T & \text{weakle in} \quad L^2(\Omega)^{)m\times m)} \quad \text{as} \quad \varepsilon_n \to 0,\\ KT_{\varepsilon_n} \to KT & \text{weakle in} \quad L^2(\Omega)^{(m\times m)} \quad \text{as} \quad \varepsilon_n \to 0,\\ \lim\limits_{\varepsilon_n \to 0}(KT_{\varepsilon_n}, T_{\varepsilon_n})_{L^2} = (KT, T)_{L^2} \end{cases} \tag{4.1}$$

for some subsequence ε_n of ε with $\varepsilon_n \to 0$. If in addition K is linear and potential and Ξ is a cone, then in fact

$$T_{\varepsilon_n} \to T \quad \text{strongly in} \quad L^2(\Omega)^{(m\times m)} \quad \text{as} \quad \varepsilon_n \to 0. \tag{4.2}$$

P r o o f. From Proposition 5 the boundness of T_ε follows, which allows us to choose a subsequence ε_n, $\varepsilon_n \to 0$, such that T_{ε_n} converges weakly to some $T \in L^2(\Omega)^{(m\times m)}$ as $\varepsilon_n \to 0$. Since $T_{\varepsilon_n} \in \Sigma_{\varepsilon_n}$, so by Proposition 3 we get $T \in \Sigma$. Further, using the inequality $(3.5)_1$ and taking into account (H.3) we can easily deduce that

$$-L^*T+f \in \partial\,\mathrm{ind}_\Xi(\bar{u}) \tag{4.3}$$

for some $\bar{u} \in \Xi$. Putting $S = T \in \Sigma \subset \Sigma_\varepsilon$ in the inequality $(3.5)_2$ and utilizing the relation (4.3) we arrive at the following inequality:

$$(KT_{\varepsilon_n}, T-T_{\varepsilon_n})_{L^2} - (T-T_{\varepsilon_n}, L\bar{u})_{L^2} \geqslant 0. \tag{4.4}$$

Proposition 5 and Hypothesis (H.2) amount to the boundness of KT_{ε_n}. Thus we can extract from KT_{ε_n} a subsequence (again denoted by KT_{ε_n}) which converges weakly to some $Z \in L^2(\Omega)^{(m\times m)}$. Passing to the limit in the relation (4.2) we obtain immediately

$$\limsup_{\varepsilon_n\to 0}\,(KT_{\varepsilon_n}, T_{\varepsilon_n}) \leqslant (Z, T)_{L^2}.$$

Hence the known argument for maximal monotone mappings yields the relations (4.1), cf. [2]. Now, let us pass to the proof of the relation (4.2). To this end let W and Ξ^* denote the potential for K and the polar cone of Ξ, respectively. It is easy to check that

$$W(S) = \frac{1}{2}(KS, S)_{L^2} \overset{\mathrm{df}}{=} ||S||_K^2, \qquad S \in L^2(\Omega)^{(m\times m)},$$

$$\alpha_{(f,\Xi)} = \mathrm{ind}_A, \qquad A = \{S \in L^2(\Omega)^{(m\times m)}: -L^*S+f \in \Xi^*\},$$

where $||\cdot||_K$ is the norm generated by K, equivalent to the usual norm $||\cdot||_{L^2}$. From the inequalities (3.5) we deduce that, [9], T_{ε_n} is a solution of the following minimization problem:

$$\inf_{S\in A\cap\Sigma_{\varepsilon_n}} \{||S||_K^2\}.$$

Hence, using Corollary 1 and Hypothesis (H.3) we get

$$||T_{\varepsilon_n}||_K^2 \leqslant ||T||_K^2.$$

Taking into account the lower semicontinuity of $||\cdot||_K^2$, we arrive at

$$\liminf_{\varepsilon_n\to 0} ||T_{\varepsilon_n}||_K^2 \geqslant ||T||_K^2.$$

It implies the following condition:

$$\lim_{\varepsilon_n\to 0} ||T_{\varepsilon_n}||_K^2 = ||T||_K^2,$$

which is equivalent to

$$\lim_{\varepsilon_n\to 0} ||T_{\varepsilon_n}||_{L^2} = ||T||_{L^2}.$$

This result together with the weak convergence of T_{ε_n} to T leads to the relation (4.2). It ends the proof.

To investigate the displacement problem let us denote by Ξ_{BD} the weak (star) closedness of Ξ in $BD(\Omega)$([2]). From Proposition 6 the boundness of u_ε in $BD(\Omega)$ follows and hence the boundness of traces $\gamma(u_\varepsilon)$ in $L^1(\Gamma)^m$ as well, [12]. Now, using the known compactness results cf. [12], we obtain.

PROPOSITION 8. Let the inequalities (3.5) be satisfied. Then there exist $u \in \Xi_{BD}$ and $du \in M(\Gamma)^m$ such that

$$\begin{aligned} u_{\varepsilon_k} &\to u \quad \text{weakly in} \quad L^{\frac{m}{m-1}}(\Omega)^m, \\ Tu_{\varepsilon_k} &\to Lu \quad \text{weakly (star) in} \quad M_1(\Omega)^{(m\times m)}, \quad \text{as} \quad \varepsilon_k \to 0, \\ \gamma(u_{\varepsilon_k})ds &\to du \quad \text{weakly (star) in} \quad M(\Gamma)^m \end{aligned}$$

for some subsequence u_{ε_k} of u_ε.

Note that without losing the generality we can assume that the subsequences ε_n and ε_k coincide.

In the sequel we assume

(H.4) Body force $b \in L^m(\Omega)^m$ and surface traction $p \in C(\Gamma)^m$.

Our main result may be formulated as follows:

THEOREM 3. *Let us suppose that Hypotheses* (H.1)–(H.4) *hold. Then there exist* $u \in \Xi_{BD}$ *and* $T \in \Sigma$ *such that*

$$(4.5) \quad \int_\Omega \mathrm{tr}[TLv]d\omega + \int_\Omega \mathrm{tr}[KT(S-T)]d\omega - \int_\Omega \mathrm{tr}[SLu] - \int_\Omega b(v-u)d\omega - \int_\Gamma p(v\,ds - du) \geqslant 0, \quad \forall v \in \Xi, \quad \forall S \in \Sigma \cap C_0(\Omega)^{(m\times m)}.$$

If $\partial\Omega$ *is a* C^1 *— manifold and* Ω *is locally on one side of* $\partial\Omega$, *then the above inequalities can be replaced by*

$$\int_\Omega \mathrm{tr}[TLv]d\omega + \int_\Omega \mathrm{tr}[KT(S-T)]d\omega - \int_\Omega \mathrm{tr}[SLu] - \int_\Gamma Sn\big(du-\gamma(u)\big)ds - \int_\Omega b(u-v)d\omega - \int_\Gamma p(v\,ds-du) \geqslant 0,$$

$$\forall v \in \Xi, \quad \forall S \in C^1(\bar{\Omega})^{(m\times m)} \cap \Sigma.$$

where n *is the unit outward normal on* Γ.

Proof of Theorem 3 follows immediately from the inequality (3.6), from Propositions 7 and 8 and from the generalized Green's formula, [12].

REMARK 2. If $u \in \Xi_{BD}$ and $T \in \Sigma$ are smooth enough that means if

$$\int_\Omega \mathrm{tr}[TLu]$$

is well defined, then the inequality (4.2) is equivalent to the following variational inequalities:

$$(4.6) \quad \begin{aligned} &\int_\Omega \mathrm{tr}[T(Lv\,d\omega - Lu)] - \int_\Omega b\cdot(v-u)d\omega - \int_\Gamma p\cdot(v\,ds-du) \geqslant 0, \\ &\int_\Omega \mathrm{tr}[KT(S-T)]d\omega - \int_\Omega \mathrm{tr}[(S-T)Lu] \geqslant 0, \end{aligned}$$

which have to hold for any $v \in \Xi$ and any $S \in \Sigma \cap C_0(\Omega)^{(m\times m)}$, respectively.

(2) A sequence v_n of $BD(\Omega)$ converges weak (star) to v iff $v_n \to v$ strongly in $L^1(\Omega)^m$ and $Lv_n \to Lv$ weakly (star) in $M_1(\Omega)^{(m\times m)}$, [12].

According to Theorem 3, it is possible to realize in $BD(\Omega)$ and $L^2(\Omega)^{m\times m}$ constraints determined by sets of the form Ξ_{BD} and Σ, in the sense of satisfying the inequality (4.2). In the particular case, if the solutions are smooth enough, then the inequality (4.2) can be separated into two variational inequalities (4.3) — one corresponding to the condition of equilibrium and the other related to the constitutive relation.

From the proof of Theorem 2, it follows that the mapping (3.1) can be regarded as an approximation problem to the inequality (4.2). This problem via Propositions 7 and 8 leads to the inequality (4.5). There is another way to get the inequality (4.5). It consists in replacing in the relations (3.1) of the mapping $\partial \mathrm{ind}_{\Sigma_\varepsilon}$ by the Yosida approximation of $\partial \mathrm{ind}_{\Sigma_\varepsilon}$, $\frac{1}{\varepsilon}(I-\mathrm{Proj}_\Sigma)$. However, this method leads only to the weak convergence of T_{ε_n} to T in the case when Ξ is a cone and K is linear and potential.

In particular, for the Hencky plasticity problem (Ξ coincides with $H_0^1(\Omega)^m$) the strong convergence of T_{ε_n} to T has been obtained.

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Received October 8, 1984.